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A CONTRIBUTION TO THE THEORY OF FINITE DIFFERENCES.

By Jos. Kaucký.

This paper contains a short demonstration of equation (8) which was treated by H. L. Smith in his paper "On the Ampère-Cauchy derived functions".*

Consider a function $f(x)$ of a single variable which we suppose has a derivative of order n. Let $x_0, x_1, \cdots x_n$ be numbers differing from each other and let us form the following sequence of functions:

$$(1)\qquad \begin{aligned} &[x_0] = f(x_0), \qquad [x_1] = f(x_1), \cdots \\ &[x_0\,x_1] = \frac{[x_0]-[x_1]}{x_0-x_1}, \qquad [x_1\,x_2] = \frac{[x_1]-[x_2]}{x_1-x_2}, \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &[x_0\,x_1\cdots x_n] = \frac{[x_0\,x_1\cdots x_{n-1}]-[x_1\,x_2\cdots x_n]}{x_0-x_n}. \end{aligned}$$

The expression $[x_0\,x_1\cdots x_n]$ may be written in the form of an integral. First it is evident that

$$[x_0\,x_1] = \int_0^1 dt_1\, f'[(1-t_1)\,x_0 + t_1\,x_1];$$

then by complete induction we get easily the general relation

$$(2)\qquad \begin{aligned} [x_0\,x_1\cdots x_n] = &\int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \\ &\int_0^{t_{n-1}} dt_n\, f^{(n)}[(1-t_1)\,x_0 + \cdots + (t_{n-1}-t_n)\,x_{n-1} + x_n\,t_n]. \end{aligned}$$

Suppose we are given n numbers $w_1, w_2, \cdots w_n$ and form the differences

$$\underset{w_1}{\Delta} f(x) = \frac{f(x+w_1)-f(x)}{w_1} \qquad (w_1 \neq 0)$$

and in general

$$(3)\qquad \underset{w_1 w_1 \cdots w_n}{\overset{n}{\Delta}} f(x) = \underset{w_n}{\Delta}\left(\underset{w_1 w_1 \cdots w_{n-1}}{\overset{n-1}{\Delta}} f(x)\right).$$

From this definition there follows first

$$\underset{w_1}{\Delta} f(x) = \int_0^1 dt_1\, f'(x + w_1\,t_1)$$

* These Annals, vol. 25 (1924), p. 124.

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and then by induction,

$$(4)\qquad \overset{n}{\underset{w_1 w_2 \cdots w_n}{\Delta}} f(x) = \int_0^1 dt_1 \int_0^1 dt_2 \cdots \int_0^1 dt_n f^{(n)}(x + w_1 t_1 + w_2 t_2 + \cdots w_n t_n).$$

Let us choose

$$(5)\qquad x_0 = x, \quad x_1 = x + w_{i_1}, \quad x_2 = x + w_{i_1} + w_{i_2}, \cdots, x_n = x + w_{i_1} + \cdots + w_{i_n},$$

where $i_1, i_2, \cdots i_n$ is a permutation of the numbers $1, 2, \cdots, n$; then

$$(1-t_1)x_0 + (t_1 - t_2)x_1 + \cdots + (t_{n-1} - t_n)x_{n-1} + t_n x_n = x + w_{i_1} t_1 + w_{i_2} t_2 + \cdots w_{i_n} t_n.$$

We shall now show that we can easily calculate the sum

$$(6)\qquad \sum_{i_1 i_2 \cdots i_n} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n f^{(n)}(x + w_{i_1} t_1 + w_{i_2} t_2 + \cdots + w_{i_n} t_n),$$

where the summation extends over all permutations of $1, 2, \cdots n$. We see first, by re-arranging the argument of the integrand, that the sum (6) goes over into

$$\sum_{i_1 i_2 \cdots i_n} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n f^{(n)}(x + w_1 t_{i_1} + w_2 t_{i_2} + \cdots + w_n t_{i_n}),$$

from which we obtain by the substitution $t_{i_r} = t'_r$ $(r = 1, 2, \cdots n)$

$$(7)\qquad \int dt_1 \int_{\Omega} dt_2 \cdots \int dt_n f^{(n)}(x + w_1 t_1 + w_2 t_2 + \cdots + w_n t_n)$$

where

$$\Omega \equiv \sum_{i_1 i_2 \cdots i_n} \text{region: } 0 \leqq t_{i_1} \leqq t_{i_2} \leqq \cdots \leqq t_{i_n} \leqq 1.$$

Consequently Ω is a n-dimensional unit cube and (7) equals

$$\int_0^1 dt_1 \int_0^1 dt_2 \cdots \int_0^1 dt_n f^{(n)}(x + w_1 t_1 + \cdots + w_n t_n).$$

We have therefore proved the relation

$$(8)\qquad \overset{n}{\underset{w_1 w_2 \cdots w_n}{\Delta}} f(x) = \sum_{i_1 i_2 \cdots i_n} [x_0 x_1 \cdots x_n],$$

where $x_0, x_1, \cdots x_n$ are given by (5).

Brno (Č. S. R.), October 1924.

ON THE ASYMPTOTIC EXPRESSIONS OF CERTAIN DEFINITE INTEGRALS.*

By J. A. Shohat (Jacques Chokhate).

Introduction. We consider in this paper integrals of the form $\int_0^1 x^n f(x)\,dx$ for n very large. This type of integral occurs very frequently. We have, for example, the following developments:

(a) $$\int_0^1 \frac{f(y)}{x-y}\,dy = \sum_0^\infty{}_n \frac{c_n}{x^{n+1}}, \qquad c_n = \int_0^1 x^n f(x)\,dx;$$

(b) $$\int_0^1 \frac{f(y)\,dy}{\sqrt{1-yt}} = \sum_0^\infty{}_n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\,c_n\,t^n.$$

The development (a) with $f(y) \geqq 0$ in (0,1) is closely connected with the theory of algebraic continued fractions and Tchebycheff's polynomials.

Darboux's method† fails to give the asymptotic expression for c_n. P. Funk‡ assuming that $f(x)$ *is continuous in* $(0, 1)$ gets

$$\lim_{n=\infty} n\int_0^1 f(x)\,x^n\,dx = f(1),$$

which expression says nothing, if $f(1) = 0$.

We give here the asymptotic expression of the integral of the form $\int_0^1 f(x)\,x^n\,dx$ under more general conditions. We apply our formulas to some definite integrals connected with the Γ-function and Tchebycheff's polynomials.

1. Consider a function $f(x)$ defined for $0 \leqq x \leqq 1$ and subject to following conditions:

(A_1) $\int_0^1 x^k |f(x)|\,dx$ exists for a certain value of $k > 0$;

(A_2) $f(x) = (1-x)^\sigma q(x)$ for $1-\delta \leqq x \leqq 1$ $\qquad (\sigma > -1)$,

where $\delta > 0$ is arbitrarily small, $q(x)$ being continuous in $(1-\delta, 1)$. Then we have, as $n \to \infty$:

* Presented to the American Mathematical Society, April 19, 1924.

† Darboux, *Mémoire sur l'approximation des fonctions de grands nombres*, Journ. des Math., 3. sér. (1878), 5–57.

‡ P. Funk, *Beiträge zur Theorie der Kugelfunktionen*, Math. Ann., Bd. 77 (1916), 136–152.

$$\text{(A)}\qquad \begin{cases} \lim_{n=\infty} n^{\sigma+1}\int_0^1 x^n f(x)\,dx = \Gamma(\sigma+1)\,q(1), \quad \text{i. e.} \\ c_n = \int_0^1 x^n f(x)\,dx = \dfrac{\Gamma(\sigma+1)\,q(1)+o(1)}{n^{\sigma+1}}. \end{cases}$$

Proof. Write c_n as follows:

$$(1)\qquad c_n = \int_0^{1-n^{-2/3}} x^n f(x)\,dx + \int_{1-n^{-2/3}}^1 x^n f(x)\,dx = i_1 + i_2.$$

Take n so large, that (using conditions $A_{1,2}$)

$$(2)\qquad n > k, \qquad n^{-2/3} < \delta.$$

The first inequality (2) assures the existence of i_1. Furthermore,

$$n^{\sigma+1}|\,i_1\,| < n^{\sigma+1}\int_0^{1-n^{-2/3}} x^{n-k}x^k\,|f(x)|\,dx < n^{\sigma+1}(1-n^{-2/3})^{n-k}\int_0^1 x^k\,|f(x)|\,dx,$$

$$(3)\qquad n^{\sigma+1}\,|\,i_1\,| < \tau\,(1-n^{-2/3})^n\,n^{\sigma+1},$$

where τ does not depend upon n.
Therefore,

$$(4)\qquad \lim_{n=\infty} n^{\sigma+1}\,i_1 = 0.$$

In order to investigate i_2, we write i_2, using $(A_2; 2)$, as follows:

$$i_2 = \int_{1-n^{-2/3}}^1 x^n\,(1-x)^\sigma\,q(1)\,dx + \int_{1-n^{-2/3}}^1 x^n\,(1-x)^\sigma\,[q(x)-q(1)]\,dx,$$

$$(5)\qquad i_2 = [q(1)+o(1)]\,i_2',$$

$$(6)\qquad i_2' = \int_{1-n^{-2/3}}^1 x^n\,(1-x)^\sigma\,dx.$$

The substitution $1-x = \dfrac{y}{n}$ gives (replacing y by x):

$$(7)\qquad n^{\sigma+1}\,i_2' = \int_0^{n^{1/3}} x^\sigma\,e^{-x}\,\psi_n(x)\,dx$$

$$(8)\qquad \psi_n(x) = e^x\left(1-\frac{x}{n}\right)^n.$$

We note, that $\psi_n'(x)$ is negative for the values of x under consideration. Therefore,

$$\left.\begin{aligned} e^{n^{1/3}}(1-n^{-2/3})^n = \psi_n(n^{1/3}) \leqq \psi_n(x) \leqq \psi_n(0) = 1 \\ 1+o(1) \leqq \psi_n(x) \leqq 1 \end{aligned}\right\} 0 \leqq x \leqq n^{1/3}.$$

We get from (7):

$$[1+o(1)]\int_0^{n^{1/8}} x^\sigma e^{-x}\,dx < n^{\sigma+1} i_2' < \int_0^{n^{1/8}} x^\sigma e^{-x}\,dx.$$

On the other hand,

$$\int_0^{n^{1/8}} x^\sigma e^{-x}\,dx = \int_0^\infty - \int_{n^{1/8}}^\infty = \Gamma(\sigma+1)+o(1);$$

therefore,

$$\lim_{n=\infty} \frac{n^{\sigma+1} i_2'}{\Gamma(\sigma+1)} = 1. \tag{9}$$

Now we have according to (1,5) $n^{\sigma+1} c_n = n^{\sigma+1} i_1 + n^{\sigma+1} i_2' [q(1)+o(1)]$, which, combined with (4,9), proves (A).

It ought to be noted, that in the proof above we use two facts only:

$$e = \lim_{\alpha\to 0} (1+\alpha)^{1/\alpha},$$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx \qquad (s>0).$$

2. Replacing in (A) x by $1-x$, we get

$$\lim_{n=\infty} n^{\sigma+1} \int_0^1 (1-x)^n f(x)\,dx = \Gamma(\sigma+1) q(0), \tag{B}$$

provided the following conditions are satisfied:

(B_1) $\int_0^1 (1-x)^k |f(x)|\,dx$ exists for a certain $k>0$;

(B_2) $f(x) = x^\sigma q(x)$ for $0 \leqq x \leqq \delta$ $\qquad (\sigma > -1)$,

$q(x)$ being continuous in the arbitrarily small interval $(0, \delta)$. Replacing in (A) x by e^{-x}, we get similarly

$$\lim_{n=\infty} n^{\sigma+1} \int_0^\infty e^{-nx} f(x)\,dx = \Gamma(\sigma+1) q(0), \tag{C}$$

provided $f(x)$ is subject to the following conditions:

(C_1) $\int_0^\infty e^{-kx} |f(x)|\,dx$ exists for a certain $k>0$;

(C_2) $f(x) = x^\sigma q(x)$ for $0 \leqq x \leqq \delta$ $\qquad (\sigma > -1$; see $B_2)$.

The case of an interval $(0, a)$ $(a>0$ finite) can be reduced by means of the substitution $x = ay$ to (A). We get

$$\text{(D)} \qquad \lim_{n=\infty} \frac{n^{\sigma+1}}{a^{n+1}} \int_0^a x^n f(x)\,dx = \Gamma(\sigma+1)\,q(a)\,a^{\sigma},$$

provided $f(x)$ is subject to following conditions:

$$\text{(D}_1\text{)} \qquad \int_0^a |f(x)|\,x^k\,dx \text{ exists for a certain } k>0;$$

$$\text{(D}_2\text{)} \qquad f(x) = (a-x)^{\sigma}\,q(x) \quad \text{for} \quad a-\delta \leqq x \leqq a \qquad (\sigma>-1),$$

δ being arbitrarily small and $q(x)$ continuous in $(a-\delta, a)$.

3. Consider now the integral $\int_{-a}^{a} x^n f(x)\,dx$ $(a>0$ finite) and suppose:

$$\text{(E}_1\text{)} \qquad \int_{-a}^{a} |x^k f(x)|\,dx \text{ exists for a certain } k>0;$$

$$\text{(E}_2\text{)} \qquad \begin{cases} f(x) = (a+x)^{\sigma_1}\,q_1(x) & \text{for} \quad -a \leqq x \leqq -a+\delta_1 \\ f(x) = (a-x)^{\sigma_2}\,q_2(x) & \text{for} \quad a-\delta_2 \leqq x \leqq a \end{cases} \qquad (\sigma_{1,2}>-1),$$

$\delta_{1,2}$ being arbitrarily small and $q_{1,2}(x)$ continuous in the corresponding interval. We get

$$\text{(E)} \qquad \begin{aligned} &\int_{-a}^{a} x^n f(x)\,dx \\ &= a^{n+1}\left[\frac{a^{\sigma_2}\,\Gamma(\sigma_2+1)\,q_2(a)+o(1)}{n^{\sigma_2+1}} + (-1)^n \frac{a^{\sigma_1}\,\Gamma(\sigma_1+1)\,q_1(-a)+o(1)}{n^{\sigma_1+1}}\right]. \end{aligned}$$

In order to prove (E), we write

$$\text{(10)} \qquad \int_{-a}^{a} x^n f(x)\,dx = \int_{-a}^{0} + \int_0^a = i_1 + i_2,$$

put in $i_2 : x = -ay$ and apply (D).

Corollaries.

$$\text{(11)} \qquad \int_{-1}^{1} f(x)\,\{x^{2n} + x^{2n+1}\}\,dx = \frac{2\,\Gamma(\sigma_2+1)\,q_2(1)+o(1)}{(2n)^{\sigma_2+1}}.$$

$$\text{(12)} \qquad \int_{-1}^{1} f(x)\,\{x^{2n} - x^{2n+1}\}\,dx = \frac{2\,\Gamma(\sigma_1+1)\,q_1(-1)+o(1)}{(2n)^{\sigma_1+1}}.$$

$$\text{(13)} \qquad \begin{aligned} &\int_{-a}^{a} \{f(x)+f(-x)\}\,x^{2n}\,dx \\ &= 2a^{2n+1}\left[\frac{a^{\sigma_2}\,\Gamma(\sigma_2+1)\,q_2(a)+o(1)}{(2n)^{\sigma_2+1}} + \frac{a^{\sigma_1}\,\Gamma(\sigma_1+1)\,q_1(-a)+o(1)}{(2n)^{\sigma_1+1}}\right]. \end{aligned}$$

$$(14)\qquad \begin{aligned}&\int_{-a}^{a}\{f(x)-f(-x)\}x^{2n+1}\,dx\\ &=2a^{2n+2}\left[\frac{a^{\sigma_2}\Gamma(\sigma_2+1)q_2(a)+o(1)}{(2n+1)^{\sigma_2+1}}-\frac{a^{\sigma_1}\Gamma(\sigma_1+1)q_1(-a)+o(1)}{(2n+1)^{\sigma_1+1}}\right].\end{aligned}$$

(11, 12) follows from (E), (13, 14) from (D).

4. We now consider the case of $f(x)$ subject to a logarithmic discontinuity. For the sake of simplicity, we suppose:

(F_1) $\qquad \int_0^1 x^k|f(x)|\,dx$ exists for a certain $k>0$;

(F_2) $\qquad f(x)=\log^\sigma(1-x^s)\,q(x)$ for $1-\delta\leqq x\leqq 1$ $\left(\begin{matrix}\sigma>0 \text{ integer}\\ s>0\end{matrix}\right)$,

$q(x)$ being continuous in the arbitrarily small interval $(1-\delta, 1)$. We get

(F) $\qquad \lim_{n=\infty}\frac{n}{\log^\sigma n}\int_0^1 x^n f(x)\,dx=(-1)^\sigma q(1).$

Proof: The proof is very similar to that given above for (A). First, we have:

$$(15)\qquad 1-x^s=(1-x)q_1(x) \text{ for } 1-\delta\leqq x\leqq 1,$$

where $q_1(x)$ is continuous, $q_1(1)=s$. On the other hand, we write

$$(16)\qquad \int_0^1 x^n f(x)\,dx=\int_0^{1-n^{-\theta}}+\int_{1-n^{-\theta}}^1=i_1+i_2 \qquad \left(\theta=\frac{2}{3}\right),$$

$$(17)\qquad n|i_1|<n(1-n^{-\theta})^{n-k}\int_0^1 x^k|f(x)|\,dx.$$

$$(18)\qquad n i_1\to 0 \text{ for } n\to\infty.$$

Using (15) we write, n being sufficiently large,

$$i_2=\int_{1-n^{-\theta}}^1 x^n\log^\sigma(1-x)\left[1+\frac{\log q_1(x)}{\log(1-x)}\right]^\sigma q(x)\,dx.$$

For the values of x under consideration we have

$$(19)\qquad \left[1+\frac{\log q_1(x)}{\log(1-x)}\right]^\sigma=1+\eta(x), \qquad \eta(x)\to 0 \text{ for } n\to\infty.$$

$$(20)\qquad \begin{aligned} i_2 &= \int_{1-n^{-\theta}}^{1} x^n \log^\sigma(1-x)\,[q(1)+q(1)\,\eta(x)+(q(x)-q(1)) \\ &\qquad +(q(x)-q(1))\,\eta(x)]\,dx \\ &= q(1)\,i_2'+q(1)\,i_2''+i_2'''+i_2^{\mathrm{IV}} \end{aligned}$$

$$(21)\qquad i_2' = \int_{1-n^{-\theta}}^{1} x^n \log^\sigma(1-x)\,dx.$$

(20) gives, using the properties of $\eta(x)$ and $q(x)$,

$$(22)\qquad i_2 = i_2'(q(1)+o(1)).$$

In order to investigate i_2', we use the substitution $x = 1-y/n$, which gives (replacing y by x):

$$(23)\qquad n\,i_2 = (-1)^\sigma \int_0^{n^{1/3}} \left(1-\frac{x}{n}\right)^n (\log n - \log x)^\sigma\,dx.$$

Since σ is a positive integer, the right-hand integral in (23) is a sum of a finite numbers of terms of the form

$$(24)\qquad \log^{\sigma-k} n\,i_{2,k} \equiv \int_0^{n^{1/3}} \left(1-\frac{x}{n}\right)^n e^x\cdot e^{-x}\log^{\sigma-k} n \log^k(x)\,dx \qquad (k=0,1,2,\cdots,\sigma)$$

with coefficients depending upon σ only. We write

$$i_{2,k} = \int_0^{n^{1/3}} e^{-x}\log^k(x)\,\psi_n(x)\,dx \quad \text{(see 8)},$$

which proves easily, that

$$(25)\qquad \lim_{n=\infty} i_{2,k} = \int_0^{\infty} e^{-x}\log^k x\,dx.$$

Therefore the right-hand of (23) is a sum of a finite number of terms of the order (with respect to n) $\log^{\sigma-k} n$ ($k=0,1,2,\cdots,\sigma$), and we get finally

$$(26)\qquad n\,i_2' = (-1)^\sigma \log^\sigma n\ (1+o(1)),$$

which, combined with (16, 18, 22), gives (F).

5. The formulas (A—F) given above enable us to find the asymptotic expressions of a great many definite integrals of different types. On the other hand, using (A—F) we obtain many important asymptotic formulas. We give here two examples only.

Example (a).

$$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}} = \frac{\Gamma(\frac{1}{2})+o(1)}{\sqrt{2n}} \qquad \text{(using } (A)\text{)}. \tag{27}$$

Put here $n = 2m$, $2m+1$ and integrate by parts, then

$$\int_0^1 \frac{x^{2m}\,dx}{\sqrt{1-x^2}} = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\,\frac{\pi}{2} = \frac{\Gamma(\frac{1}{2})+o(1)}{\sqrt{4m}} \tag{28}$$

$$\int_0^1 \frac{x^{2m+1}\,dx}{\sqrt{1-x^2}} = \frac{2\cdot 4\cdot 6\cdots 2m}{1\cdot 3\cdot 5\cdots(2m+1)} = \frac{\Gamma(\frac{1}{2})+o(1)}{\sqrt{4m+4}}. \tag{29}$$

Now, if we divide (28) by (29), we get Wallis' formula:

$$\frac{\pi}{2} = \left\{\frac{2\cdot 4\cdot 6\cdots 2m}{1\cdot 3\cdot 5\cdots(2m-1)}\right\}^2 \cdot \frac{1}{2m+1}(1+o(1)),$$

even without using the value of $\Gamma(\frac{1}{2})$, and we have

$$\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m} = \frac{1}{\sqrt{\pi m}}(1+o(1)),$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Example (b).

$$\int_0^1 x^{n-1}(1-x)^{\alpha-1}\,dx = \frac{\Gamma(\alpha)\,(1+o(1))}{n^\alpha} \quad \text{(using A)} \quad (\alpha > 0). \tag{30}$$

Replacing the left-hand member by

$$\frac{\Gamma(\alpha)\,\Gamma(n)}{\Gamma(\alpha+n)} = \frac{\Gamma(\alpha)\,\Gamma(n)}{\alpha(\alpha+1)\cdots(\alpha+n-1)\,\Gamma(\alpha)},$$

we get from (30):

$$\Gamma(\alpha) = \lim_{n=\infty} \frac{\Gamma(n)\,n^\alpha}{\alpha(\alpha+1)\cdots(\alpha+n-1)}$$

and, for n integer,

$$\Gamma(\alpha) = \lim_{n=\infty} \frac{1\cdot 2\cdot 3\cdots(n-1)\,n^\alpha}{\alpha(\alpha+1)\cdots(\alpha+n-1)} \qquad (\alpha > 0).$$

6. We apply now the same method to find the asymptotic expression for n very large of the integral

$$i = \int_0^1 f(x)\,[1-(x-\xi)^2]^n\,dx \qquad (0 < \xi < 1) \tag{31}$$

under the following conditions:

(G_1) $$\int_0^1 |f(x)|\,dx \quad \text{exists;}$$

(G_2) $$f(x) = |x-\xi|^\sigma q(x) \quad \text{for} \quad \xi-\delta \leqq x \leqq \xi+\delta \qquad (\sigma > -1),$$

$q(x)$ being continuous in the arbitrarily small interval $(\xi-\delta, \xi+\delta)$. Then, we have the formula:

(G) $$\lim_{n=\infty} n^{\frac{\sigma+1}{2}} \int_0^1 f(x)\,[1-(x-\xi)^2]^n\,dx = \Gamma\left(\frac{\sigma+1}{2}\right) q(\xi).$$

Proof. Write (31) as follows:

(32) $$\begin{cases} i = \int_0^{\xi-n^{-\theta}} f(x)[1-(x-\xi)^2]^n\,dx + \int_{\xi-n^{-\theta}}^{\xi+n^{-\theta}} \cdots + \int_{\xi+n^{-\theta}}^1 \cdots = i_1+i_2+i_3 \\ \theta < \dfrac{1}{2}, \qquad n^{-\theta} < \delta \quad \text{for} \quad n \geqq n_0. \end{cases}$$

We find easily

(33) $$\lim_{n=\infty} n^h\, i_1 = 0$$

(34) $$\lim_{n=\infty} n^h\, i_3 = 0$$

$(h$ arbitrary$)$.

We consider now i_2. Here we have $|x-\xi|^2 \leqq n^{-2\theta}$. Therefore, putting $x-\xi = y$ and replacing y by x, we get

(35) $$i_2 = \int_{-n^{-\theta}}^{n^{-\theta}} |x|^\sigma (1-x^2)^n\, q_1(x)\,dx \qquad (q_1(0) = q(\xi))$$

$q_1(x) \equiv q(x+\xi)$ being continuous for $-n^{-\theta} \leqq x \leqq n^{-\theta}$,

(36) $$i_2 = i_2'[q_1(0)+o(1)], \qquad i_2' = 2\int_0^n x^\sigma (1-x^2)^n\,dx.$$

The substitution $x^2 = y$ gives (replacing y by x)

$$i_2' = \int_0^{n^{-2\theta}} x^{\frac{\sigma-1}{2}} (1-x)^n\,dx = \int_{1-n^{-2\theta}}^1 x^n (1-x)^{\frac{\sigma-1}{2}}\,dx$$

$$n^{\frac{\sigma+1}{2}} i_2' = \Gamma\left(\frac{\sigma+1}{2}\right) + o(1) \quad \text{(see (9))},$$

which, combined with (32–36), proves (G).

We want to apply (G) to the case $f(x) \geqq 0$ for $0 \leqq x \leqq 1$. Denote by $\varphi_n(x)$ $(n = 0, 1, 2 \cdots)$ the orthogonal and normal system of Tchebycheff's polynomials corresponding to the interval $(0, 1)$ with the characteristic function $f(x)$. We have then*

$$(37)\quad \frac{1}{K_n(\xi)} \equiv \frac{1}{\sum_0^n{}_i \varphi_i^2(\xi)} = \min \int_0^1 f(x)[1 + X_1(x-\xi) + \cdots + X_n(x-\xi)^n]^2 dx.$$

Evidently,

$$\min \int_0^1 f(x)[1 + X_1(x-\xi) + \cdots + X_n(x-\xi)^{2n}]^2 dx \leqq \int_0^1 f(x)[1-(x-\xi)^2]^{2n} dx,$$

which proves, in combination with (G) and (37), that

$$K_{2n}(\xi) > \tau (2n)^{\frac{\sigma+1}{2}},$$

where τ does not depend upon n. Since $K_{2n+1}(\xi) \geqq K_{2n}(\xi)$, we get

$$(38)\qquad K_n(\xi) = \sum_0^n{}_i \varphi_i^2(\xi) > \tau n^{\frac{\sigma+1}{2}} \qquad (0 < \xi < 1),$$

provided the characteristic function vanishing at $x = \xi$ satisfies there the condition (G_2).

The quantity $K_n(x)$ occurs frequently in the theory of Tchebycheff's polynomials.

* The proof is very similar to that given for the case $\xi = 0$ in my paper, *Sur le développement de l'intégrale* $\int_a^b \frac{p(y)}{x-y}\, dy$ *en fraction continue et sur les polynomes de Tchebycheff*, Rend. Circ. Mat. di Palermo, vol. 47 (1923), 25–46; p. 41.

UNIVERSITY OF CHICAGO,
March 1924.

THE FREQUENCY LAW OF A FUNCTION OF VARIABLES WITH GIVEN FREQUENCY LAWS.

By Edward L. Dodd.

1. **Introduction.** The aim of this paper is to give a somewhat general treatment of one of the important problems of "geometric" or "continuous" probability, the determination of the frequency law of a function of variables with given frequency laws. A good deal of literature* can be found dealing with different phases of this subject.

The treatment of the subject in this paper may be characterized as follows: (1) The n events whose probabilities are considered need not be independent (Theorem I); (2) From the data an exact expression for the required probability is obtained, valid—except when a restriction is specified—for any n, large or small, not merely an approximate or asymptotic expression; (3) It is not assumed that first or second moments exist for the given frequency functions; (4) These functions are characterized by general properties. Thus the discussion is not confined to the Gaussian probability function

$$z = \frac{h}{\sqrt{\pi}} e^{-h^2x^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}, \tag{1}$$

or to specified simple functions; (5) To reduce solutions to simple forms, making evaluation by quadrature possible, use is made of the Dirichlet† "discontinuity factor"

$$\operatorname{sgn}(\nu) = \frac{2}{\pi}\int_0^\infty \sin \nu t \frac{dt}{t} = \begin{cases} +1, & \nu > 0 \\ 0, & \nu = 0 \\ -1, & \nu < 0. \end{cases} \tag{2}$$

(6) For changes in the order of integration, which accompany the use of this factor, sufficient conditions are given. This is essential for a critical treatment of the subject. Although this factor has been used extensively

* See references in Czuber, *Theorie der Beobachtungsfehler*, Leipzig. Also: Mayr, *Wahrscheinlichkeitsfunctionen und ihre Anwendungen*, Monat. f. Math. u. Phys., vol. 30 (1920), pp. 17-43; R. von Mises, *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Math. Zeit., vol. 4 (1919), pp. 1-97; Dodd, *Functions of measurements under general laws of error*, Skandinavisk Aktuarietidskrift, vol. 5 (1922), pp. 133-158; Rietz, *Frenquency distributions obtained by certain transformations of normally distributed variables*, these Annals, ser. 2, vol. 23 (1922), pp. 292-300.

† *Sur une nouvelle méthode pour la determination des integrales multiples*, Journ. de Math., (1), vol. 4 (1839), pp. 164-174.

in individual probability problems—usually in connection with the Gaussian function—a systematic and critical treatment seems to be lacking.

2. The law of frequency for a continuous function of n variables.

THEOREM I. *Let*

$$y = F(x_1, x_2, \cdots, x_n) \tag{3}$$

be real and continuous in n-dimensional space D and let $y = u$ have boundary of zero content for every u. Let the frequency of $x_1, \cdots, x_n$ be

$$z = \Phi(x_1, x_2, \cdots, x_n), \tag{4}$$

where z is continuous and subject to the usual conditions: $z \geqq 0$,

$$\int_D z\, dx = 1, \tag{5}$$

this being the multiple integral over D. Set

$$\begin{aligned} C = C(t) &= \int_D z \cos y t\, dx; \qquad S = S(t) = \int_D z \sin y t\, dx; \\ R = R(t) &= +\sqrt{C^2+S^2}; \quad \Theta = \Theta(t) = \arcsin S/R = \arccos C/R. \end{aligned} \tag{6}$$

Let E denote any "rectangular" portion of D defined by $x_j' \leqq x_j \leqq x_j''$, $j = 1, 2, \cdots, m$. Suppose that there exist positive constants $\varkappa$ and λ such that for D and for every E

$$\left| \int z \sin (u - y) t\, dx \right| < \varkappa t^{-\lambda}. \tag{7}$$

Then the probability $P(u)$ that $y \leqq u$ is given by

$$P(u) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty R \sin (ut - \Theta) \frac{dt}{t}. \tag{8}$$

If, furthermore, $R(t)$ is integrable in $(0, \infty)$,

$$p(u) = \frac{dP}{du} = \frac{1}{\pi} \int_0^\infty R \cos (ut - \Theta)\, dt. \tag{9}$$

Proof. The required probability $P(u)$ is the integral of z over that portion A of D for which $y \leqq u$. Let B be determined by $y > u$; then

$$P = \int_A z\,dx = \frac{1}{2}\int_A z\,dx + \frac{1}{2}\int_B z\,dx + \frac{1}{2}\int_A z\,dx - \frac{1}{2}\int_B z\,dx$$
$$= \frac{1}{2}\int_D z\,dx + \frac{1}{2}\int_D z\,\mathrm{sgn}\,(u-y)\,dx. \tag{10}$$

The boundary $y = u$ may, indeed, be ignored in the integration, since this, by hypothesis, has zero content and $\mathrm{sgn}\,(0) = 0$. Thus, from (2) and (5),

$$P(u) = \frac{1}{2} + \frac{1}{\pi}\int_D z\,dx\int_0^\infty \sin(u-y)\,t\,\frac{dt}{t}. \tag{11}$$

Inverting the order of integration, we write in a formal manner:

$$Q(u) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{dt}{t}\int_D z\sin(u-y)\,t\,dx. \tag{12}$$

A sufficient condition for $P(u) = Q(u)$ is the existence of the corresponding $(n+1)$-fold multiple integral now to be considered and the existence of $Q(u)$.

Let a "rectangular" region G in $(n+1)$-fold space be defined by the inequalities $0 \leqq t \leqq t_1$; $x_j' \leqq x_j \leqq x_j''$, $j = 1, 2, \cdots, n$. Then, over G, the multiple integral of $z[\sin(u-y)\,t]/t$ exists, since the integrand is the product of two continuous functions. Then

$$\int_G z\,[\sin(u-y)\,t]/t\,dt\,dx = \int_0^{t_1} dt\int_E z\,[\sin(u-y)\,t]/t\,dx. \tag{13}$$

For in the right member integration with repect to dx yields a continuous function of t, continuous indeed for $t = 0$, since in E the function y is continuous and hence bounded and, as t approaches zero, $\sin(u-y)\,t/t$ approaches $u-y$ uniformly. Now, by (7), for each positive ϵ, there is a positive t_1 such that

$$\left|\int_{t_1}^{t_2}\frac{dt}{t}\int_E z\sin(u-y)\,t\,dx\right| < \frac{\epsilon}{2}, \tag{14}$$

is valid for every $t_2 > t_1$. Furthermore,

$$\left|\int_0^{t_1}\sin(u-y)\,t\,\frac{dt}{t}\right| = \left|\int_0^{(u-y)t_1}\sin v\,\frac{dv}{v}\right| < 3, \tag{15}$$

and, since by (5) the integral of z exists, we can find a "rectangular" region E_1 so that, if E encloses E_1, and E_2 is that portion of E not in E_1,

$$\left|\int_{E_2} z\,dx\right| < \frac{\epsilon}{6}. \tag{16}$$

Thus

$$(17)\quad \left|\int_0^{t_1}\frac{dt}{t}\int_{E_2} z\sin(u-y)t\,dx\right| = \left|\int_{E_2} z\,dx\int_0^{t_1}\sin(u-y)t\,\frac{dt}{t}\right| < \frac{\epsilon}{2}.$$

Hence, since t_2 and E may now increase without limit, (14) and (17) show the convergence of the $(n+1)$-fold integral of $z[\sin(u-y)]/t$. But, since a sine is not numerically greater than unity, the integral over D of $z[\sin(u-y)t]/t$ exists for every positive t. Hence

$$\int_0^\infty \frac{dt}{t}\int_D z\sin(u-y)t\,dx$$

exists, being equal to the corresponding multiple integral whose existence has just been proven. This establishes the equality of $P(u)$ and $Q(u)$ in (11) and (12), and the substitutions (6) then yield (8).

We have just used the theorem that, if the multiple integral and a corresponding iterated integral both exist, they are equal. This theorem is also useful in the application of Theorem I, because integration is usually effected in such a case as an iterated process.

To prove (9), we note that

$$(18)\quad \frac{\Delta P}{\Delta u} = \frac{1}{\pi}\int_0^\infty R\cos(ut-\Theta+\delta)\cdot(\sin\delta/\delta)\,dt,$$

where $\delta = t\Delta u/2$. Here the two coefficients of R are numerically not greater than unity; hence, with t_1 large enough, the integral from t_1 to ∞ may be made less than $\epsilon/3$, since, by hypothesis, R is integrable. In the interval from 0 to t_1 these coefficients approach their limits uniformly. Thus dP/du exists (finite) as given by (9).

Instead of postulating the convergence of R, we may lay down the condition,

$$(19)\quad \left|\int_D z\sin(u-y)t\,dx\right| < \varkappa t^{-1-\lambda},$$

resembling (7) somewhat, and a similar condition with sine replaced by cosine.

In passing, it may be noted that $C(t)$, $S(t)$, $R(t)$, and $\Theta(t)$ are continuous, $\Theta(t)$ being multiple valued. C and R are even functions; S and Θ are odd functions of t. C and S are the "probable values" or "expectations" of $\cos yt$ and $\sin yt$ respectively. Hence, by a well known theorem,

$$(20)\quad R^2 = C^2+S^2 \leqq \int_D z(\cos^2 yt+\sin^2 yt)\,dx = 1.$$

In fact, $R<1$ unless $t=0$.

To obtain approximately the point of maximum or "mode" for $p(u)$ in (9), we may first find

$$M = \int_{-\infty}^{\infty} x\, y\, dx, \tag{21}$$

if this exists. Then, referring to (6), we would in general have approximately for small values of t, $S/t = M$, and $\Theta = S = Mt$. The mode for $p(u)$ would then be expected near the value $u = M$; for then the cosine in (9) would approximate unity for small values of t.

3. The law of frequency for a sum.

THEOREM II. *Let $\varphi_j(x_j)$ be frequency functions for x_j, with $j = 1, 2, \cdots, n$, these functions thus being non-negative and integrable from $-\infty$ to $+\infty$ with result unity. Set*

$$c_j = \int_{-\infty}^{\infty} \varphi_j(x) \cos tx\, dx; \quad s_j = \int_{-\infty}^{\infty} \varphi_j(x) \sin tx\, dx;$$

$$\varrho_j = +\sqrt{c_j^2 + s_j^2}; \qquad \theta_j = \arcsin s_j/\varrho_j = \arccos c_j/\varrho_j; \tag{22}$$

$$R = \prod_1^n \varrho_j; \qquad \Theta = \sum_1^n \theta_j.$$

Let $\varphi_1(x_1)$ be of limited variation, and let each φ be either continuous or of limited variation. Then the probability $P(u)$ that

$$x_1 + x_2 + \cdots + x_n \leqq u,$$

is

$$P(u) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} R \sin(ut - \Theta) \frac{dt}{t}. \tag{23}$$

If, furthermore, $\varphi_2(x_2)$ is of limited variation,

$$p(u) = \frac{dP}{du} = \frac{1}{\pi} \int_0^{\infty} R \cos(ut - \Theta)\, dt. \tag{24}$$

Moreover, in case each $\varphi_j(-x_j) = \varphi_j(x_j)$, then, in the above,

$$\Theta = 0; \qquad R = \prod_1^n c_j. \tag{25}$$

In case the φ's are all identical, we have—dropping subscripts—

$$\Theta = n\theta; \qquad R = \varrho^n. \tag{26}$$

Proof. As this theorem closely resembles Theorem I, the argument may be abbreviated. We wish to show that (7) is satisfied under the conditions of Theorem II. Now, $\sin(u-x_1-x_2-\cdots-x_n)t$ may be expanded into a sum of products of sines and cosines, so that, for integration, the variables, $x_1, x_2, \cdots, x_n$ separate. Each such product contains either $\sin x_1 t$ or $\cos x_1 t$. In the plot of these functions as functions of x_1, a "wave"—a half-period— of width π/t above the x_1-axis is followed by a wave of the same width below this axis. The contribution of such a pair of intervals towards the integral of $\varphi_1(x_1)\sin x_1 t\,dx_1$ is thus numerically not greater than $2\omega/t$, where ω is the variation of $\varphi_1(x_1)$ in this pair of intervals. For any part of a full period, the contribution is not greater than the product of $2\pi/t$ by the upper bound of $|\varphi_1(x_1)|$. Hence, inasmuch as $\varphi_1(x_1)$ is of limited variation, the integral of $\varphi_1(x_1)\sin x_1 t$ over any portion of the x_1-axis is numerically less than $|k/t|$, where k is some constant. Moreover, for any j, the integral of $\varphi_j(x_j)\sin x_j t$ is numerically not greater than unity. This establishes (7) and (8),—indeed, (22) and (23), with the aid* of

$$e^{i\theta} = \cos\theta + i\sin\theta; \qquad i = \sqrt{-1}.$$

For the proof of (24), we note that, if any two φ's are of limited variation, t^2 will appear in the denominator to insure the convergence of the integral with respect to t.

THEOREM III. *Let $\varphi_j(x_j)$ be frequency functions for x_j, leading to* (22). *Let each φ have but a limited number of points of maximum and points of discontinuity, the latter, when the discontinuity is infinite, being such that in the vicinity of a point of discontinuity a (omitting subscripts)*

$$|x-a|^{\alpha}\varphi(x) < K, \tag{27}$$

where α and K are positive constants $\alpha < 1$. Then, $P(u)$ is given by (23). *If, furthermore, $n > 1/(1-\alpha)$, $p(u)$ is given by* (24).

Proof. For each φ, the x-axis may be divided into a finite number of portions in each of which φ is univariant. In passing from a finite maximum to a finite minimum, or vice versa, φ has limited variation, and the discussion of Theorem II is applicable. Likewise, near $\pm\infty$ $\varphi(x)$ decreases with increasing $|x|$, and has limited variation.

Suppose, now, that $\varphi(a) = +\infty$ and that b is the first point of minimum

* Cf. Catalan, *Note sur une formule relative aux intégrales multiples*, Journ. de Math., (1), vol. 8 (1843), pp. 239-240. Here no attempt is made to justify the change in the order of integration.

to the right of a. Then, in the interval from a to b, $\varphi(x)$ does not increase. From (27) it follows that, if δ is small enough,

$$\int_a^{a+\delta} \varphi(x)\, dx < \frac{K}{1-\alpha}\, \delta^{1-\alpha}. \tag{28}$$

Let δ be determined from

$$\delta = \frac{2\pi}{t}. \tag{29}$$

Then δ will be small for large values of t, in which we are interested for the convergence proof. Moreover, for the curve $y = \sin xt$, a wave of width π/t will exist in the interval from a to $a+\delta$. Let ε_1 be the integral of $\varphi(x) \sin xt$ corresponding to this wave, ε_2 the integral corresponding to the next wave, and so on. The series $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots$, continuing to the vicinity of b, is alternating in sign, and the numerical values of the terms do not increase. The numerical value of the sum is thus not greater than $|\varepsilon_1|$, which in turn by (28) and (29) is less than $K'/|t|^{1-\alpha}$, where K' is some positive constant. The other points of infinite discontinuity may be treated in the same way; and, by hypothesis, there are only a limited number of them. Thus, the requirement (7) is satisfied, and (23) is valid. To obtain (24), we note that the integral of each $\varphi_j(x_j) \sin x_j t$ puts into the denominator t raised a power at least equal to $1-\alpha$. Convergence of the integral with respect to t is then secured if $n(1-\alpha)>1$.

4. **Illustrations.** Some problems will now be given showing the applications of the foregoing theorems.

It should be noted* that if β is a single-valued continuous univariant function, $\beta = g(x)$, with a derivative $d\beta/dx = g'(x)$, and if $\psi(\beta)$ is the frequency function for β, then

$$\varphi(x) = \psi[g(x)]\, |g'(x)| \tag{30}$$

is the frequency for x. If, indeed, $g(x)$ is an increasing function, $\varphi(x)dx = \psi(\beta)d\beta$.

The sum of projections. Suppose that a needle of length l is thrown n times upon a rectangular table. Let β be the inclination of the needle to one side of the table, where $0 \leqq \beta < \pi$. Then $x = l \cos \beta$ is the orthogonal projection of the needle. The assumption that all inclinations are equally likely leads to the frequency function $\psi(\beta) = 1/\pi$ if $0 \leqq \beta < \pi$; otherwise $\psi(\beta) = 0$. Then the frequency function for x is

$$\varphi(x) = \frac{1}{\pi\sqrt{l^2 - x^2}}, \quad |x| < l; \qquad \varphi(x) = 0, \quad |x| > l. \tag{31}$$

* Cf. Mayr, loc. cit., p. 20.

Theorem III is applicable with $\alpha = 1/2$; and, if subscripts are used for the individual throws, the probability $P(u)$ that $l \cos \beta_1 + l \cos \beta_2 + \cdots + l \cos \beta_n \leqq u$ is

$$(32)\quad P(u) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin u t\, dt}{t}\left[\frac{2}{\pi}\int_0^\infty (l^2 - x^2)^{-1/2} \cos tx\, dx\right]^n$$

which may be computed by quadratures.

Again, suppose that the above table is ruled with parallel lines at a distance b from each other, as in a diffraction grating. Let x now represent the projection upon these lines of a segment of the needle between two adjacent intersections. Then $x = b \cot \beta$ where $0 \leqq \beta < \pi$ as before. This leads to

$$(33)\quad \varphi(x) = \frac{b}{\pi(b^2 + x^2)}; \qquad s(t) = 0 = \theta(t); \qquad \varphi^n(t) = e^{-nbt}.$$

Then the probability that $u \leqq x_1 + x_2 + \cdots + x_n \leqq u + du$ is

$$(34)\quad P'(u)\, du = \frac{n b\, du}{\pi(u^2 + n^2 b^2)}.$$

The sum of cubes. Suppose that the weight w of certain solids, such as oranges or eggs, varies as the cube of a certain axis or length ξ and that ξ is subject to the Gaussian law

$$(35)\quad \psi(\xi) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(\xi - a)^2/2\sigma^2}.$$

What is the probability that the weight of n of these solids will be at least equal to W?

It is given that $w = \gamma \xi^3$ where γ is a constant. Set $v = \xi/\sigma$; $b = a/\sigma$; $x = v^3$. Then the frequency law* for x is

$$(36)\quad \varphi(x) = \frac{x^{-2/3}}{3\sqrt{2\pi}}\, e^{-(x^{1/3} - b)^2/2}.$$

For this function, Theorem III is applicable with $\alpha = 2/3$. And, as x is a simple increasing function of v, we may transform from x back to v to obtain $c(t)$. Thus

$$(37)\quad c(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(v - b)^2/2} \cos t v^3\, dv.$$

* Cf. Rietz, loc. cit., p. 294.

If in (37) the cosine is replaced by sine, $s(t)$ is obtained. Then, with $\varrho = (c^2+s^2)^{1/2}$, $\theta = \arcsin s/\varrho = \arccos c/\varrho$, the required probability that $\sum x \geqq W/\gamma\sigma^3$ is

$$\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \varrho^n \sin\left(\frac{Wt}{\gamma\sigma^3} - n\theta\right)\frac{dt}{t}, \tag{38}$$

where the negative sign appears in front of the integral because $\sum w$ is to be greater than W, or at least equal to it.

If a table giving θ and $\log\varrho$ were to be prepared to expedite the solution of problems of this character, one argument would naturally be $100/b$, the coefficient of variability, and the other argument t. Small values of the variability would not need consideration because for such values the distribution would be approximately Gaussian.

5. **Conclusion.** Analytic probability is essentially a part of function theory. The need of the strict methods of function theory is becoming more apparent with the use of frequency functions having infinite discontinuities. This increases the chance that a change of variable or a change in the order of integration may lead to erroneous results if performed indiscriminately. The attempt has been made in this paper to set forth rather broad general conditions under which certain useful operations may be employed safely in problems of probability.

UNIVERSITY OF TEXAS,
September 11, 1924.

THE FOUR TERM DIOPHANTINE ARCCOTANGENT RELATION.

By A. A. Bennett.

The determination of integers, x, y_1, y_2, to satisfy the relation

$$\operatorname{arccot} x = \operatorname{arccot} y_1 + \operatorname{arccot} y_2,$$

is important* in determining simple formulas to serve in Gregory's series to compute π. This equation may be shown to be equivalent to the set, $uv = x^2+1$, $y_1 = x+u$, $y_2 = x+v$, where all the arguments are integers. In connection with the corresponding four-term relation for which some infinite systems of solutions are here appended, the theorem is not so obvious. We have however the following

Theorem. *The determination of integers, x_1, x_2, y_1, y_2, to satisfy the relation*

$$\operatorname{arccot} x_1 + \operatorname{arccot} x_2 = \operatorname{arccot} y_1 + \operatorname{arccot} y_2, \qquad (x_1 + x_2 \neq 0), \tag{1}$$

is equivalent to the determination† of integers to satisfy simultaneously the conditions,

$$u_1 v_1 \equiv u_1 v_2 \equiv u_2 v_1 \equiv u_2 v_2, \pmod{x_1 + x_2}, \tag{2}$$

$$u_1 v_1 = x_1^2 + 1, \qquad u_2 v_2 = x_2^2 + 1. \tag{3}$$

Proof. With (2) and (3) satisfied, the following expressions are integers,

$$\begin{aligned}
&\frac{u_1 v_2 - u_1 v_1}{x_1 + x_2} + x_1 = \frac{u_1 v_2 - (x_1^2+1)}{x_1 + x_2} + x_1 = \frac{x_1 x_2 - 1 + u_1 v_2}{x_1 + x_2} = y_1', \\
&\frac{u_1 v_2 - u_2 v_2}{x_1 + x_2} + x_2 = \frac{u_1 v_2 - (x_2^2+1)}{x_1 + x_2} + x_2 = \frac{x_1 x_2 - 1 + u_1 v_2}{x_1 + x_2} = y_1', \\
&\frac{u_2 v_1 - u_1 v_1}{x_1 + x_2} + x_1 = \frac{u_2 v_1 - (x_2^2+1)}{x_1 + x_2} + x_1 = \frac{x_1 x_2 - 1 + u_2 v_1}{x_1 + x_2} = y_2', \\
&\frac{u_2 v_1 - u_2 v_2}{x_1 + x_2} + x_2 = \frac{u_2 v_1 - (x_2^2+1)}{x_1 + x_2} + x_2 = \frac{x_1 x_2 - 1 + u_2 v_1}{x_1 + x_2} = y_2'.
\end{aligned} \tag{4}$$

* For references, cf. Hobson, *Plane Trigonometry*, p. 310.

† This theorem is utilized in determining the complete table of solutions, within the limits specified, given at the close of this paper.

Now equation (1) may be expressed as

$$\frac{x_1 x_2 - 1}{x_1 + x_2} = \frac{y_1 y_2 - 1}{y_1 + y_2}, \qquad (x_1 + x_2 \neq 0). \tag{5}$$

But the expression, $(y_1' y_2' - 1)/(y_1' + y_2')$, reduces formally by use of (3) and (4) to the left hand member of (5), save for a common extraneous factor in numerator and denominator of the form, $2(x_1 x_2 - 1) + u_1 v_2 + u_2 v_1$, and this cannot vanish. For if it were zero, we should have $4(x_1 x_2 - 1)^2 = (u_1 v_2 + u_2 v_1)^2$, whence also $-4(x_1 + x_2)^2 = (u_1 v_2 - u_2 v_1)^2$, which is impossible for real numbers unless both members vanish, contrary to the condition of (1). Thus the y_1' and y_2' of (4) satisfy the conditions of (5) for y_1 and y_2. Thus (2) and (3) lead to (1).

Conversely, we have from (5) for y_1 the following:

$$y_1 = \frac{y_2(x_1 x_2 - 1) + (x_1 + x_2)}{y_2(x_1 + x_2) - (x_1 x_2 - 1)}. \tag{6}$$

If we denote this denominator by d, and solve for y_2 in this relation, we obtain $y_2 = [(x_1 x_2 - 1) + d]/(x_1 + x_2)$. Substituting in (6), we get

$$y_1 = \frac{(x_1^2 + 1)(x_2^2 + 1) + d(x_1 x_2 - 1)}{d(x_1 + x_2)}. \tag{7}$$

Since y_1 is an integer, the two algebraic factors, d, and $x_1 + x_2$, of the denominator must be factors of the numerator. Thus in particular, d is a factor of $(x_1^2 + 1)(x_2^2 + 1)$, not excluding the case of d being unity. It is therefore possible in at least one way to express d as $u_2 v_1$ where u_2 is a factor, perhaps merely ± 1, of $x_2^2 + 1$, and v_1 is a factor, perhaps trivial, of $x_1^2 + 1$. Let the other factor of $x_1^2 + 1$ be u_1, and the other factor of $x_2^2 + 1$ be v_2. With this convention, (7) reduces to

$$y_1 = \frac{u_1 v_1 u_2 v_2 + u_2 v_1 (x_1 x_2 - 1)}{u_2 v_1 (x_1 + x_2)} = \frac{(x_1 x_2 - 1) + u_1 v_2}{x_1 + x_2}, \tag{8}$$

which is of the form given in (4), where also the conditions (2) are satisfied. Substituting (8) in (5), we have after reduction the following relation for y_2,

$$y_2 = \frac{(x_1 x_2 - 1)^2 + (x_1 x_2 - 1) u_1 v_2 + (x_1 + x_2)^2}{(x_1 + x_2) u_1 v_2} = \frac{(x_1^2 + 1)(x_2^2 + 1) + (x_1 x_2 - 1) u_1 v_2}{(x_1 + x_2) u_1 v_2}$$

$$= \frac{u_1 v_1 u_2 v_2 + (x_1 x_2 - 1) u_1 v_2}{(x_1 + x_2) u_1 v_2} \qquad = \frac{(x_1 x_2 - 1) + u_2 v_1}{x_1 + x_2},$$

which is of the form given for y_2' in (4). Since y_1 and y_2 are integers it is necessary that each of the expressions, $y_1 - x_1$, $y_1 - x_2$, $y_2 - x_1$, $y_2 - x_2$, be integers, that is, as in (4), that $(u_1 v_2 - u_1 v_1)/(x_1 + x_2)$, $(u_1 v_2 - u_2 v_2)/(x_1 + x_2)$, $(u_2 v_1 - u_1 v_1)/(x_1 + x_2)$, $(u_2 v_1 - u_2 v_2)/(x_1 + x_2)$, be integers, whence (2). This completes the proof.

For the relation (1) there are, beside the trivial solutions, $x_1 = y_1$, $x_2 = y_2$, and $x_1 = y_2$, $x_2 = y_1$, certain other solutions of (1) that might be regarded as degenerate in that they may be obtained by use of the three term relation. Thus for any two factorizations of z^2+1 as $r_1 s_1$, and $r_2 s_2$, we have

$$\text{(9)} \quad \operatorname{arccot}(z + r_1) + \operatorname{arccot}(z + s_1) = \operatorname{arccot}(z + r_2) + \operatorname{arccot}(z + s_2).$$

We have also certain useful infinite systems of solutions of (1) which are not in this sense degenerate. For example,

$$\text{(10)} \quad \begin{aligned} &\operatorname{arccot}[a(a+b)c] + \operatorname{arccot}[b(a+b)c] \\ &\qquad = \operatorname{arccot}(abc) + \operatorname{arccot}([a^2 b^2 (a+b)^2 c^3 + a^2 + ab + b^2]c). \end{aligned}$$

For x and t such that $4x(x^2+1)$ is exactly divisible by t^2+1 with quotient, s, we have the formula,

$$\text{(11)} \quad \operatorname{arccot} x + \operatorname{arccot}(s - x) = \operatorname{arccot}(2x + t) + \operatorname{arccot}(2x - t).$$

This gives in particular for $t = 0$ and 1, respectively,

$$\text{(11')} \qquad \operatorname{arccot} x + \operatorname{arccot}(4x^3 + 3x) = 2 \operatorname{arccot}(2x),$$

$$\text{(11'')} \quad \operatorname{arccot} x + \operatorname{arccot}(2x^3 + x) = \operatorname{arccot}(2x + 1) + \operatorname{arccot}(2x - 1).$$

For x of the form $5k$, $5k+2$, or $5k+3$, we may take t to be either 2 or 3, and so on for special forms of x. More generally, for any x, a, b for which $ab(a+b)$ is exactly divisible by $ab - (x^2+1)$ with quotient, c, we have

$$\text{(12)} \quad \operatorname{arccot} x + \operatorname{arccot}(x + a + b - c) = \operatorname{arccot}(x + a) + \operatorname{arccot}(x + b).$$

A complete table of all solutions of (1) for which $0 < x_1$, x_2, and $0 < x_1 + x_2 < 25$ is here appended save only for the omission of the trivial solutions, $x_1 = y_1$, or $x_1 = y_2$. These solutions were obtained by use of the congruences (2) and (3). The parentheses denote the values, (x_1, x_2, y_1, y_2), while the tabular integer on the left is the value of $x_1 + x_2$.

2. —
3. (1, 2, 0, −3),
4. (2, 2, 1, 7), (1, 3, 0, −2),
5. —
6. (2, 4, 1, −13), (3, 3, 1, −7),
7. (2, 5, 1, −8),
8. (1, 7, 2, 2), (3, 5, 2, 18), (4, 4, 2, 38),
9. (3, 6, 2, 43), (4, 5, 2, −47),
10. (3, 7, 1, −3),
11. (3, 8, 2, −57), (4, 7, 2, −13),
12. (3, 9, 2, −32), (4, 8, 3, 21), (5, 7, 3, 57), (5, 7, 2, −8), (6, 6, 3, 117),
13. (5, 8, 4, 13), (6, 7, 3, −68),
14. (5, 9, 3, −73), (6, 8, 3, −31),
15. (2, 13, 1, −4), (3, 12, 2, −17), (4, 11, 3, 72), (7, 8, 4, 47), (7, 8, 2, −5), (7, 8, 3, −18),
16. (4, 12, 3, 157), (7, 9, 5, 132), (8, 8, 4, 268),
17. (4, 13, 5, 8), (7, 10, 4, −293),
18. (4, 14, 3, −183), (5, 13, 3, −21), (6, 12, 4, 302),
19. (4, 15, 3, −98), (6, 13, 4, −327),
20. (2, 18, 3, 5), (3, 17, 2, −12), (7, 13, 2, −4), (7, 13, 5, 47), (7, 13, 4, −38), (8, 12, 5, 99), (9, 11, 5, 255), (10, 10, 5, 515),
21. (8, 13, 5, 666), (9, 12, 5, −278),
22. (4, 18, 3, −47), (8, 14, 5, −577),
23. —
24. (3, 21, 4, 8), (5, 19, 4, 200), (7, 17, 5, 307), (11, 13, 6, 438), (12, 12, 6, 882).

ON WEYL'S TREATMENT OF THE PARALLEL DISPLACEMENT OF A VECTOR AROUND AN INFINITESIMAL CLOSED CIRCUIT IN AN AFFINELY CONNECTED MANIFOLD.*

By Tracy Yerkes Thomas.

In a recent article in the Annals† J. L. Synge has treated the problem of the parallel displacement of a vector around an infinitesimal closed circuit in a manner which appears to be most satisfactory. But apart from this Synge has adversely criticized the work of certain other writers who have also treated this problem, among these Eddington and Weyl. Synge's objection to Eddington's method is undoubtedly correct, and in fact last year at the University of Chicago I criticized Eddington's treatment on similar grounds. But his characterization of the work of Weyl as loose and unrigorous does not appear to me to be correct, although it is possible that many readers have experienced difficulty regarding Weyl's treatment of this problem. I thought therefore that it might be desirable to give an exposition of the rigorous method, involving derivatives rather than differentials, by which Weyl considers the curvature of an affinely connected manifold.‡ A complete description of this method, applied however in a slightly different connection, is to be found in Weyl's Spanish lectures.§

We consider a point P of our space through which a surface of parametric representation‖

$$x^i = x^i(s, t) \qquad i = 1, 2, \cdots, n \tag{1}$$

passes, the parameters s and t being so chosen that $s = 0$, $t = 0$ is the point P itself. Let $(\xi^i)_0$ denote an arbitrary vector at P. Then the equations

$$\frac{d\xi^i}{ds} + \Gamma^i_{\alpha\beta}\,\xi^\alpha \frac{dx^\beta}{ds} = 0, \tag{2}$$

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* National Research Fellow in Mathematical Physics.

† Ann. of Math., vol. 25, p. 181.

‡ Cf. *Raum, Zeit, Materie,* 5th ed., § 16, p. 120, where Weyl remarks that the differential processes may be made rigorous by the use of derivatives.

§ *Mathematische Analyse des Raumproblems* (1923), Anhang 3, p. 64.

‖ We assume without special mention the analytic character of the surface (1) necessary for the subsequent processes.

expressing the condition that the vectors ξ^i along the parametric line $t = 0$ be parallel, admit one and only one solution $\xi^i(s, 0)$ such that

$$\xi^i(0, 0) = (\xi^i)_0. \tag{3}$$

Similarly the equations

$$\frac{d\xi^i}{dt} + \Gamma^i_{\alpha\beta}\,\xi^\alpha \frac{dx^\beta}{dt} = 0, \tag{4}$$

which express the condition that the vectors ξ^i along the parametric line $s = 0$ be parallel, admit one and only one solution $\xi^i(0, t)$ satisfying the above condition (3). The vectors $\xi^i(s, 0)$ and $\xi^i(0, t)$ for arbitrary values of s and t are said to arise from the vector $(\xi^i)_0$ at P by parallel displacement along the parametric lines $t = 0$ and $s = 0$ respectively.

When we consider the functional vectors $\xi^i(s, 0)$ and $\xi^i(0, t)$, we see that there are two possibilities of extending these vectors by parallel displacement to the other points of the surface (1). For a given value of $t = t_0$ we may displace the vector $\xi^i(0, t)$ in a parallel manner along the line $t = t_0$. Allowing t_0 to assume all values we obtain a surface field of vectors which we denote by $\xi^i(s, t)$. The similar process, by which the vector $\xi^i(s, 0)$ is displaced parallel to itself along the line $s = s_0$, leads to a second surface field of vectors which we denote by $\xi^i_*(s, t)$. The vector fields $\xi^i(s, t)$ and $\xi^i_*(s, t)$ are each identical with $\xi^i(s, 0)$ and $\xi^i(0, t)$ along the parametric lines $t = 0$ and $s = 0$ respectively. In view of this fact

$$\xi^i_* - \xi^i = s \cdot t \left\{ \left(\frac{\partial^2(\xi^i_* - \xi^i)}{\partial s\, \partial t} \right)_{00} + \cdots \right\}, \tag{5}$$

where the derivatives in the right member are evaluated at the point $s = 0$, $t = 0$ and where the dots indicate terms multiplied by s, t. Hence

$$\lim_{s,t \to 0} \frac{(\xi^i_* - \xi^i)}{s \cdot t} = \left(\frac{\partial^2(\xi^i_* - \xi^i)}{\partial s\, \partial t} \right)_{00}. \tag{6}$$

Since $\xi^i_* - \xi^i$ is a vector, it follows that the limit in (6) is a vector and hence

$$\left(\frac{\partial^2(\xi^i_* - \xi^i)}{\partial s\, \partial t} \right)_{00} \tag{7}$$

is also a vector.

The vector $\xi^i_* - \xi^i$, or more fully $\xi^i_*(s, t) - \xi^i(s, t)$, represents the change of the vector $\xi^i(s, t)$ which results from parallel displacement around the closed circuit indicated by (s, t); $(s, 0)$; $(0, 0)$; $(0, t)$; (s, t).

From its method of formation the surface field $\xi^i(s,t)$ satisfies the equations

$$\frac{\partial \xi^i}{\partial s} + \Gamma^i_{\alpha\beta}\,\xi^\alpha \frac{\partial x^\beta}{\partial s} = 0 \tag{8}$$

identically. If we differentiate these equations with respect to t, evaluate at the point $s=0$, $t=0$, and then make use of the fact that

$$\frac{\partial \xi^i}{\partial t} + \Gamma^i_{\alpha\beta}\,\xi^\alpha \frac{\partial x^\beta}{\partial t} = 0 \quad \text{when} \quad s=0,\ t=0, \tag{9}$$

we obtain

$$\left(\frac{\partial^2 \xi^i}{\partial s\,\partial t}\right)_{00} = -\frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^\gamma}\,\xi^\alpha \frac{\partial x^\beta}{\partial s}\frac{\partial x^\gamma}{\partial t} - \Gamma^i_{\alpha\beta}\,\xi^\alpha \frac{\partial^2 x^\beta}{\partial s\,\partial t} + \Gamma^i_{\mu\beta}\,\Gamma^\mu_{\alpha\gamma}\,\xi^\alpha \frac{\partial x^\beta}{\partial s}\frac{\partial x^\gamma}{\partial t}, \tag{10}$$

an equation in which both members are evaluated at the point $s=0$, $t=0$. Similarly

$$\left(\frac{\partial^2 \xi^i_*}{\partial s\,\partial t}\right)_{00} = -\frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^\gamma}\,\xi^\alpha \frac{\partial x^\beta}{\partial t}\frac{\partial x^\gamma}{\partial s} - \Gamma^i_{\alpha\beta}\,\xi^\alpha \frac{\partial^2 x^\beta}{\partial t\,\partial s} + \Gamma^i_{\mu\beta}\,\Gamma^\mu_{\alpha\gamma}\,\xi^\alpha \frac{\partial x^\beta}{\partial t}\frac{\partial x^\gamma}{\partial s}. \tag{11}$$

Hence

$$\left(\frac{\partial^2(\xi^i_* - \xi^i)}{\partial s\,\partial t}\right)_{00} = B^i_{\alpha\beta\gamma}\,\xi^\alpha \frac{\partial x^\beta}{\partial s}\,\frac{\partial x^\gamma}{\partial t}, \tag{12}$$

where

$$B^i_{\alpha\beta\gamma} = \frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial \Gamma^i_{\alpha\gamma}}{\partial x^\beta} + \Gamma^i_{\mu\gamma}\,\Gamma^\mu_{\alpha\beta} - \Gamma^i_{\mu\beta}\,\Gamma^\mu_{\alpha\gamma} \tag{13}$$

is the curvature tensor.* The equations (12), in which all quantities except $B^i_{\alpha\beta\gamma}$ are known to be vectors, give the rigorous proof of the tensor character of $B^i_{\alpha\beta\gamma}$. From (6) and (12) we have

$$\lim_{s,t\to 0} \frac{(\xi^i_* - \xi^i)}{s\cdot t} = B^i_{\alpha\beta\gamma}\,\xi^\alpha \frac{\partial x^\beta}{\partial s}\,\frac{\partial x^\gamma}{\partial t}, \tag{14}$$

where the quantities in the right member are evaluated at the point $s=0$, $t=0$.

We now make the transition to differentials. Putting

$$\Delta \xi^i = \xi^i_* - \xi^i \tag{15}$$

* This is the negative of the curvature tensor used by Weyl.

and denoting an infinitesimal displacement from the point P, i. e., $s = 0$, $t = 0$, along the parametric line $t = 0$ by dx^i, a similar displacement along the parametric line $s = 0$ by δx^i, we have

$$\Delta \xi^i = B^i_{\alpha\beta\gamma}\, \xi^\alpha\, dx^\beta\, \delta x^\gamma \tag{16}$$

as the differential equations corresponding to (14). In the equations (16), $\Delta \xi^i$ represents the change in the vector ξ^i on being carried by parallel displacement around the infinitesimal circuit $(dx, \delta x)$. We should, strictly speaking, on the basis of the preceding work, consider the left member of (16), i. e., the increment $\Delta \xi^i$, as taken at the vertex of the circuit $(dx, \delta x)$ opposite to the point P, and the right member of (16) as taken at the point P itself.

In regard to the finite circuit Weyl has stated that this can be considered as the sum of the infinitesimal circuits into which a surface bounded by the circuit is decomposed by its parametric lines.* The remark is perhaps unfortunate, for, as Synge has pointed out, it is possibly not correct. But this remark in no way vitiates the character of the actual analysis.

* See Raum, Zeit, Materie, p. 118.

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CONCERNING THE SUBSETS OF A PLANE CONTINUOUS CURVE.*

By Harry Merrill Gehman.

I. Introduction.

In the present paper we shall discuss plane continuous curves† with reference to certain problems concerning the subsets of a continuous curve. It will be understood that the point sets considered lie in a Euclidean space of two dimensions. In part II of this paper is given a method for the construction of a continuous curve from any bounded, closed, and connected point-set by the addition of a countable infinity of arcs, thus showing that any bounded, closed and connected point-set may be a subset of a continuous curve. In part III are given two conditions which are necessary and sufficient, and one condition which is sufficient but not necessary, that every closed and connected subset of a continuous curve be a continuous curve. In part IV is given a characterization of simple continuous curves by means of one of the conditions introduced in part III.

In conclusion, we wish to express our thanks to Professor John Robert Kline, who first suggested the problems which are discussed in this paper, and who, by his never failing encouragement, has materially assisted in their solution.

II.

In this part we shall consider this problem: given a bounded, closed and connected point-set K, which is not a continuous curve; can we always construct a bounded, connected point-set L, such that $K+L$ is a continuous curve? If so, what conditions must the set L satisfy?

The first question is answered in the affirmative by Theorem II, while Theorem I and its corollary state that L must be a set which cannot be expressed as the sum of a finite number of arcs.

Theorem I. *If K is a bounded, closed and connected point-set, which is not a continuous curve, and if L is any arc, then $K+L$ is not a continuous curve.*

Proof. Since K is not a continuous curve, it contains a continu of condensation W, such that K is not connected im kleinen at any point of W.‡

* Various parts of this paper were presented to the American Mathematical Society, April 19, October 25, and December 30, 1924.

† For definitions and theorems concerning continuous curves, see R. L. Moore: *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 289-302. This paper will hereafter be referred to as "Report".

‡ Report: pp. 296-97.

Suppose L is an arc which does not contain every point of W, and let P be a point of W which is not a point of L. Since L is closed, about P we can put a circle C, not containing or enclosing any points of L. Since the only points of $K+L$ which C contains or encloses are points of K only, and since K is by hypothesis not connected im kleinen at P, it follows that $K+L$ is not connected im kleinen at P, and therefore that $K+L$ is not a continuous curve.

Suppose L is an arc which contains every point of W. Since W is closed and connected, W is an arc, $\overset{\frown}{AB}$, which is a sub-arc of L. Let P be a point of $\underset{\smile}{AB}$.* Since $L-\underset{\smile}{AB}$ is closed, we can put about P a circle C, not containing any points of $L-\underset{\smile}{AB}$. C will contain no points of L that are not also points of K, and therefore, as in the preceding paragraph, $K+L$ is not connected im kleinen at P, and is not a continuous curve.

We have shown that no matter how the arc L be constructed, $K+L$ is not a continuous curve. The following corollary can be proved by a repeated application of Theorem I.

COROLLARY. *If K is a bounded, closed and connected point-set, which is not a continuous curve, and if L is any point-set which can be expressed as the sum of a finite number of arcs, then $K+L$ is not a continuous curve.*

Note that the next theorem shows the necessity of assuming in this Corollary that the number of arcs into which L is separated, is finite.

THEOREM II. *If K is a bounded, closed and connected point-set, which is not a continuous curve, there exists a bounded connected point-set L consisting of a countable infinity of arcs, such that $K+L$ is a continuous curve.*

PROOF. Let us construct in the plane of K a pair of coördinate axes, and let us select our unit of length so that the diameter of K is greater than 2. Since K is bounded, there is some positive integer n, such that every point of K is at a distance less than n from the origin.

Let L_1 be the sum of the straight line intervals from $(-n, i)$ to (n, i), $(i = 0, \pm 1, \pm 2, \cdots, \pm n)$, and the straight line intervals from (j, n) to $(j, -n)$, $(j = 0, \pm 1, \pm 2, \cdots, \pm n)$.

Let W denote the set of all points of K at which K is not connected im kleinen, plus all limit points of such points. In each square of unit side formed by intervals of L_1, that contains or encloses a point of W, let us join the mid points of opposite sides by segments of lines parallel to the X and Y axes. Let L_2 be the sum of the segments thus added.

In general, let L_i $(i = 3, 4, 5, \cdots)$, be the set of segments added to $L_1+L_2+\cdots+L_{i-1}$, by joining mid points of opposite sides of each square

* If $\overset{\frown}{XYZ}$ is an arc, $\underset{\smile}{XYZ} = \overset{\frown}{XYZ} - X - Z$.

of side $1/2^{i-2}$ formed by intervals of $L_1+L_2+\cdots+L_{i-1}$, that contains or encloses a point of W.

Let $L=L_1+L_2+\cdots+L_i+\cdots$. Since each L_i consists of a finite number of arcs, L will consist of a countable infinity of arcs. The set L is bounded and connected. L is not necessarily closed, but every limit point of L which is not in L is a point of W. Therefore $K+L$ is bounded, closed and connected.

The set L (and therefore $K+L$) is connected im kleinen at all points, which are not points of K. The set K (and therefore $K+L$) is connected im kleinen at all points, save points of W. It remains to be shown that $K+L$ is connected im kleinen at every point of W.

Let P be a point of W. Let us put about P as center a circle C, of arbitrary radius ε. There exists an integer j such that $1/2^j<\varepsilon$. By our method of formation of L, P is enclosed by a square formed of arcs of $L_1+L_2+\cdots+L_{j+3}$, the diameter of which is $\sqrt{2}/2^{j+1}$ or less and therefore less than $1/2^j$. The points of $K+L$, lying within and on this square form a connected set N containing P and lying within the circle C. Hence, if we construct a circle C' having P as center and lying within the square, any point of $K+L$ within the circle C' is joined to P by the connected set N which lies within C. Therefore $K+L$ is connected im kleinen at P, where P is any point of W.

Therefore $K+L$ is a continuous curve, since it is connected im kleinen at every point. This completes the proof of Theorem II.

Another method for the construction of a set L, which is such that $K+L$ is a continuous curve, would be to let L be all points (x, y), such that $|x|\leqq n$, and $|y|\leqq n$. In this case, however, L would consist of an uncountable infinity of arcs, and the continuous curve $K+L$ would be everywhere dense within the square whose vertices are $(\pm n, \pm n)$.

The method outlined in Theorem II is such that $L+W$ is nowhere dense. For suppose the interior of some circle C contained only points of $L+W$. If D, the interior of C, consisted entirely of points of W, then K would be connected im kleinen at all these points, contrary to the definition of W. Some points of D are therefore not points of W. Let P be a point of D which is not a point of W, and therefore is a point of L. Let us construct about P a circle C' within C, and enclosing no points of W. As in the previous discussion, there exists an integer $j>2$, such that C' encloses a square of side $1/2^{j-2}$ or less, formed from lines of $L_1+L_2+\cdots+L_j$, which encloses P. The square encloses no points of W, and therefore the supposition that it encloses a point P of L, contradicts our definition of the construction of L.

We have thus proved that, if L is constructed as in Theorem II, it

follows that, if K is nowhere dense, $K+L$ is nowhere dense, while, if K is dense in any parts, $K+L$ is dense only in those parts in which K is dense.

We can include Theorem I, its Corollary, Theorem II, and this additional property of $K+L$ in a new theorem:

THEOREM III. *If K is any bounded, closed and connected point-set which is not a continuous curve, a point-set L consisting of a countable set of arcs, necessarily infinite in number, can be so constructed that $K+L$ is a continuous curve which is dense only in those parts in which K is dense.*

III.

It is well known that a continuous curve may have the property that each of its closed and connected subsets is also a continuous curve, while another continuous curve may contain closed and connected subsets which are not continuous curves. In the first class is any continuous curve which contains no simple closed curve;* in the second class is any continuous curve which is dense in any part. In this section we obtain necessary and sufficient conditions that a continuous curve be in the first class, i. e., that every closed and connected subset be a continuous curve.

We shall first introduce and discuss several definitions.

Definition. If M and W are point-sets, then M is said to be *connected im kleinen relative to* W at a point P, if for every positive number ε, there exists a positive number δ, such that, if X and Y are any two points of M at a distance less than δ from P, then they lie in a connected subset of M, which is such that either (1) every point is at a distance less than ε from P, or (2) every point is at a distance less than ε from a point of W.

Definition. If M and W are point-sets, then M is said to be *uniformly connected im kleinen relative to* W, if for every positive number ε, there exists a positive number δ, such that if X and Y are any two points of M at a distance apart less than δ, then X and Y lie in a connected subset of M which is such that either (1) it lies within a circle of radius ε, or (2) every point is at a distance less than ε from a point of W.†

If M is a closed and bounded set and is connected im kleinen relative to W, then it is uniformly connected im kleinen relative to W, but this is not necessarily true in case M is open or unbounded.

The property of being *relatively uniformly connected im kleinen* is weaker

* S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), p. 123.

† This notion is somewhat akin to that of relatively uniform convergence of functions, due to E. H. Moore. See *The New Haven Mathematical Colloquium*.

than that of being *uniformly connected im kleinen*, and the property of being *relatively connected im kleinen* is weaker than that of being *connected im kleinen*. The properties of being *relatively uniformly connected im kleinen* and of being *connected im kleinen* are independent of each other; a set may have either property without having the other. For example, if W is a point on a simple closed curve J and M is $J-W$, then M is connected im kleinen but is not uniformly connected im kleinen relative to W. On the other hand, if M is the curve $y=\sin\frac{1}{x}$, for $0<x\leqq 1$ plus the interval of the Y-axis between $y=1$ and $y=-1$, and if W is the portion of the Y-axis included in M, then M is not connected im kleinen but it is uniformly connected im kleinen relative to W.

For the special case where W is closed and has no points in common with M, the property of being *relatively uniformly connected im kleinen* is stronger than that of being *connected im kleinen*. We have given above an example in which M is connected im kleinen but not uniformly connected im kleinen relative to W. If, however, a set M is uniformly connected im kleinen relative to W, where W is closed and has no points in common with M, then M is connected im kleinen. For if it were not, it would fail to be connected im kleinen at a point P. Let ϵ be less than one-third the distance from P to any point of W. For any number δ, there are points in M at a distance from P less than δ that cannot be joined to P either by a connected set in M which lies within a circle of radius ϵ or by a connected set which is such that every point is at a distance less than ϵ from a point of W. The set M is therefore not connected im kleinen relative to W, either uniformly or non-uniformly, contrary to hypothesis.

Therefore in case W is closed and has no points in common with M, the property of being *relatively uniformly connected im kleinen* lies between the properties of being *connected im kleinen* and of being *uniformly connected im kleinen*. The property S defined by R. L. Moore* is also a property which lies between these two properties. The following examples show, however, that, if W is closed and has no points in common with M, the property of being *relatively uniformly connected im kleinen* and the property S are independent of each other. For if W is a point on a simple closed curve J, and M is $J-W$, then M has property S, but is not uniformly connected im kleinen relative to W. If M is the curve $y=\sin\frac{1}{x}$, for $0<x\leqq 1$, and W is $x=0$ for $-1\leqq y\leqq 1$, then M is uniformly connected im kleinen relative to W but does not have property S.

* Report, p. 296. "*A point set M has property S* if, for every positive number ϵ, M is the sum of a finite number of connected subsets each of diameter less than ϵ."

Definition. An R—set is a closed and connected point set N, containing a continu of condensation W, such that $N-W$ is uniformly connected im kleinen relative to W.

THEOREM IV. *If M is a continuous curve, a necessary and sufficient condition that every closed and connected subset of M be a continuous curve, is that M contain no R-sets.*

Before proceeding with the proof of Theorem IV, we shall prove two lemmas which are needed in the proof.

LEMMA *A. If A is a connected set, and if B is a connected subset of A, such that $A-B$ is disconnected, and if $A-B=H_1+H_2$ is any method of expressing $A-B$ as the sum of two mutually separated* sets, then $B+H_1$ and $B+H_2$ are connected.*

PROOF. Suppose $B+H_1$ were disconnected. Then $B+H_1$ can be expressed as the sum of two mutually separated sets T_1+T_2. Since B is connected, B lies entirely in one of these sets, say T_1. In that case, T_2 is a subset of H_1. Now the set $A=(B+H_1)+H_2=T_1+T_2+H_2=(T_1+H_2)+T_2$. But T_1 and T_2 are mutually separated, and so are H_2 and T_2, because H_1 and H_2 are mutually separated and T_2 is a subset of H_1. Therefore A it expressed as the sum of the two mutually separated sets: (T_1+H_2) and T_2, and A is therefore disconnected, contrary to hypothesis.

Therefore $B+H_1$ is connected. A similar proof shows that $B+H_2$ is connected.

LEMMA *B. Any maximal† connected subset of a continuous curve contained in a simple closed curve plus its interior, is also a continuous curve.*

PROOF. Let M be a continuous curve; J a simple closed curve; I the interior of J; K a maximal connected subset of M lying in $J+I$. We shall prove that K is a continuous curve.

The set K is closed and connected. We shall next show that K is connected im kleinen at every point. Let P be a point of K interior to J. Since the continuous curve M is connected im kleinen at P, if we construct any circle J_1 interior to J and having P as center, there can be constructed another circle J_2, interior to J_1 and having P as center, such that every point of M interior to J_2 can be joined to P by an arc in M interior to J_1. As K is a maximal connected set all such points and all points of such arcs lie in K, and therefore we can say that corresponding to the circle J_1 there exists the circle J_2 such that every point of K interior to J_2 can be joined to P by an arc in K interior to J_1.

* Two sets are said to be *mutually separated* if neither set contains a point or a limit point of the other set.

† A set A is said to be a *maximal connected subset* of a set B, if A is a connected subset of B which is not a proper subset of any other connected subset of B.

In case a circle J_3 is constructed about P as center but not interior to J, we can construct a circle J_1 about P as center which is interior to both J_3 and J. As above, a circle J_2 exists such that every point of K interior to J_2 can be joined to P by an arc in K interior to J_1 and therefore interior to J_3. Therefore K is connected im kleinen at P, where P is any point interior to J.

Let Q be a point of K on J. If K is not connected im kleinen at Q, it fails to be connected im kleinen at any point of a continu of condensation L. Since K is connected im kleinen at all points interior to J, it follows that L is a subset of J, and is therefore either an arc or the simple closed curve J. Let Q_1 be any point of L, but not an end point in case L is an arc. As Q_1 is a point at which K fails to be connected im kleinen, there is a circle J_4 about Q_1 as center such that no circle J_5 can be constructed within J_4 and with center at Q_1 and such that every point of K within J_5 can be joined to Q_1 by an arc in K lying within J_4. We shall obtain the desired contradiction by showing that such a circle J_5 can be constructed.

If $Q_2 Q_1 Q_3$ be an arc in L and interior to J_4, we can construct an arc joining Q_2 and Q_3, interior to J_4, and interior to J save for its end points. We can also construct an arc joining Q_2 and Q_3, interior to J_4, and exterior to J save for its end points. These two arcs form a simple closed curve J_6, which contains Q_2 and Q_3 and encloses $Q_2 Q_1 Q_3$, and which is interior to J_4. Since the continuous curve M is connected im kleinen at Q_1, corresponding to J_6 is a circle J_5 interior to J_6 and having Q_1 as center such that every point of M interior to J_5 can be joined to Q_1 by an arc in M interior to J_6. If any point of M interior to J and to J_5 is joined to Q_1 by an arc interior to J_6, and having points exterior to J, this arc has a first point in common with $Q_2 Q_1 Q_3$, since $Q_2 Q_1 Q_3$ is the subset of J which separates the interior of J_6 into points interior to J and points exterior to J. Therefore any point of M interior to J and to J_5 can be joined to Q_1 by an arc lying in $J+I$, and is therefore a point of K, as are all points on these arcs. Therefore all points of K interior to this circle J_5 can be joined to Q_1 by an arc in K interior to J_6 and therefore interior to J_4. Thus our supposition that K is not connected im kleinen at Q has lead to a contradiction.

Therefore K is connected im kleinen at every point, and is therefore a continuous curve.

PROOF OF THEOREM IV. We shall first prove that the given condition is necessary by showing that an R-set either is not a continuous curve, or contains a closed and connected subset which is not a continuous curve. That sets of the first type exist may be seen by the aid of the following

example in which the set N consists of W, which is the interval of the Y-axis between $y = 1$ and $y = -1$, and $N-W$, which is the curve, $y = \sin\frac{1}{x}$, for $0 < x \leqq 1$.

Let us suppose then that N is an R-set which is a continuous curve. Since $N-W$ is uniformly connected im kleinen relative to W, given $\varepsilon_1 > 0$, there exists a number $\delta_1 > 0$, such that if two points of $N-W$ are at a distance apart less than δ_1, they can be joined by a connected subset of $N-W$, which either lies entirely within a circle of radius ε_1, or is such that every point is at a distance less than ε_1 from a point of W. They can therefore be joined by an arc in $N-W$, satisfying one of the two conditions.*

If we put about each point of W as center a circle of radius $\delta_1/4$, by the Heine-Borel Theorem we can select from this infinite set a finite subset of circles, $C_1, C_2, \cdots, C_n$, covering W. Let P_i $(i = 1, 2, \cdots, n)$ be a point of $N-W$ lying within C_i. Since W is connected, each circle of $C_1 + C_2 + \cdots + C_n$ will intersect at least one other circle. We will arrange the set of circles in some order, $C_{n_1}, C_{n_2}, \cdots, C_{n_k}$, such that (1) each circle occurs at least once, and (2) each circle intersects both the preceding and the following circles in this arrangement. If C_i and C_j intersect $(i, j = 1, 2, \cdots, n)$, P_i and P_j are at a distance apart less than δ_1, and can therefore be joined by an arc in $N-W$, every point of which is at a distance less than either $(2\varepsilon_1 + \frac{1}{4}\delta_1)$ or ε_1, and therefore less than $3\varepsilon_1$, from a point of W.

If we construct the arcs, $P_{n_1}P_{n_2}$, $P_{n_2}P_{n_3}$, $\cdots$, $P_{n_{k-1}}P_{n_k}$, their sum is a continuous curve K_1, having no points in common with W, and such that every point of K_1 is at a distance less than $3\varepsilon_1$ from some point of W, and every point of W is at a distance less than $\delta_1/2$ (and therefore less than $\varepsilon_1/2$) from some point of K_1.

If $3\varepsilon_2$ is less than the distance from any point of W to any point of K_1, we will construct a continuous curve K_2 corresponding to ε_2 in the same way that K_1 corresponds to ε_1.

By this process we obtain an infinite sequence of continuous curves, $K_1, K_2, K_3, \cdots$, each one having no points in common with any other one or with W, and such that every point of K_i $(i = 1, 2, 3, \cdots)$ is at a distance less than $3\varepsilon_i$ from some point of W, and every point of W is at a distance less than $\varepsilon_i/2$ from some point of K_i. For $i \geqq 2$, $\varepsilon_i < \varepsilon_{i-1}/6$. From these considerations, it follows that the only limit points of $(K_1 + K_2 + K_3 + \cdots)$ are in W, and every point of W is a limit point of $(K_1 + K_2 + K_3 + \cdots)$.

* Report, p. 294. See also: R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeit., vol. 15 (1922), p. 255.

Let P be a point of W. Let α be a positive number such that there are points of W at a distance greater than α from P. Since N is connected im kleinen at P, given $\alpha > 0$, there exists a number $\beta > 0$ such that any point of N at a distance less than β from P can be joined to P by an arc in N of diameter less than α. Only a finite number of the sets, $K_1, K_2, K_3, \cdots$, have no points closer to P than β, and therefore there exists an integer m such that, for $i \geqq m$, K_i has points at a distance less than β from P. Let $Q_i\ (i \geqq m)$ be a point of K_i at a distance less than β from P. Let us join each of the points, $Q_m, Q_{m+1}, Q_{m+2}, \cdots$, to P by an arc in N lying within a circle of radius α having P as center, which circle we shall denote by C.

We will now show that $K = W + (K_m + K_{m+1} + \cdots) + (\widehat{Q_m P} + \widehat{Q_{m+1} P} + \cdots)$ + limit points is a subset of N which is closed and connected, but which is not a continuous curve because it is not connected im kleinen. The set K is connected, by construction, because it consists of a countable infinity of connected sets [W, and $(K_i + \widehat{Q_i P})$], each of which contains P, forming a connected set which remains connected upon the addition of its limit points. But K is also closed, since we have included in K the limit points of $(\widehat{Q_m P} + \cdots)$, all of which lie within or on the circle C.

Let A be a point of W exterior to C. If ε is less than the distance from A to any point of C, then although there are points of K [i. e., points of $(K_m + K_{m+1} + \cdots)$] arbitrarily close to A, these points can be joined to A only by a connected set in K which lies partly within C, and therefore is of diameter greater than ε.

Therefore K is not connected im kleinen at A, and is therefore not a continuous curve.

Continuing our proof of Theorem IV, we shall next prove that the given condition is sufficient by showing that, if some closed and connected subset K of a continuous curve M is not a continuous curve, then M contains an R-set.

By means of R. L. Moore's characterization of closed and connected sets which are not continuous curves,* it is evident that K must satisfy the following conditions: there exists a square $ABCD$ in the plane, and a countable infinity of closed and connected sets, $W, K_1, K_2, K_3, \cdots$, such that (1) each of these sets is a subset of K, has at least one point on AD and at least one point on BC, has no points on AB or CD, and is a subset of the set H consisting of $ABCD$ and its interior, (2) no two of these sets have a point in common, and no one of them (save possibly W) is a proper subset of any connected point-set which is common to K and H,

* Report, pp. 296–97.

(3) the set W is the sequential limiting set* of the sequence of sets, $K_1, K_2, K_3, \cdots$.

From (1) and (2), if K' is any one of the sets, $W, K_1, K_2, K_3, \cdots$, then $H - K'$ is disconnected into at least two connected parts, of which one contains AB and a second contains CD, and no connected part save these two contains any point of $W + K_1 + K_2 + K_3 + \cdots$. In the case of $H - W$, either H_1, the connected part containing AB, or H_2, the connected part containing CD, contains an infinite number of sets of $K_1, K_2, K_3, \cdots$. Suppose H_1 contains an infinite number.

Let E be a point of W interior to $ABCD$. If we put a circle of radius ϵ about E as center, ϵ being chosen so that the circle is interior to $ABCD$, then, since M is a continuous cnrve, there exists a number δ, such that any point of M within a circle of radius δ about E as center can be joined to E by an arc in M within the circle of radius ϵ. By (3) the circle of radius δ contains a point F of K_a, where K_a is one of the sets, $K_1, K_2, K_3, \cdots$, which lie in H_1. The arc EF in M and within the circle of radius ϵ contains as a subset an arc E_1F_1 such that E_1 is in W, F_1 is in K_a, $\underset{\frown}{E_1F_1}$ has no points in common with either W or K_a, E_1F_1 lies, except for E_1, in H_1.

Let N_1 be the maximal connected subset of M lying in H_1 and containing F_1. Evidently N_1 will contain $\underset{\frown}{E_1F_1}$, the set K_a, and an infinite number of other sets of $K_1, K_2, K_3, \cdots$. Since any limit point of H_1 not contained in H_1 is contained in W, all limit points of N_1 which are not in N_1 are in W. Since N_1 contains an infinite number of the sets: $K_1, K_2, K_3 \cdots$, and since W is their sequential limit, it follows that every point of W is a limit point of N_1. Therefore $N_1 + W$ forms a closed set, which we shall designate by N.

We shall now show that N is an R-set. N is closed and connected, and W is a continu of condensation of N. We shall next prove that $N - W$ is uniformly connected im kleinen relative to W.

Let us select a positive number ϵ. From (3), corresponding to ϵ there is a positive integer n such that, for $i > n$, every point of K_i and every point of that connected part S of $H_1 - K_i$ which has limit points in W

* The point-set M is said to be the *limiting set of the sequence of point-sets* $M_1, M_2, M_3, \cdots$ provided that (a) each point of M is the sequential limit point of an infinite subsequence of some sequence of points, $P_1, P_2, P_3, \cdots$, such that, for every n, P_n belongs to M_n, (b) if $P_1, P_2, P_3, \cdots$ is a sequence of points such that, for every n, P_n belongs to M_n, then M contains the sequential limit point of every subsequence of $P_1, P_2, P_3, \cdots$ that has a sequential limit point. If the further condition is satisfied that every infinite subsequence of the sequence, $M_1, M_2, M_3, \cdots$, has the same limiting set M, then M is said to be the *sequential limiting set* of the sequence $M_1, M_2, M_3, \cdots$.

is at a distance less than ϵ from some point of W. Let a number i be chosen such that $i > n$ and such that K_i is in N. We shall denote $S + K_i$ by H_3.

The set $N - W$ is connected, because $N - W = N_1$, and N_1 is connected by definition. Therefore any two points X and Y of $N - W$ in H_3 can be joined by a connected set in $N - W$. If this connected set has points in $H_1 - H_3$, the removal of K_i disconnects it into the subsets lying in $H_3 - K_i$ and in $H_1 - H_3$. By Lemma A, the subset in $H_3 - K_i$ and the set K_i form a connected set joining X and Y in H_3. Therefore any two points in H_3 can be joined by a connected set in H_3, and therefore such that every point is at a distance less than ϵ from a point of W.

In the set $S = H_3 - K_i$ let us construct an arc $C_1 D_1$ such that C_1 is on $\overset{\smile}{BC}$, D_1 is on $\overset{\smile}{AD}$, and $C_1 D_1$ is interior to $ABCD$. The arc $C_1 D_1$ and the arcs $D_1 A$, $\overset{\smile}{AB}$, and $\overset{\smile}{BC_1}$ of $ABCD$ form a simple closed curve J which contains or encloses all points of K_i, and which contains W in its exterior. Let L be the maximal connected set of M lying in J plus its interior and containing K_i. Since J plus its interior is a subset of H_1, L is a subset of N. By Lemma B, L is a continuous curve.

Since L is a continuous curve, L is uniformly connected im kleinen. Therefore corresponding to our given ϵ, there exists a positive number δ_1 such that, if the distance between any two points of L is less than δ_1, the two points lie in a connected subset of L which lies in a circle of radius ϵ.

No point of L save points on $C_1 D_1$ can be a limit point of points of $N - L$. Therefore there is a positive number δ_2 which is less than the distance from any point of K_i to any point of $N - L$. Note also that all points of $N - L$ lie in $S + W = H_3 - K_i + W$ and none lie in $H_1 - H_3$.

We shall now show that, if δ is selected so as to satisfy the inequalities $\delta < \delta_1$, $\delta < \delta_2$, then, if X and Y are any two points of $N - W$ at a distance apart less than δ, they can be joined by a connected subset of $N - W$ which is such that either it lies within a circle of radius ϵ, or every point is at a distance less than ϵ from a point of W.

If X and Y are points of $N - W$ in H_3, we have shown that they can be joined by a connected set such that every point is at a distance less than ϵ from a point of W. If X is a point of $N - W$ in $H_1 - H_3$, it is a point of L and, since $\delta < \delta_2$, Y is also a point of L. Then, since $\delta < \delta_1$, X and Y can be joined by a connected subset of L (and therefore of $N - W$) lying in a circle of radius ϵ.

Therefore N is an R-set, which completes the proof of Theorem IV.

THEOREM V. *If M is a continuous curve, a necessary and sufficient condition that every closed and connected subset of M be a continuous curve is*

that, given any positive number ε, M contains at most a finite number of mutually-exclusive closed and connected sets of diameter greater than ε.

PROOF. The given condition is sufficient. For if some closed and connected subset N of M is not a continuous curve, then from R. L. Moore's characterization of closed and connected sets which are not continuous curves it follows that N will contain an infinite number of closed and connected sets of diameter greater than some fixed number ε, and no two of the sets will have any points in common.

To show that the condition is necessary, we shall show that, if every closed and connected subset of M is a continuous curve, the supposition that M contains an infinite number of mutually-exclusive closed and connected sets of diameter greater than some fixed number ε leads to a contradiction. Let these sets be $M_1, M_2, \cdots$.

In $M_i (i = 1, 2, 3, \cdots)$ let A_i and B_i be two points whose distance apart is more than ε. Since M_i is a continuous curve, by hypothesis, there is an arc $A_i B_i$ in M_i whose diameter is more than ε. Let A be a limit point of the points $A_1, A_2, \cdots$. From the set of points $A_1, A_2, \cdots$, select a subset $A_{n_1}, A_{n_2}, \cdots$, approaching A as a sequential limit point. If B is a limit point of the corresponding set $B_{n_1}, B_{n_2}, \cdots$, then we can select a subset of $B_{n_1}, B_{n_2}, \cdots$, say $B_{j_1}, B_{j_2}, \cdots$, approaching B as a sequential limit point. If we replace A_{j_i} by C_i $(i = 1, 2, \cdots)$ and B_{j_i} by D_i $(i = 1, 2, \cdots)$, we have this state of affairs: an infinite sequence of arcs $C_1 D_1, C_2 D_2, \cdots$ in M, no pair having any points in common, such that $C_1, C_2, \cdots$ approach A as their sequential limit point, the distance from C_i to D_i $(i = 1, 2, \cdots)$ being more than ε.

An arc $C_1 A$ can be passed through all the points of $C_1, C_2, \cdots$, and an arc $D_1 B$ can be passed through $D_1, D_2, \cdots$.* Moreover, these arcs can be drawn so that they have no points in common and so that they have only C_1 and D_1 in common with the arc $C_1 D_1$ of M. Each of the arcs $C_2 D_2, C_3 D_3, \cdots$, of M contains a subarc, $E_2 F_2, E_3 F_3, \cdots$, such that E_i is on $C_1 A$, F_i is on $D_1 B$, and $E_i F_i$ has no other points in common with $C_1 A$ or $D_1 B$.

From the set of arcs $E_2 F_2, E_3 F_3, \cdots$, we can select a subset $E_{n_1} F_{n_1}, E_{n_2} F_{n_2}, \cdots$ such that, if $E_j F_j$ and $E_k F_k$ are any two distinct arcs of the subset, then either $E_j F_j$ is interior (save possibly for its end points) to the simple closed curve $E_k F_k D_1 C_1$, or $E_k F_k$ is interior (save possibly for its end points) to the simple closed curve $E_j F_j D_1 C_1$.

Let E be a limit point of the set $E_{n_1}, E_{n_2}, \cdots$. The point E will be on the arc $C_1 A$. Let $G_1, G_2, \cdots$ be a subset of the set $E_{n_1}, E_{n_2}, \cdots$ which

* R. L. Moore and J. R. Kline, *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, Ann. of Math., vol. 20 (1919), p. 218.

approaches E as a sequential limit point, and such that G_i lies between G_j and E on the arc $C_1 A$, if $j<i$. Except possibly for a finite number of points, the corresponding points $H_1, H_2, \cdots$ of the set $F_2, F_3, F_4, \cdots$ will occur on $D_1 B$ in the same order as the points $G_1, G_2, \cdots$ occur on $C_1 A$. Suppose this holds true for all points after H_n. If F is a limit point of the set $H_n, H_{n+1}, \cdots$, it cannot lie between any two of them, nor can H_n lie between F and H_{n+1}, and F is therefore a sequential limit point of $H_n, H_{n+1}, \cdots$ and H_i lies between H_j and F on the arc $D_1 B$ if $n \leqq j<i$.

We have now an infinite sequence of arcs $G_n H_n, G_{n+1} H_{n+1}, \cdots$, such that G_i is on $C_1 A$, and H_i is on $D_1 B$, and such that the simple closed curve $G_i H_i H_j G_j$ contains and encloses the arc $G_k H_k$ if $i<k<j$. Let W denote the set of limit points of $G_1 H_1, G_2 H_2, \cdots$; W is closed and contains E and F. If W were not connected, we could enclose V, the largest connected part of W that contains E, by a simple closed curve J not containing any points of W and having points of $W-V$ in its exterior.*

Let P be a point of W exterior to J. A circle J' exterior to J about P encloses points of infinitely many of the arcs $G_n H_n, G_{n+1} H_{n+1}, \cdots$. Since J encloses E, J encloses points of all but a finite number of the arcs $G_n H_n, \cdots$. There are therefore an infinite number of arcs having points interior to J and also points exterior to J (i. e., interior to J'), and therefore having points on J. Some limit point of $G_n H_n, G_{n+1} H_{n+1}, \cdots$ therefore lies on J; that is, some point of W is on J, contrary to our hypothesis on J. Therefore W is connected.

The set W, being a closed and connected subset of M, is a continuous curve. Draw the arc EF in W. If $W-EF \neq 0$, every point of $W-EF$ is enclosed by the simple closed curve $EFH_n G_n$. Note also that no one of the arcs $G_n H_n, G_{n+1} H_{n+1}, \cdots$ has any points in common with W, otherwise our hypotheses on $G_n H_n, G_{n+1} H_{n+1}, \cdots$, are contradicted.

The set M is connected im kleinen at E. Therefore, if we put a circle about E such that F lies in its exterior, there exists a number $k \geqq n$, such that, if $i \geqq k$, an arc $G_i E$ in M lies within the circle.

Let N be $G_k H_k + G_{k+1} H_{k+1} + \cdots$ plus $G_k E + G_{k+1} E + \cdots$ plus W plus limit points of $G_k E + G_{k+1} E + \cdots$ all of which lie within the circle about E. The set N is a closed and connected subset of M, which is therefore a continuous curve. But N is not connected im kleinen at F because, if a circle is put about F having no points in common with the one about E, it will contain points of N arbitrarily close to F, these points being points of $G_k H_k + G_{k+1} H_{k+1} + \cdots$, and, since each of these arcs is distinct and has no points in common with W, points on different arcs can be joined

* L. Zoretti, *Sur les fonctions analytiques uniformes*, Journ. de Math., 6e ser., vol. 1 (1905), p. 10.

in N only by connected sets having points within the circle about E and therefore exterior to the circle about F.

We have thus obtained the desired contradiction and the proof of Theorem V is completed.

THEOREM VI. *If M is a continuous curve, a sufficient (but not necessary) condition that every closed and connected subset of M be a continuous curve is that, given any positive number ϵ, M can be expressed as the sum of a finite number of closed and connected point-sets each of diameter less than ϵ, and each pair of sets having at most a finite number of points in common.*

PROOF. Let $M = M_1 + M_2 + \cdots + M_n$, each a closed and connected set of diameter less than ϵ, and let the points of M common to two or more of these sets be $P_1, P_2, \cdots, P_k$. We shall determine the maximum possible number of mutually exclusive closed and connected sets of diameter greater than ϵ that are contained in M.

Any such set does not lie entirely in any one of the sets $M_1, M_2, \cdots, M_n$ and, being connected, must contain at least one of the points $P_1, P_2, \cdots, P_k$. No two of the sets of diameter greater than ϵ can contain the same point, P_i, as the sets are mutually exclusive. Therefore k is the maximum number of such sets.

Therefore, by Theorem V, every closed and connected subset is a continuous curve.

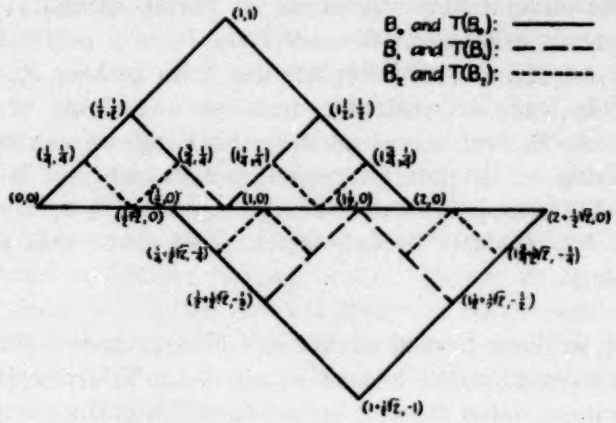
The following example shows that the condition is not necessary. Let B_0 be the straight line intervals from $(0, 0)$ to $(2, 0)$, from $(2, 0)$ to $(1, 1)$, and from $(1, 1)$ to $(0, 0)$. Let B_1 be the straight line intervals from $(1, 0)$ to $(\frac{1}{2}, \frac{1}{2})$; from $(1, 0)$ to $(1\frac{1}{2}, \frac{1}{2})$. Let B_2 be the straight line intervals from $(\frac{1}{2}, 0)$ to $(\frac{1}{4}, \frac{1}{4})$; from $(\frac{1}{2}, 0)$ to $(\frac{3}{4}, \frac{1}{4})$; from $(1\frac{1}{2}, 0)$ to $(1\frac{1}{4}, \frac{1}{4})$; from $(1\frac{1}{2}, 0)$ to $(1\frac{3}{4}, \frac{1}{4})$. In general, the intervals of $B_i (i = 1, 2, 3, \cdots)$ form with the intervals of $B_0 + B_1 + \cdots + B_{i-1}$ a set of 2^i isosceles triangles, each of which is such that (1) the base is on the X-axis and is of length $1/2^{i-1}$, (2) the x-coördinates of the end points of the base occur among the numbers $\frac{0}{2^{i-1}}, \frac{1}{2^{i-1}}, \frac{2}{2^{i-1}}, \frac{3}{2^{i-1}}, \cdots, \frac{2^i}{2^{i-1}}$, (3) the sides have the slopes $+1$ and -1, (4) the y-coördinate of the vertex is $1/2^i$. Given B_i, B_{i+1} is the set of 2^{i+1} intervals formed by joining, in each triangle, the mid point of the base to the mid point of each side.

Let $A_1 = B_1 + B_2 + B_3 + \cdots$. Let A_2 be the set (x', y') obtained by subjecting the points (x, y) of A_1 to the transformation $x' = x + \frac{1}{2}\sqrt{2}, y' = -y$. Let $M = A_1 + A_2$. (See Figure.)

That every closed and connected subset of M is a continuous curve is best seen by means of Theorem IV. We shall next show that there exists

no finite set of points in M such that M can be expressed as the sum of a finite number of closed and connected sets each of diameter less than 1/2 and each pair having in common only points of this finite set.

Suppose such a finite set of points did exists in M. Call them $P_1, P_2, \cdots, P_k$. On the straight line interval from $(\frac{1}{2}\sqrt{2}, 0)$ to $(2, 0)$, which is common to A_1 and A_2, there occur at least two points of the set, say P_1 and P_2. If the x-coördinate of P_1 is not of the form $a/2^b$, where a and b are integers, P_1 is the sequential limit of a sequence of isosceles triangles in A_1, whose bases contain P_1, each triangle in the sequence containing and enclosing the triangles that follow. Similarly, unless the x-coördinate of P_1 is of the form $a/2^b + \frac{1}{2}\sqrt{2}$, where a and b are integers, P_1 is the sequential limit of a similar sequence of triangles in A_2. Since the



x-coördinate of P_1 cannot be of both these forms, one of the above cases holds for any point P_1. Suppose in this case that the x-coördinate of P_1 is not of the form $a/2^b$.

One of the sets $M_1, M_2, \cdots, M_n$ contains P_1 and an interval of the X-axis having one end at P_1 and the other end between $(0, 0)$ and P_1. Call this set M_1. One of the sets, say M_2, contains P_1 and an interval of the X-axis having one end at P_1 and the other end between P_1 and $(2 + \frac{1}{2}\sqrt{2}, 0)$. Since by hypothesis the x-coördinate of P_1 is not of the form $a/2^b$, there is an infinite sequence of triangles in A_1, one end of whose base lies in M_1, and the other end in M_2. If α is a number less than the distance from P_1 to any of the points $P_2, P_3, \cdots, P_k$, there is a triangle in A_1 of diameter less than α whose base contains P_1 and whose other two sides join a point of M_1 to a point of M_2 by an arc not containing any points of $P_1 + P_2 + \cdots + P_k$. Therefore $M_1 = M_2$.

If the x-coordinate of P_1 is also not of the form $a/2^b + \frac{1}{2}\sqrt{2}$, P_1 is interior to a simple closed curve in M of diameter less than α, and therefore entirely in M_1. In this case none of the sets $M_2, M_3, \cdots, M_n$ contain P_1.

If the x-coordinate of P_1 is of the form $a/2^b + \frac{1}{2}\sqrt{2}$, P_1 is an end point of two straight line intervals of slope $+1$ and -1 in A_2. Denote one of these by P_1Q. Let M_3 denote the set which contains P_1 and an interval of P_1Q having one end at P_1. There is an infinite set of line intervals in A_2, perpendicular to P_1Q, joining P_1Q to the X-axis, and approaching P_1 as a sequential limit. As before, there are an infinite number of these at a distance less than α from P_1 and therefore joining a point of M_1 to a point of M_3 by an arc not containing any points of $P_1 + P_2 + \cdots + P_k$. Therefore $M_1 = M_3$. Therefore, if M_1 contains P_1, it contains an interval of each of the straight line intervals in A_2 having an end at P_1. In this case also there is a minimum distance from P_1 to a point of any of the sets $M_2, M_3, \cdots, M_n$, and therefore none of them contain P_1.

Therefore P_1 can be eliminated from our finite set of points and $P_2 + P_3 + \cdots + P_k$ will serve as well. Similarly we can eliminate any other point lying on the interval from $(\frac{1}{2}\sqrt{2}, 0)$ to $(2, 0)$, in which case this interval lies entirely in one of the sets, say M_1. M_1 is then of diameter greater than $1/2$, contrary to hypothesis. This shows that the condition is not necessary.

IV.

In part III we have introduced the idea of a point-set being *uniformly connected im kleinen relative to another point-set.* This new idea gives us a characterization of the class of point-sets which R. L. Moore has called simple continuous curves,* namely the simple continuous arc, the simple closed curve, the ray, and the open curve,—the point-sets which are in continuous $(1-1)$ correspondence† respectively with an interval of a straight line, a circle, a half-line, and a line.

THEOREM VII: *If M is a closed and connected plane point-set, such that $M-W$ is uniformly connected im kleinen relative to W, where W is any closed and connected subset of M consisting of more than a single point, then M is a simple continuous arc, a simple closed curve, a ray, or an open curve.*

PROOF. We have shown in part III that if $M-W$ is uniformly connected im kleinen relative to W, where W is closed, then $M-W$ is connected im kleinen. Since W is arbitrary, it follows that M is connected im kleinen

* R. L. Moore, *Concerning simple continuous curves*, Trans. Amer. Math. Soc., vol. 21 (1920), pp. 333–47.

† A correspondence T which sends M into $T(M)$ is said to be *continuous* if, in case the point P of M is a limit point of N, a subset of M, then $T(P)$ is a limit point of $T(N)$.

at every point, and is therefore a continuous curve. There are two cases to be considered.

CASE I. M contains a simple closed curve J. Since by hypothesis $M-J$ is uniformly connected im kleinen relative to J, no point of J is a common limit point of a set of points of M interior to J and a set exterior to J.

If M contains an arc PQR, such that P and R are on J, and all other points of the arc are either interior to J or exterior to J, then M contains three arcs, PQR, PSR, PTR, having only P and R in common. One arc, say PSR, lies within the domain bounded by the other two. Since $M-(PQR+PTR)$ is uniformly connected im kleinen relative to $(PQR+PTR)$, and since P is a limit point of points on the arc PSR interior to the simple closed curve $(PQR+PTR)$, there exists a positive number ε_1, which is less than the distance from P to any point of M exterior to the simple closed curve $(PQR+PTR)$. In the same way we can show the existence of positive numbers ε_2 and ε_3 which are less respectively than the distance from P to any point of M interior to $(PQR+PSR)$ and the distance from P to any point of M interior to $(PSR+PTR)$. Let us now select a positive number α which is less than the smallest of the numbers ε_1, ε_2, ε_3 and also less than the distance from P to R. Let us also select a point U, on the arc PSR, such that the diameter of the arc PU of PSR is less than $\alpha/4$. We shall now show that $M-PU$ is not uniformly connected im kleinen relative to PU. For, if we let $\varepsilon = \alpha/4$, no matter what value δ is given we can always select a point X, different from P on PQR, and a point Y, different from P on PTR, such that the distances from P to X and P to Y are less than $\delta/2$ and also less than $\alpha/4$. The distance from X to Y is less than δ, and yet any connected subset N of M containing X and Y either contains R or contains a point not on PQR, PSR, or PTR (in either case, N contains a point whose distance from P is more than α) and therefore N cannot lie within a circle of radius $\alpha/4$, nor is every point of N at a distance less than $\alpha/4$ from a point of PU. We have thus shown that the assumption that the arc PQR exists, leads to a contradiction.

If there are points of M not on J, and there are no arcs in M joining two points of J, save arcs that lie wholly in J, then any point Q of $M-J$ lies in a maximal connected subset T of $M-J$, which is closed save for a single limit point P on J. Let R be another point of J, and let us select a positive number α which is less than the distance from P to R. Let us also select a point U of T such that an arc PU exists in T of diameter less than $\alpha/4$. Then we shall show that $M-PU$ is not uniformly connected im kleinen relative to PU by a proof exactly the same as in the preceding paragraph, save that N in this case necessarily con-

tains R. We have thus shown that the assumption that there are points of M not on J leads to a contradiction. Therefore, in case I, $M = J$.

CASE II. M contains no simple closed curve.* Let AB be an arc in M. If AB contains a point P, $A \neq P \neq B$, which is a limit point of $M - AB$, and if T is any maximal connected subset of $M - AB$ having P as a limit point, then $M - (T + P)$ is disconnected, one maximal connected subset of $M - (T + P)$ containing $AP - P$, another containing $PB - P$. Since P is a common limit point of these two subsets of $M - (T + P)$, it is evident that $M - (T + P)$ is not uniformly connected im kleinen relative to $(T + P)$; for if we select X and Y according to our previous method, there is no connected subset of $M - (T + P)$ containing both X and Y. We have thus shown that the assumption that the point P exists leads to a contradiction. The points A and B may be limit points of $M - AB$ so that M consists of a set of arcs having only their end points in common. Therefore, in case II, M is a simple continuous arc, a ray, or an open curve, and the theorem is proved.

In conclusion, I wish to point out that the theorem is not true if *$M - W$ is uniformly connected im kleinen relative to W* is replaced by either the weaker condition, *$M - W$ is connected im kleinen*, or the independent condition *$M - W$ has the S-property*. For example, under these hypotheses M might consist of three arcs having only their end points in common.

* For a discussion of this type of continuous curve see S. Mazurkiewicz, loc. cit. pp. 119–130, and R. L. Wilder, Bull. Amer. Math. Soc., vol. 29 (1923), p. 118.

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ON THE OSCILLATIONS OF A LINE NEAR A POSITION OF EQUILIBRIUM.

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If a deformable line of finite length, and possessing a continuous linear density, is fixed at its end-points in the space of three dimensions, while its remaining points are in equilibrium under the action of forces which vary from point to point, then a small displacement from this position will give rise, in general, to a motion of the whole line which can be called an oscillation. It is the object of this paper to study the corresponding motion in n dimensions by two types of approximation; first, by supposing the equations of motion reduced to a linear form; and second, by considering certain related dynamical systems each possessing a finite number of degrees of freedom. In both cases a fundamental role is played by a system of orthogonal functions which has been studied by Hilbert.*

A deformable line fixed at two points may be considered as a dynamical system with an infinite number of degrees of freedom; just as in the case of a finite number of degrees of freedom it is convenient to transform the quadratic forms in the kinetic and potential energies to sums of squares and to adopt as new variables those which appear in these sums, so it is convenient in the present case to transform the corresponding integrals or *functionals* into infinite series of squares; the variables which appear directly in these series are the generalized Fourier coefficients in the expansion of the coördinates of an arbitrary point on the line, in terms of certain orthogonal functions of the arc length in a "normal" state. In the first approximation the generalized equations of variation of the system are set up and integrated without difficulty; in the second, the coördinates of an arbitrary point on the moving line are considered as represented by expansions of order n in terms of the orthogonal functions, while the equations are not restricted to the linear form. The degree of the resulting approximation is not taken up here.

1. Suppose $W(x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n', s)$ continuous together with its partial derivatives of the first three orders with respect to all its arguments in a region (D), and let C be a curve whose points are interior points of (D), defined by the equations $x_i = x_i^0(s)$, $0 \leqq s \leqq 1$, $i = 1, 2, \cdots, n$;

* D. Hilbert: *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (1912), pp. 205–212.

suppose the first and second derivatives to exist and be continuous on this interval, and assume the equations

$$(1) \qquad \frac{\partial W}{\partial x_i} - \frac{d}{ds}\left(\frac{\partial W}{\partial x_i'}\right) = 0, \qquad x_i' = \frac{dx_i}{ds}, \qquad i = 1, \cdots, n,$$

satisfied identically along C. If s denotes arc length along C from an initial point, and $g(s)$ the linear density along C, assumed positive and continuous, $0 \leqq s \leqq 1$, the motion of a curve $\bar{C}$ having the same fixed end-points and such that C can be continuously deformed into $\bar{C}$ can be represented by a system of functions $\bar{x}_i = \bar{x}_i(s, t)$, $i = 1, 2, \cdots, n$. Let $\bar{s}$, $\bar{g}$, denote the arc length from the initial point and the linear density, respectively, along $\bar{C}$; then the kinetic energy and the potential energy of the curve $\bar{C}$ will be assumed to be equal,* respectively to the integrals

$$(2) \qquad \begin{aligned} T &= \frac{1}{2}\int_0^{\bar{l}} \bar{g}\left[\sum_{i=1}^{n}\left(\frac{\partial \bar{x}_i}{\partial t}\right)^2\right] d\bar{s} = \frac{1}{2}\int_0^1 g\left[\sum_{i=1}^{n}\left(\frac{\partial \bar{x}_i}{\partial t}\right)^2\right] ds \\ V &= \int_0^1 W\left(\bar{x}_1, \cdots, \bar{x}_n, \frac{\partial \bar{x}_1}{\partial s}, \cdots, \frac{\partial \bar{x}_n}{\partial s}, s\right) ds. \end{aligned}$$

The equations of motion of the line will be assumed such that Hamilton's principle is satisfied; in addition, it will be assumed that $\bar{x}_i(s, t) = x_i^0(s) + \varepsilon x_i(s, t)$, where $x_i(s, t)$ possesses continuous partial derivatives of the first and second orders with respect to both variables, and $x_i(0, t) = x_i(1, t) = 0$. Consequently,

$$(3) \qquad \begin{aligned} T &= \varepsilon^2 \bar{T}, & \bar{T} &= \frac{1}{2}\int_0^1 \left[g\sum_{i=1}^{n}\left(\frac{\partial x_i}{\partial t}\right)^2\right] ds, \\ V &= V^0 + \varepsilon^2 \bar{V}, & \bar{V} &= \frac{1}{2}\int_0^1 Q\left(x_1 \cdots x_n, \frac{\partial x_1}{\partial s}, \cdots, \frac{\partial x_n}{\partial s}, s\right) + \varepsilon \bar{V}^1; \end{aligned}$$

$dV/d\varepsilon = 0$ as a consequence of equations (1), and Q represents the sum of the terms of the second degree in the Taylor expansion of W in powers of ε. From the equation

* If the energy is given directly in terms of $\bar{x}_i$, $\partial \bar{x}_i/\partial \bar{s}$, it can be thrown into the form above by means of the relations

$$\frac{d\bar{s}}{ds} = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial \bar{x}_i}{\partial s}\right)^2} \qquad \frac{\partial \bar{x}_i}{\partial \bar{s}} = \frac{\partial \bar{x}_i}{\partial s}\Big/\frac{d\bar{s}}{ds} \qquad i = 1, \cdots, n.$$

The equation of continuity, $g\,ds = \bar{g}\,d\bar{s}$, is satisfied because C *can* be deformed into $\bar{C}$, even though this deformation may not actually take place during the motion, and any such continuous deformation must satisfy the equation of continuity.

$$\delta\int_{t_0}^{t_1}(T-V)\,dt = 0, \tag{4}$$

we obtain

$$\delta\int_{t_0}^{t_1}(\bar{T}-\bar{V})\,dt = 0, \tag{5}$$

and a first approximation to the motion can be obtained by setting $\epsilon = 0$. The equations become

$$g(s)\frac{\partial^2 x_i}{\partial t^2} = \frac{1}{2}\left\{\frac{\partial}{\partial s}\left[\frac{\partial Q}{\partial \frac{\partial x_i'}{\partial s}}\right] - \frac{\partial Q}{\partial x_i}\right\}, \qquad i = 1, \cdots, n. \tag{6}$$

Particular solutions of (6) can be found by assuming a solution of the form $x_k = e^{i\mu t}u_k(s)$; the equations being linear, we obtain

$$L_k(u) = \frac{1}{2}\left[\frac{d}{ds}\left(\frac{\partial Q}{\partial u_k'}\right) - \frac{\partial Q}{\partial u_k}\right] = -\mu^2 g(s)u_k, \qquad k = 1, \cdots, n. \tag{7}$$

These equations are in the form studied by Hilbert, (l. c., p. 207) for $n = 2$, by a method which can be applied without change for n arbitrary, if the Hessian $\left|\frac{\partial^2 Q}{\partial u_i' \partial u_j'}\right|$ does not vanish on the given interval, as will be assumed; and if, in addition,* the point $s = 1$ is not a conjugate point of $s = 0$.

The solution of the equations

$$L_i(u) = f_i(s), \qquad u_i = 0, \quad s = 0, \quad s = 1, \qquad i = 1, \cdots, n, \tag{8}$$

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* The system of equations $L_i(u) = f_i$, $i = 1, \cdots, n$, is equivalent to the system of order $2n$,

$$\frac{du_i}{ds} = u_{n+i}, \quad \frac{du_{n+i}}{ds} + \sum_{k=1}^{2n} a_{ik}u_k = \varphi_i \qquad i = 1, \cdots, n, \tag{a}$$

where the functions φ_i are linear combinations of the functions f_i. If the $2n$ solutions of the homogeneous system obtained from (a), $(u_1^{(k)}, u_2^{(k)}, \cdots, u_{2n}^{(k)})$, $k = 1, \cdots, n$, form a linearly independent set, and if $(\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_{2n})$ is a particular solution of (a), the general solution can be written

$$u_i = \bar{u}_i + \sum_{k=1}^{2n} C_k u_i^{(k)} \qquad i = 1, \cdots, 2n.$$

If a particular solution be required to vanish at $s = 0$, $s = 1$, the resulting equations

$$\bar{u}_i(s) + \sum_{k=1}^{2n} C_k u_i^{(k)}(s) = 0, \qquad s = 0, \quad s = 1$$

can be written in the form

$$u_i(s) = \int_0^1 \Big[\sum_{r=1}^{n} G_{ir}(s, z) f_r(z) \Big] dz, \qquad i = 1, \cdots, n, \tag{9}$$

the functions $G_{ik}(s, z)$ satisfying the relations $G_{ik}(s, z) = G_{ki}(z, s)$, $G_{ik}(s, 0) = G_{ik}(s, 1) = 0$, and $\frac{d}{ds} G_{ik}(s, z)$ is discontinuous at $s = z$. Hence the function

$$H(s, z) = -G_{ik}(s - i + 1,\ z - k + 1), \qquad \begin{matrix} i - 1 \leqq s \leqq i, \\ k - 1 \leqq z \leqq k, \end{matrix}$$
$$(i, k = 1, \cdots, n)$$

is symmetric in s and z. Then if

$$u(s) = u_i(s - i + 1), \quad g(s) = g(s - i + 1), \quad i = 1, \cdots, n, \tag{10}$$
$$(i - 1 \leqq s \leqq i),$$

we obtain

$$u(s) = \mu^2 \int_0^1 H(s, z) g(z) u(z) dz; \tag{11}$$

and, if $\lambda = \mu^2$, $x(s) = \sqrt{g(s)}\, u(s)$, $K(s, z) = \sqrt{g(s) g(z)}\, H(s, z)$, equations (9) become,

$$x(s) = \lambda \int_0^1 K(s, z) x(z) dz. \tag{12}$$

Since the kernel $K(s, z) = K(z, s)$, the classical theorems of Hilbert and Schmidt can be applied; there exists at least one value of λ for which (12) has a solution not identically zero, and all the characteristic numbers λ_k

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possess a unique solution for the constants C_k if the determinant

$$D = \begin{vmatrix} u_1'(0) & u_1''(0) & \cdots & u_1^{(2n)}(0) \\ u_2'(0) & u_2''(0) & \cdots & u_2^{(2n)}(0) \\ \cdots & \cdots & \cdots & \cdots \\ u_n'(0) & u_n''(0) & \cdots & u_n^{(2n)}(0) \\ u_1'(1) & u_1''(1) & \cdots & u_1^{(2n)}(1) \\ \cdots & \cdots & \cdots & \cdots \\ u_n'(1) & u_n''(1) & \cdots & u_n^{(2n)}(1) \end{vmatrix}$$

does not vanish. The assumption $D \neq 0$ is equivalent to the assumption that the point $s = 1$ is not a conjugate point of $s = 0$; (see Poincaré, *Les Méthodes Nouvelles*, t. 3, p. 262). If $D = 0$, the integral in (9) cannot give the general solution of (8), since (9) implies that when $f_i = 0$, $i = 1, \cdots, n$, then $u_i = 0$, $i = 1, \cdots, n$; while in this case there exists a solution of the homogeneous equations which vanishes at the end-points. In applying Hilbert's formulae it is necessary, therefore, to assume $D \neq 0$.

are real. From (10) the corresponding particular solutions of (6) are obtained immediately.

With the aid of the orthogonal functions the integral

$$J = \frac{1}{2}\int_0^1 Q(x_1 \cdots x_n, x_1' \cdots x_n', s)\,ds \tag{13}$$

can be given a simple form, if the functions $x_i(s)$ are continuous, together with their first and second derivatives, and vanish for $s = 0$, $s = 1$. Integrating by parts,

$$J = -\int_0^1 \left[\sum_{k=1}^{n} x_k L_k(x)\right] ds \tag{14}$$

and from (19), if $L_k(x) = v_k(s)\sqrt{g(s)}$, $k = 1, \cdots, n$,

$$J = -\int_0^1 \left[\sum_{k=1}^{n} \sqrt{g(s)}\, v_k(s) \int_0^1 \left(\sum_{r=1}^{n} G_{kr}(s, z)\sqrt{g(z)}\, v_r(z)\right) dz\right] ds. \tag{15}$$

If $v(s) = v_k(s-k+1)$, $k = 1, \cdots, n$,

$$J = \int_0^n \int_0^n H(s,z)\sqrt{g(s)g(z)}\, v(s)\, v(z)\, ds\, dz = \int_0^n \int_0^n K(s,z)\, v(s)\, v(z)\, ds\, dz. \tag{16}$$

Suppose $\varphi_k(s)$ the normalized solutions of the equation

$$u_k(s) = \lambda_k \int_0^n K(s,z)\, u_k(z)\, dz; \tag{17}$$

if $|\lambda_1| \leqq |\lambda_2| \leqq \cdots \leqq |\lambda_n| \cdots$, then from the theorems of Hilbert and Schmidt,

$$J = \sum_{k=1}^{\infty} \frac{c_k^2}{\lambda_k}, \qquad c_k = \int_0^n \varphi_k(s)\, v(s)\, ds$$

or if $\varphi_k = \sqrt{g}\,\psi_k(s)$ and

$$a_k = \int_0^n g(s)\, x(s)\, \psi_k(s)\, ds$$

then

$$J = \sum_{k=1}^{\infty} \lambda_k\, a_k^2. \tag{18}$$

From the last equation it follows that J can take on both positive and negative values if the values of λ_k are not all of the same sign; for if $x(s) = \psi_k(s)$ then $J = \lambda_k$.

The existence of an infinite number of characteristic values λ_k follows from the fact that if there were only a finite number, m,

$$(19) \qquad K(s,z) = \sum_{k=1}^{m} \frac{\varphi_k(s)\,\varphi_k(z)}{\lambda_k}, \qquad H(s,z) = \sum_{k=1}^{m} \frac{\psi_k(s)\,\psi_k(z)}{\lambda_k},$$

it would follow that the function $H(s,z)$ would possess a continuous derivative with respect to s, and the same would be true of the functions $G_{ik}(s,z)$, $0 < s < 1$ while the derivatives of these functions are discontinous for $s = z$.

If $\psi_k^{(i)} = \psi_k(s-i+1)$, $i = 1, \cdots, n$, a particular solution of (6) is the set of functions

$$x_r = e^{i\sqrt{\lambda_k}t}\,\psi_k^{(r)}(s), \qquad r = 1, \cdots, n,$$

or, since equations (6) are linear, the set of finite sums

$$(20) \qquad x_r = \sum_{k=1}^{m} C_k\, e^{i\sqrt{\lambda_k}t}\,\psi_k^{(r)}(s), \qquad r = 1, \cdots, n,$$

is also a solution.

A certain approximation to the solution of the non-linear equations can be obtained by supposing the coördinates x_i in the form

$$(21) \qquad x_r(s,t) = \sum_{k=1}^{m} C_k(t)\,\psi_k^{(r)}(s), \qquad r = 1, \cdots, n.$$

If these functions are substituted in (4), we obtain at once

$$(22) \qquad T = \frac{1}{2}\sum_{k=1}^{m} C_k'^2, \qquad V = \overline{\overline{V}}(C_1, \cdots, C_m)$$

and consequently the equations (4) become, after the usual transformations,

$$(23) \qquad \frac{d^2 C_k}{dt^2} = -\frac{\partial \overline{\overline{V}}}{\partial C_k}, \qquad k = 1, \cdots, m$$

as a result of the relations

$$\int_0^n g(s)\,\psi_i(s)\,\psi_j(s)\,ds = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

If the integral V of equations (2) has a minimum for the curve C, the second variation J must be positive or zero for the functions admitted,* and hence the values λ_k must be all positive; in this case the dynamical system of order m is *stable* in a certain neighborhood of C, for m arbitrary. If the values of λ_k are not all of the same sign, there will exist systems of order m which possess asymptotic solutions approaching C as t becomes infinite.

* Bolza, *Vorlesungen über Variationsrechnung* (1909), p. 620.

NOTE ON TRINOMIAL CONGRUENCES AND THE FIRST CASE OF FERMAT'S LAST THEOREM.*

By H. S. Vandiver.

Let us consider the relation

$$x^p + y^p + z^p = 0 \tag{1}$$

where x, y and z are integers all different from zero and prime to the odd prime p. In this case there have been various criteria given for the solution of (1) *which involve p only*, such as $2^{p-1} \equiv 1 \pmod{p^2}$, and $p \nmid (q-1)/2$ where q is prime. An examination of the literature shows that *all* the criteria heretofore published which involve p only† may be derived from the theorems:

Theorem‡ I. (Sophie Germain.) *If there exists an odd prime q such that*

$$\xi^p + \eta^p + \zeta^p \equiv 0 \pmod{q} \tag{2}$$

has no set of integral solutions ξ, η, ζ, each not divisible by q, and such that p is not the residue of the pth power of any integer modulo q, then (1) *has no set of integral solutions each prime to q.*

Theorem§ II. (Kummer, Mirimanoff.) *If* (1) *is satisfied in integers prime to p then*

$$f_{p-1}(t) \equiv 0, \qquad B_{(p-i)/2} f_i(t) \equiv 0 \pmod{p}$$

($i = 3, 5, \cdots, p-2$) where

$$f_i(t) = \sum_{n=1}^{p-1} n^{i-1} t^n,$$

B_i is the ith Bernoulli number, and

$$-t = x/y,\ y/x,\ x/z,\ z/x,\ y/z,\ z/y.$$

There is, however, another theorem due to Furtwängler‖ which, although it has not yielded heretofore any criteria of the type we are considering which cannot also be derived from I and II, will be used here:

* Presented to the American Mathematical Society, December, 1924. The author was enabled to carry out this investigation through a grant from the Heckscher Research Foundation, established by August Heckscher of Cornell University.

† Excepting those involving ideals in an algebraic field in their statement.

‡ Legendre, *Mem. Acad. R. Sc. de l'Institut de France*, 6, 1823, 1-60.

§ Abh. Akad. Wiss. Berlin (Math.) 1857, 1858, 41-74. Jour. für Math., 128, 1905, 45-68.

‖ Wiener Bericht, Abt. IIa, 121, 1912, 589-592.

THEOREM III. *If* (1) *is satisfied in integers prime to the odd prime* p *then every integral division of* xyz (x, y, *and* z *prime to each other*) *satisfies*

$$r^{p-1} \equiv 1 \pmod{p^2}. \tag{3}$$

In the present note we shall derive Theorem IV (below) from Theorem III, and show that the former yields new criteria. Also we use Theorem I, IV, and V to prove VI.

We shall now prove the following theorem.

THEOREM IV. *If there exists an odd prime* q *such that* (2) *has no set of integral solutions each not divisible by* q *and such that* $q \not\equiv 1 \pmod{p^2}$ *then* (1) *has no solutions each prime to* p.

Assume that (1) holds under the conditions of the theorem; then it follows from the assumption regarding (2) that one of the integers x, y, and z, say z, is divisible by q. Also, if (2) does not hold, then $q = 1+mp$. Since z is prime to p in (1), we have by Theorem III,

$$q^{p-1} \equiv 1 \pmod{p^2},$$

or

$$\begin{aligned} (1+mp)^p &\equiv 1+mp \pmod{p^2}, \\ mp &\equiv 0 \pmod{p^2}, \\ m &\equiv 0 \pmod{p}, \end{aligned}$$

contrary to the assumption that $q \not\equiv 1 \pmod{p^2}$.* Wendt† proved

THEOREM V. *If* (2) *has no solutions prime to* q, *and* $q = 1+mp$, $m = 2^v p^k$, v *not divisible by* p, *then* (1) *has no solutions prime to* p.

Wendt's proof is a bit complicated. We shall now show that the result follows easily from Theorem I. If we assume that (1) holds and (2) has no solutions prime to q, then $p \equiv \varepsilon^p \pmod{q}$. From this it follows that $p^m \equiv 1 \pmod{q}$ and $m^m \equiv 1 \pmod{q}$, or $(2^v p^k)^m \equiv 1 \pmod{q}$. The latter gives $2^{vm} \equiv 1 \pmod{q}$. Since v is prime to p, there is an integer a such that $av = 1+kp$, k an integer. Also $2^{avm} \equiv 1 \equiv 2^{m(1+kp)} \equiv 2^m \pmod{q}$. Let r_1 be a primitive root of q; then $2 \equiv r_1^{sp} \equiv r^p \pmod{q}$, since otherwise $2^m \not\equiv 1 \pmod{q}$, and therefore $2-1-1 = 0$ gives $r^p + (-1)^p + (-1)^p \equiv 0 \pmod{q}$, contrary to our assumption regarding (2).

Lastly, with the use of the above results, we may prove

THEOREM VI. *If* (2) *has no set of integral solutions each prime to* q, *where* $q = 1+mp$ *and* $m < 10p$, *then* (1) *has no solutions in integers each prime to* p.

* Cf. also Furtwängler l. c., p. 542.

† Jour. für Math. 113, 1894, 335-347.

If m is prime to p this follows immediately from IV. If $m \equiv 0 \pmod 3$ then (2) has solutions for $p > 3$. Since p and q are odd, m is even. If $m = 2p$, $4p$ or $8p$ then by V Theorem VI holds. Hence the proof is complete.

For $p = 5$ there are four values q and for $p = 7$, three values of q such that $m < 10p$ and (2) has no solutions.* It would be interesting to carry out an extended induction to see if the number of values of q tend to increase with p.

By the use of Theorem IV much of the computation involved in Dickson's proof that (1) is impossible in integers prime to p for $p < 7000$ may be avoided.

As is well-known† the necessary and sufficient condition that the congruence (2) has solutions is that the circulant

$$D_m = \begin{vmatrix} \binom{m}{1}, & \binom{m}{2}, & \cdots, & \binom{m}{m-1}, & 1 \\ \binom{m}{2}, & \binom{m}{3}, & \cdots, & 1 & \binom{m}{1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1, & \binom{m}{1}, & \cdots, & \binom{m}{m-2}, & \binom{m}{m-1} \end{vmatrix}$$

is divisible by q, where $q = 1 + mp$. Also

$$D_m = \prod_{i=1}^{m} ((1 + \varrho^i)^m - 1)$$

where ϱ is a primitive root of $x^m = 1$. Using either of these expressions for D_m, we may state Theorem VI in another form.

* Dickson, Jour. für Math., 135, 1909, 134-141.

† Bachmann, Niedere Zahlentheorie, II, p. 471.

DEFINITE LINEAR DEPENDENCE.

By Lloyd L. Dines.

The m sets of n constants each

$$(1)\qquad \begin{array}{llll} a_{11}, & a_{12}, & \cdots, & a_{1n}, \\ a_{21}, & a_{22}, & \cdots, & a_{2n}, \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}, & a_{m2}, & \cdots, & a_{mn}, \end{array}$$

are commonly said to be linearly dependent if m constants $c_1, c_2, \cdots, c_m$, not all zero, exist such that

$$(2)\qquad c_1 a_{1r} + c_2 a_{2r} + \cdots + c_m a_{mr} = 0, \qquad (r = 1, 2, \cdots, n).$$

In the present paper we shall assume that the constants a_{ir} are all real, and we shall say that the m sets (1) are *definitely linearly dependent* if there exist n identities of form (2) in which not all the c's are zero, and all the non-zero ones have *the same sign.*

In a recent paper* Professor Carver has shown that this particular type of linear dependence is a necessary aod sufficient condition for the non-existence of solutions of the system of inequalities

$$(3)\qquad a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n > 0, \quad (i = 1, 2, \cdots, m).$$

In an earlier paper† the writer found conditions for the existence of solutions of the system (3) in terms of a concept which he called the *I-rank* of the matrix of elements (1).

From a comparison of the results of the two papers cited, one easily arrives at the logical equivalence of the three conditions:

1°. *That the m sets of constants* (1) *be definitely linearly dependent.*

2°. *That the system of inequalities* (3) *admit no solution.*

3°. *That the I-rank of the matrix of* (1) *be zero.*

The object of the present paper, in addition to emphasizing the notion of definite linear dependence, is to give a somewhat more direct approach to the equivalence of the three conditions 1°, 2°, 3°, than is furnished

* Annals of Math., vol. 23 (1922), p. 212.

† Annals of Math., vol. 20 (1919), p. 191.

by the two papers cited. Incidentally a helpful geometric interpretation is introduced.

It may be noted here that the notions considered in this paper admit analogies in other fields and, to some extent at least, generalizations in which the matrix of (1) is replaced by a function $A(p, q)$ of two variables on general ranges. The author hopes to consider these in a later paper.

1. **Vector terminology.** A set of constants $(a_1, a_2, \cdots, a_n)$ may be interpreted geometrically as a point in space of n dimensions or as a vector $\alpha = (a_1, a_2, \cdots, a_n)$ with initial point at the origin and with the n constants as components. Adopting the latter interpretation, we may think of the system (1) as a system of vectors

$$S: \qquad \alpha_i = (a_{i1}, a_{i2}, \cdots, a_{in}), \qquad (i = 1, 2, \cdots, m).$$

The sum of two such vectors is a vector whose components are obtained by adding corresponding components of the two vectors. The product of a vector by a scalar number c is the vector whose components are the components of the given vector multiplied respectively by c. The definite linear dependence of the system of vectors S would therefore consist in the existence of constants $c_1, c_2, \cdots, c_m$, of which not all are zero and the non-zero ones have the same sign, such that

$$c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_m \alpha_m = \omega$$

where ω is the zero vector

$$\omega = (0, 0, \cdots, 0).$$

The scalar product of two vectors

$$\alpha = (a_1, a_2, \cdots, a_n), \qquad \xi = (x_1, x_2, \cdots, x_n),$$

is the scalar number defined by

$$\alpha \cdot \xi = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

The system of inequalities (3) can therefore be written in the form

$$(4) \qquad \alpha_i \cdot \xi > 0, \qquad (i = 1, 2, \cdots, m).$$

For vectors in three dimensions (and after appropriate definitions in n dimensions) the scalar product $\alpha \cdot \xi$ is equal to the product of the lengths

of the vectors and the cosine of the angle between them. Hence it is positive, zero, or negative according as this angle is acute, right, or obtuse. The condition 2° may therefore be interpreted as meaning that the system of vectors S is such that there is no vector ξ which makes an acute angle with all of them. It evidently has to do with the distribution of the vectors in space. It will be convenient to say that when the inequalities (4) admit no solution, the system of vectors S is *definitely distributed.**

2. Definite linear combinations of vectors. A system of vectors

$$S': \qquad \alpha_i' = c_{i1}\alpha_1 + c_{i2}\alpha_2 + \cdots + c_{im}\alpha_m, \qquad (i = 1, 2, \cdots, m),$$

in which not all of the c's with a common subscript i are zero and in which those c's which are not zero have the same sign will be called a *definite linear combination* of the system S.

THEOREM I. *If the system S' is definitely linearly dependent, so is the system S; and if the system S' is definitely distributed, so is the system S.*

The first statement of the theorem is an obvious consequence of the definition of definite linear dependence. To prove the second statement indirectly, suppose the system S were not definitely distributed. Then there would be a vector $\bar{\xi}$ such that

$$\alpha_i \cdot \bar{\xi} > 0, \qquad (i = 1, 2, \cdots, m),$$

and the m' scalar products $\alpha_i' \cdot \bar{\xi}$ $(i = 1, 2, \cdots, m')$ would all have the same sign; for

$$\alpha_i' \cdot \bar{\xi} = \sum_{j=1}^{m} c_{ij}\,\alpha_j \cdot \bar{\xi}, \qquad (i = 1, 2, \cdots, m'),$$

and the c_{ij} are all of one sign. Hence either $\bar{\xi}$ or its negative would satisfy the system of inequalities $\alpha_i' \cdot \xi > 0$ $(i = 1, 2, \cdots, m')$. In either case the system S' would not be definitely distributed. This contradiction of the hypothesis shows that our supposition was incorrect and S is definitely distributed.

The converse of the above theorem is not in general true. A simple case in which it is, however, vacuously true is that in which, for some definite r, all the rth components of the vectors S have the same sign. The system S' will clearly have the same property, and it is easily seen that neither system can be definitely linearly dependent nor definitely distributed.

* It is easily seen that two vectors in a plane are definitely distributed if and only if they are collinear and oppositely directed. Three vectors in ordinary space are definitely distributed if and only if they are coplanar and their distribution is such that no circular sector with vertex at the origin and with angle less than 180° contains all of them.

3. Reduction of a system of vectors. In this section we will assume that not all the rth components of the system S have the same sign, and we will describe an important definite linear combination for which the converse of the preceding theorem is true.

Let us divide the rth components of S into three classes, viz.,

$$\begin{aligned} &\text{those which are positive:} && a_{ir}, && (i = i_1, i_2, \cdots, i_P),\\ &\text{those which are negative:} && a_{jr}, && (j = j_1, j_2, \cdots, j_N),\\ &\text{those which are zero:} && a_{kr}, && (k = k_1, k_2, \cdots, k_Z), \end{aligned}$$

the number of elements in the respective classes being denoted by P, N, and Z.

Then the following system of vectors ($P \cdot N + Z$ in number)

$$S_r: \qquad \begin{aligned} \alpha'_{ij} &= a_{ir}\,\alpha_j - a_{jr}\,\alpha_i, && \left\{\begin{matrix} i = i_1, i_2, \cdots, i_P \\ j = j_1, j_2, \cdots, j_N \end{matrix}\right\},\\ \alpha'_k &= \alpha_k, && (k = k_1, k_2, \cdots, k_Z), \end{aligned}$$

is a definite linear combination of the system S with the noteworthy property that the rth component of each vector is 0. The system S_r will be said to be obtained from S by *reduction with respect to the* rth *component.*

THEOREM II: *The status of a system of vectors with respect to definite linear dependence and definite distribution is unchanged by reduction with respect to any component.*

Since by definition S_r is a definite linear combination of S it follows from Theorem I that S_r cannot be definitely linearly dependent or definitely distributed unless S has these respective properties.

To prove the theorem we need therefore only to show *first*, that if S is definitely linearly dependent, the same is true of S_r; and *second*, if S is definitely distributed, the same is true of S_r.

To prove the former, suppose there exist constants $c_1, c_2, \cdots, c_m$, none negative and at least one positive, such that

$$\sum_{s=1}^{m} c_s\,\alpha_s = \omega. \tag{5}$$

Limiting our consideration to the rth components, we may write

$$\sum_i c_i\,a_{ir} + \sum_j c_j\,a_{jr} + \sum_k c_k\,a_{kr} = 0,$$

where the summation index i extends over the positive elements, j over the negative, and k over the zero elements. Hence, since

$$\sum_k c_k a_{kr} = 0,$$

we have

$$\sum_j c_j a_{jr} = -\sum_i c_i a_{ir}.$$

Furthermore, unless every $c_i (i = i_1, i_2, \cdots, i_P)$ is zero, we may assume on account of the homogeneity of the linear relation that

$$\sum_i c_i a_{ir} = 1, \qquad \sum_j c_j a_{jr} = -1. \tag{6}$$

We can now establish the definite linear dependence of the system S_r by showing that the expression

$$\sum_{\substack{i=i_1, i_2, \cdots, i_P \\ j=j_1, j_2, \cdots, j_N}} c_i c_j (a_{ir} \alpha_j - a_{jr} \alpha_i) + \sum_k c_k \alpha_k$$

vanishes identically. This expression may be written in the form

$$\sum_i c_i a_{ir} \sum_j c_j \alpha_j - \sum_j c_j a_{jr} \sum_i c_i \alpha_i + \sum_k c_k \alpha_k.$$

In the exceptional case when every c_i equals zero, and hence also every c_j equals zero, this reduces to the left side of (5), and so the expression vanishes. In the general case, in view of (6), the expression reduces to

$$\sum_i c_i \alpha_i + \sum_j c_j \alpha_j + \sum_k c_k \alpha_k$$

and therefore vanishes by virtue of (5). The system S_r is therefore definitely linearly dependent.

To prove the *second* statement, our hypothesis is that there exists no vector ξ such that

$$\alpha_i \cdot \xi > 0, \qquad (i = 1, 2, \cdots, m), \tag{7}$$

and we are to prove that there is no vector ξ such that

$$\alpha_i' \cdot \xi > 0 \tag{8}$$

for every vector α' of the system S_r.

We will prove it indirectly by supposing that there were a vector $\bar{\xi}$ satisfying (8) for every vector α' of S_r. Since the rth component of every

such vector α' is 0, the rth component of $\bar{\xi}$ would be arbitrary, and $\bar{\xi}$ would have the form

$$\bar{\xi} = (\bar{x}_1, \cdots, \bar{x}_{r-1}, x_r, \bar{x}_{r+1}, \cdots, \bar{x}_n) \tag{9}$$

with x_r entirely arbitrary.

The fact that $\alpha' \cdot \bar{\xi} > 0$ for every α' in S_r may be expressed by the system of inequalities

$$\sum_{t=1}^{n}{}'' (a_{ir}\, a_{jt} - a_{jr}\, a_{it})\, \bar{x}_t > 0, \qquad \begin{Bmatrix} i = i_1, i_2, \cdots, i_P \\ j = j_1, j_2, \cdots, j_N \end{Bmatrix}, \tag{10}$$

$$\sum_{t=1}^{n}{}'' a_{kt}\, \bar{x}_t > 0, \qquad (k = k_1, k_2, \cdots, k_z), \tag{11}$$

where the apostrophe ' above the summation sign indicates that the value r is to be omitted from the range of the summation index t.

Since a_{ir} is positive and a_{jr} is negative, the product $-a_{ir}\, a_{jr}$ is positive, and upon dividing through by this product we may write the inequalities (10) in the form

$$\sum_{t=1}^{n}{}'' \frac{a_{jt}}{-a_{jr}}\, \bar{x}_t > \sum_{t=1}^{n}{}'' \frac{a_{it}}{-a_{ir}}\, \bar{x}_t, \qquad \begin{Bmatrix} i = i_1, i_2, \cdots, i_P \\ j = j_1, j_2, \cdots, j_N \end{Bmatrix}. \tag{12}$$

Now we choose for x_r any number between the least of the N sums represented by the left side of (12) and the greatest of the P sums represented by the right side of (12). That is,

$$\sum_{t=1}^{n}{}'' \frac{a_{jt}}{-a_{jr}}\, \bar{x}_t > x_r > \sum_{t=1}^{n}{}'' \frac{a_{it}}{-a_{ir}}\, \bar{x}_t, \qquad \begin{Bmatrix} i = i_1, i_2, \cdots, i_P \\ j = j_1, j_2, \cdots, j_N \end{Bmatrix}. \tag{13}$$

Then (13) affords the inequalities

$$\sum_{t=1}^{n}{}'' a_{jt}\, \bar{x}_t + a_{jr}\, x_r > 0, \qquad (j = j_1, j_2, \cdots, j_N),$$

$$\sum_{t=1}^{n}{}'' a_{it}\, \bar{x}_t + a_{ir}\, x_r > 0, \qquad (i = i_1, i_2, \cdots, i_P),$$

which together with (11) express the fact that the vector (9) satisfies the system of inequalities (7). This contradicts the hypothesis. Hence our supposition was incorrect and there is no vector $\bar{\xi}$ satisfying (8). That is, the system S_r is definitely distributed. This completes the proof of Theorem II.

4. The equivalence of conditions 1°, 2° and 3°. The system of vectors S will be said to be *I-definite with respect to the rth component,* if the rth components of all the vectors have the same sign. We will say simply that the system is *I-definite* if there is any component with respect to which it is I-definite. The system S is of *I-rank zero* if it is not I-definite, and if no sequence of "reductions" of the type described in § 3 will lead to an I-definite system.*

THEOREM III. *With reference to a system of vectors*

$$S \qquad \alpha_i = (a_{i1}, a_{i2}, \cdots, a_{in}), \qquad (i = 1, 2, \cdots, m),$$

the three conditions

1°. *That the system be definitely linearly dependent,*

2°. *That the system be definitely distributed,*

3°. *That the system be of I-rank zero,*

are logically equivalent.

Suppose, first, that the system S is of I-rank zero. Then by a sequence of $n-1$ "reductions" there can be obtained from it a system

$$\alpha'_i = (0, 0, \cdots, 0, a'_{in}), \qquad (i = 1, 2, \cdots, m') \tag{14}$$

in which each of the first $n-1$ components of every vector is zero, while not all of the nth components have the same sign. Suppose these last components are classified as follows:

$$\begin{aligned} &\text{positive:} \quad a'_{in}, && (i = i_1, i_2, \cdots, i_P) \\ &\text{negative:} \quad a'_{jn}, && (j = j_1, j_2, \cdots, j_N) \\ &\text{zero:} \quad\ a'_{kn}, && (k = k_1, k_2, \cdots, k_Z). \end{aligned}$$

Let c_1, c_2 and c_3 be any three non-negative numbers† satisfying the equation

$$c_1 \sum_i a'_{in} + c_2 \sum_j a'_{jn} + \sum_k c_3\, a'_{kn} = 0.$$

Then the identity

$$\sum_i c_1\, \alpha'_i + \sum_j c_2\, \alpha'_j + \sum c_3\, \alpha'_k = \omega$$

will also hold, and the system (14) is definitely linearly dependent. Hence by Theorem II the system S is definitely linearly dependent.

Furthermore, the system (14) is definitely distributed. For the existence of a vector

* The definitions of *I-definite* and *I-rank* here given are equivalent to those given by the author in his earlier paper, where however they had reference to a matrix rather than to a system of vectors.

† At least one of the three can and shall be chosen positive.

$$\xi = (x_1, x_2, \cdots, x_n)$$

satisfying the inequalities

$$a_i' \cdot \xi > 0, \qquad (i = 1, 2, \cdots, m'),$$

would necessitate the consistency of the inequalities

$$a_{in}' x_n > 0, \qquad (i = 1, 2, \cdots, m');$$

and this is impossible, since not all of the a_{in}' are positive nor are all negative. Therefore, by Theorem II, the system S is definitely distributed.

We have now proved that, if the I-rank of the system S is zero, the system is definitely linearly dependent, and definitely distributed.

Suppose, on the other hand, that the system S is not of I-rank zero. Then a sequence of "reductions" will afford a system

$$a_i'' = (a_{i1}'', a_{i2}'', \cdots, a_{in}''), \qquad (i = 1, 2, \cdots, m''), \tag{15}$$

which is I-definite with respect to some component, say the rth. It is clear that this system cannot be definitely linearly dependent, since no definite linear combination of the rth components could vanish.

Nor can the system (15) be definitely distributed, since the system of inequalities

$$a_i'' \cdot \xi > 0, \qquad (i = 1, 2, \cdots, m''),$$

admits solutions $\xi = (x_1, x_2, \cdots, x_n)$ in which $x_j = 0$, $(j \neq r)$, while x_r satisfies the conditions

$$a_{ir}'' x_r > 0, \qquad (i = 1, 2, \cdots, m'').$$

Since the system (15) is not definitely linearly dependent nor definitely distributed, the same is true of the system S.

We have then proved that if condition 3° is satisfied, conditions 1° and 2° are also, while if 3° is not satisfied, neither 1° nor 2° is satisfied. This establishes Theorem III.

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LEAST DISTANCE FROM A POINT TO A LINEAR $(n-k)$-SPACE, BOTH IN A LINEAR n-SPACE.

By H. S. Uhler.

In A. N. Whitehead's "Treatise on Universal Algebra", vol. 1 (1898), article 226 pertains to distances from points to subregions. A formula for the least distance of any point a from any subregion P_ϱ of $\varrho-1$ dimensions is derived in 226(2). Article 227 deals with shortest distances between subregions and it includes a formula for the distance from the point x in Q_σ to another subregion P_ϱ, where Q_σ is a subregion determined by σ independent points $q_1, q_2, \cdots, q_\sigma$. Since Whitehead's formulae are stated for elliptic space and in Grassman's notation, they are not adapted to the purposes for which the formulae developed below are designed.

The object in the present paper is to do the same thing for ordinary analytic geometry as article 226 just cited does for elliptic geometry. The chief result obtained will amount to a generalization of Hesse's normal form* for the length of the perpendicular dropped from a point in flat space of n-dimensions, S_n, to a subregion of $(n-1)$ dimensions, S_{n-1}.

Incidentally, the writer was forced to solve this obvious problem in order to determine rigorously the parameters of the linear equations of a S_{n-k} so situated in S_n as to make the sum of the squares of the perpendiculars dropped upon it from a large number of random points a minimum (or a maximum, etc.). The general minimum flat subregion when equal weights (or finite weights) are assigned to all of the n coördinates has not been solved heretofore. The references given in the foot-note† will enable the reader to obtain a fair perspective of this phase of the Method of Least Squares.

The success of the following analysis is due to the fact that in Euclidean space the perpendicular distance from a point to a S_{n-k} is identical with the minimum distance. In the derivation of the fundamental formula (7) the generalized concepts of angle, projection, trigonometric function, etc., have been avoided intentionally and the processes kept as purely algebraic as possible.

Let the rectangular coördinates of any point in S_{n-k} be $x_1, x_2, \cdots, x_n$; and let the equations defining S_{n-k} be

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*P. H. Schoute, *Mehrdimensionale Geometrie*, 1. Teil, p. 138.

†Karl Pearson, Phil. Mag., vol. 2, pp. 559-572 (1901). R. Meldrum Stewart, Phil. Mag., vol. 40, pp. 217-227 (1920). H. S. Uhler, Jour. Opt. Soc. Amer., vol. 7, pp. 1043-1066 (1923). H. S. Uhler, Phil. Mag., vol. 49, pp. 1260-1271 (1925).

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + a_{1n+1} &= 0,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + a_{2n+1} &= 0,\\
\cdots\cdots\cdots

Next multiply the equations of set (4) in order by the successive coefficients of the x's with their signs changed in the first equation of set (1), that is, by $-a_{11}, -a_{12}, \cdots, -a_{1n}$, and add the results to obtain the first equation of set (6) below. In general, multiply the equations of set (4) by $-a_{j1}, -a_{j2}, \cdots, -a_{jn}$ $(j = 1, 2, \cdots, k)$ and add to get the jth equation of set (6).

$$
\begin{aligned}
&F'_1 m_0 + \left(\sum_1^n a_{1j}\, a_{1j}\right) m_1 + \left(\sum_1^n a_{1j}\, a_{2j}\right) m_2 + \cdots + \left(\sum_1^n a_{1j}\, a_{kj}\right) m_k = 0 \\
&F'_2 m_0 + \left(\sum_1^n a_{2j}\, a_{1j}\right) m_1 + \left(\sum_1^n a_{2j}\, a_{2j}\right) m_2 + \cdots + \left(\sum_1^n a_{2j}\, a_{kj}\right) m_k = 0 \qquad (6) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&F'_k m_0 + \left(\sum_1^n a_{kj}\, a_{1j}\right) m_1 + \left(\sum_1^n a_{kj}\, a_{2j}\right) m_2 + \cdots + \left(\sum_1^n a_{kj}\, a_{kj}\right) m_k = 0.
\end{aligned}
$$

Elimination of the m's from the homogeneous equations (5) and (6) gives

$$
\begin{vmatrix}
\Delta_0^2 & F'_1 & F'_2 & \cdots & F'_k \\
F'_1 & c_{11} & c_{12} & \cdots & c_{1k} \\
F'_2 & c_{21} & c_{22} & \cdots & c_{2k} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
F'_k & c_{k1} & c_{k2} & \cdots & c_{kk}
\end{vmatrix} = 0, \qquad (7)
$$

where $c_{r,s} \equiv \sum_{j=1}^{j=n} a_{r,j}\, a_{s,j} = c_{s,r}$.

Equation (7), the left member of which is a symmetrical bordered determinant of order $k+1$, constitutes the formal solution of the problem. In any numerical case all of the a's and F''s would have known values so that equation (7) could be solved for Δ_0^2 directly. Incidentally when $k = 1$, that is for S_{n-1}, formula (7) reduces to the well-known form

$$\Delta_0 = \pm F'_1 \Big/ \sqrt{\textstyle\sum a_{1j}^2}$$

which may be transformed immediately into the normal form of Hesse. For $k > 1$ formula (7) shows that, in general, Δ_0 is not a rational function of the F's.

The facts stated below in italics facilitate very appreciably the application of equation (7) to practical algebraic problems. Let this equation be solved for Δ_0^2 and written in the form $D\Delta_0^2 - N = 0$, where D denotes the symmetrical determinant of order k bordered by the F's in (7).

D equals the arithmetical sum of the squares of all the $n!/[k!\,(n-k)!]$ determinants of order k that can be formed from the matrix M. This

fundamental property of D is a consequence of the well-known process of compounding the matrix M with its conjugate.*

N equals the arithmetical sum of the squares of $n!/[(k-1)!\,(n-k+1)!]$ determinants of order k in every one of which the first column read downward consists of $F_1', F_2', \cdots, F_{k-1}', F_k'$ and the remaining $k-1$ monomial columns constitute collectively one of the combinations of the n columns of the matrix M taken $k-1$ at a time. This fact may be proved by first employing Cauchy's theorem† for the expansion of a determinant in products in pairs of the constituents of a row and column, and by then showing that the homogeneous quadratic function of all the F''s thus formed may have its terms collected into squared determinants as a consequence of the symmetrical nature of the complement of $c_{r,s}$ in M, that is, the coefficients of $F_r' F_s'$ and $F_s' F_r'$ are algebraically equal, etc.

For a straight line (S_1) in 5-space (S_5) $k=4$, $n=5$, and

$$\begin{aligned}\Delta_0^2 = \{&|F_1', a_{21}, a_{32}, a_{43}|^2 + |F_1', a_{21}, a_{32}, a_{44}|^2 + |F_1', a_{21}, a_{32}, a_{45}|^2 \\ &+ |F_1', a_{21}, a_{33}, a_{44}|^2 + |F_1', a_{21}, a_{33}, a_{45}|^2 + |F_1', a_{21}, a_{34}, a_{45}|^2 \\ &+ |F_1', a_{22}, a_{33}, a_{44}|^2 + |F_1', a_{22}, a_{33}, a_{45}|^2 + |F_1', a_{22}, a_{34}, a_{45}|^2 \\ &+ |F_1', a_{23}, a_{34}, a_{45}|^2\} / \{|a_{11}, a_{22}, a_{33}, a_{44}|^2 + |a_{11}, a_{22}, a_{33}, a_{45}|^2 \\ &+ |a_{11}, a_{22}, a_{34}, a_{45}|^2 + |a_{11}, a_{23}, a_{34}, a_{45}|^2 + |a_{12}, a_{23}, a_{34}, a_{45}|^2\}.\end{aligned}$$

Further reduction of any determinant involved in N may be effected by substituting for all the F''s their values in terms of the x''s and all the a's conformably to the equations of set (1).

In conclusion attention should be called to the fact that the equations of set (6) can be solved for the k ratios of the m's to any finite one of their number. The expressions for these ratios may then be substituted for them in the equations of set (4). Hence the coördinates $(x_1, x_2, \cdots, x_n)$ are uniquely determined so that there is only one minimum distance from a point to a linear $(n-k)$-space in n-space. The angular or directional characteristic of perpendicularity usually employed is implied by the first equation of set (3).

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* R. F. Scott and G. B. Mathews, *Theory of Determinants*, 2d Ed., p. 53, Art. 6.

† Scott and Mathews, p. 47, Arts. 24 and 25.

YALE UNIVERSITY,
December 13, 1924.

NOTES ON THE AMPERE-CAUCHY DERIVED FUNCTIONS.

By H. L. Smith.

In this paper will be given some notes supplementary to the writer's article* "On the Ampere-Cauchy Derived Functions". All page references in the present paper are to this article.

1. **A mean value theorem.** Let the numbers $x_0, \cdots, x_n$ be on $\overline{a\,b}$ and consist of r distinct ones $a_1, \cdots, a_r$ repeated respectively $i_1, \cdots, i_r$ times $(i_1 + \cdots + i_r = n+1)$, and suppose that a_k is a root of $\varphi(x)$ of multiplicity i_k (at least) $(k = 1, \cdots, r)$ in the sense defined on p. 136. $\varphi(x)$ has thus $(n+1)$ roots (at least) on $\overline{a\,b}$. On writing down the expression for $\varphi^{[n]}(x_0, \cdots, x_n)$ in terms of partial fractions by aid of equation (15), p. 128, it is seen that $\varphi^{[n]}(x_0, \cdots, x_n) = 0$. We thus have

Lemma 1. *If $\varphi(x)$ has $(n+1)$ roots (at least) on $\overline{a\,b}$ there are numbers $x_0, \cdots, x_n$ on $\overline{a\,b}$ such that $\varphi^{[n]}(x_0, \cdots, x_n) = 0$.*

We also have

Lemma 2. *If $\varphi(x)$ is continuous on $\overline{a\,b}$ and has there $(n+1)$ roots (at least) and $\varphi'(x)$ exists (finite or infinite) on $\overline{a\,b}$, then there are numbers*

$$x_0 \leqq x_1 \leqq \cdots \leqq x_{n-1} \leqq x_n$$

$$x_{01} \leqq x_{12} \leqq \cdots \leqq x_{n-1,n}$$

such that $x_i \leqq x_{i,i+1} \leqq x_{i+1}$ $(i = 0, \cdots, n-1)$ and

$$\varphi^{[n]}(x_0, \cdots, x_n) = \varphi'^{[n-1]}(x_{01}, \cdots, x_{n-1,n}) = 0.$$

This lemma follows from Lemma 1 and Rolle's theorem which shows that φ' has n roots distributed like the $x_{i,i+1}$ of the lemma.

Now let $f(x)$, $P(x)$, have the meanings assigned on p. 135 and let $f(x)$ be continuous on $\overline{a\,b}$ and $f'(x)$ exist (finite or infinite) there. Then as stated on p. 136, $\varphi(x) = f(x) - P(x)$ has $(n+1)$ roots there. On applying Lemma 2 we get the

Mean value theorem. *If $f(x)$ is continuous on $\overline{a\,b}$, $f'(x)$ exists (finite or infinite) on $\overline{a\,b}$ and $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ are on $\overline{a\,b}$, then there*

* These Annals, (2) vol 25, pp. 123–136.

are numbers $x_{01}, \cdots, x_{n-1,n}$ *where* $x_i \leqq x_{i,i+1} \leqq x_{i+1}$ $(i = 0, \cdots, n-1)$ *such that*

$$f^{[n]}(x_0, \cdots, x_n) = \frac{f'^{[n-1]}(x_{01}, \cdots, x_{n-1,n})}{n}.$$

More generally, if $f', f'', \cdots, f^{(m-1)}$ *are continuous on* $\overline{ab}$ *and* $f^{(m)}$ *exists (finite or infinite), then there are numbers* $x_{0k}\ x_{1k+1}, \cdots, x_{n-k,n}$ *where* $x_i \leqq x_{i,i+k} \leqq x_{i+k}$ $(i = 0, \cdots, n-k)$ $(k = 1, \cdots, m)$ *such that*

$$f^{[n]}(x_0, \cdots, x_n) = \frac{f^{(k)[n-k]}(x_{0k}, \cdots, x_{n-k,n})}{n(n-1)\cdots(n-k+1)} \qquad (k = 1, \cdots, m).$$

For by Lemma 2

$$0 = \varphi'^{[n-1]}(x_{01}, \cdots, x_{n-1,n}) = f'^{[n-1]}(x_{01}, \cdots, x_{n-1,n}) - nf^{[n]}(x_0, \cdots, x_n),$$

which is the first part of the theorem; the generalization follows by induction.

We may apply this mean value theorem to secure various forms of the remainder term in Newton's interpolation formula. Thus

$$R(x) = (x - x_0)\cdots(x - x_n)\frac{f^{(m+1)[n-m]}(\xi_{0x}, \cdots, \xi_{n-m,x})}{(n+1)\cdots(n-m+1)}$$

where $\xi_{0x}, \cdots, \xi_{n-m,x}$ are between the smallest and the largest of $x_0, \cdots, x_n, x$ and where there is an interval $\overline{ab}$ containing $x_0, \cdots, x_n, x$ such that $f, f', \cdots, f^{(m)}$ are continuous on $\overline{ab}$ and $f^{(m+1)}$ is finite or infinite on $\overline{ab}$.

2. On functions whose Ampere-Cauchy derived functions are bounded. We now prove

LEMMA 3. *Let* f *be continuous on* $\overline{ab}$ *and* f' *exist (finite or infinite) on* $\overline{ab}$. *Then, if* $|f^{[n]}(x_0, \cdots, x_n)| \leqq M$ *for all distinct* $x_0, \cdots, x_n$ *on* $\overline{ab}$, *it follows that* $|f'^{[n-1]}(x_0, \cdots, x_{n-1})| \leqq nM$ *for all distinct* $x_0, \cdots, x_{n-1}$ *on* $\overline{ab}$. *Conversely, if* $|f'^{[n-1]}(x_0, \cdots, x_{n-1})| \leqq nM$ *for all distinct* $x_0, \cdots, x_{n-1}$ *on* $\overline{ab}$, *it follows that* $|f^{[n]}(x_0, \cdots, x_n)| \leqq M$ *for all distinct* $x_0, \cdots, x_n$ *on* $\overline{ab}$.

More generally, let $f, f', \cdots, f^{(m-1)}$ *be continuous on* $\overline{ab}$ *and* $f^{(m)}$ *exist (finite or infinite) on* $\overline{ab}$. *Then, if* $|f^{[n]}(x_0, \cdots, x_n)| \leqq M$ *for all* $x_0, \cdots, x_n$ *on* $\overline{ab}$, *it follows that* $|f^{(k)[n-k]}(x_0, \cdots, x_{n-k})| \leqq n(n-1)\cdots(n-k+1)M$ *for all distinct* $x_0, \cdots, x_{m-k}$ *on* $\overline{ab}$ $(k = 1, \cdots, m)$. *Conversely, if* $|f^{(k)[n-k]}(x_0, \cdots, x_{n-k})| \leqq n(n-1)\cdots(n-k+1)M$ *for all distinct* $x_0, \cdots, x_{n-k}$ *on* $\overline{ab}$, *it follows that* $|f^{[n]}(x_0, \cdots, x_n)| \leqq M$ *for all distinct* $x_0, \cdots, x_n$ *on* $\overline{ab}$.

To prove this first part note that by (2) and (6), p. 124, $f'^{[n-1]}(x_0, \cdots, x_{n-1})$ can be expressed as the limit as h approaches 0 of the sum

$$f^{[n]}(x_0+h, x_0, \cdots, x_{n-1})+f^{[n]}(x_0+h, x_1+h, x_1, \cdots, x_{n-1})+\cdots$$
$$+f^{[n]}(x_0+h, \cdots, x_{n-1}+h, x_{n-1})$$

which is less than nM. The converse follows from the mean value theorem of the previous section. The generalization then follows by induction.

LEMMA 4. *If* $|f^{[n]}(x_0, \cdots, x_n)| \leqq M$ *for every* $x_0, \cdots, x_n$ *distinct on* $\overline{ab}$, *then*

$$|f^{[n-1]}(x'_0, \cdots, x'_{n-1})-f^{[n-1]}(x''_0, \cdots, x''_{n-1})| \leqq (b-a)\,nM$$

for every $x'_0, \cdots, x'_{n-1}$ *distinct on* $\overline{ab}$ *and* $x''_0, \cdots, x''_{n-1}$ *distinct on* $\overline{ab}$.

If some of the x' coincide with some of the x'', we may suppose the notation such that they are the first r; thus $x'_0 = x''_0 = c_0$ say, $\cdots$, $x'_{r-1} = x''_{r-1} = c_{r-1}$ say, and $c_0, \cdots, c_{r-1}, x'_r, \cdots, x'_{n-1}, x''_r, \cdots, x''_{n-1}$ are distinct. We can then express $f^{[n-1]}(x'_0, \cdots, x'_{n-1})-f^{[n-1]}(x''_0, \cdots, x''_{n-1})$ as the sum

$$(x'_r-x''_r)\,f^{[n]}(c_0, \cdots, c_{r-1}\,x'_r\,x''_r\,x'_{r+1}, \cdots, x'_{n-1})$$
$$+(x'_{r+1}-x''_{r+1})\,f^{[n]}(c_0, \cdots, c_{r-1}\,x''_r\,x'_{r+1}\,x''_{r+1}\,x'_{r+2}, \cdots, x'_{n-1})$$
$$+\cdots+(x'_{n-1}-x''_{n-1})\,f^{[n]}(c_1, \cdots, c_{r-1}\,x''_r, \cdots, x''_{n-2}\,x'_{n-1}\,x''_{n-1})$$

which is less in absolute value than $(b-a)(n-r)M$ and a fortiori less then $(b-a)nM$.

LEMMA 5. *If* $|f^{[n]}(x_0, \cdots, x_n)| \leqq M$ *for all distinct* $x_0, \cdots, x_n$ *on* $\overline{ab}$, *then there are numbers* $M_0, \cdots, M_{n-1}$ *such that* $|f^{[r]}(x_0, \cdots, x_r)| \leqq M_r$ *for all* $x_0, \cdots, x_r$ *distinct and on* ab $(r = 0, \cdots, n-1)$.

Let us prove this lemma for $r = n-1$; by induction the lemma will then be true in complete generality. Set $a^i = a + \dfrac{i}{n-1}(b-a)$ $(i = 0, \cdots, n-1)$. Then by Lemma 4,

$$|f^{[n-1]}(x_0, \cdots, x_{n-1})| \leqq |f^{[n-1]}(a_0, \cdots, a_{n-1})|+(b-a)\,nM \leqq M_{n-1}$$

if M_{n-1} is taken as $|f^{[n-1]}(a_0, \cdots, a_{n-1})|+(b-a)\,nM$.

THEOREM. *If* $|f^{[n]}(x_0, \cdots, x_n)| \leqq M$ *for all* $x_0, \cdots, x_n$ *distinct on* $\overline{ab}$, *then* $f', \cdots, f^{(n-1)}$ *exist and are continuous on* $\overline{ab}$ *and* $|f^{(k)[n-k]}(x_0, \cdots, x_{n-k})| \leqq n(n-1)\cdots(n-k+1)M$ *for every* $x_0, \cdots, x_{n-k}$ *distinct on* $\overline{ab}$ $(k = 1, \cdots, n-1)$.

We prove first that f' exists on $\overline{ab}$. Our hypothesis implies (by Lemma 5) that $|f^{[2]}(x_0\,x_1\,x_2)| \leqq M_2$ for all distinct $x_0\,x_1\,x_2$ on $\overline{ab}$. Hence

$$|f^{[1]}(x_0\,x_1)-f^{[1]}(x_0\,x_2)| = |x_1-x_2|\,|f^{[2]}(x_0\,x_1\,x_2)| \leqq |x_1-x_2|\,M_2.$$

The limit as x_1 and x_2 simultaneously approach x_0 of $|f^{[1]}(x_0 x_1) - f^{[1]}(x_0 x_2)|$ is therefore 0, which is the Cauchy condition for the existence of $f'(x_0)$.

Moreover f' is continuous. For, since

$$\lim_{k \to 0} f(x; hk) = \frac{f'(x+h) - f'(x)}{h},$$

we have

$$|f'(x+h) - f'(x)| = |h| \, |\lim_{k \to 0} f(x; hk)|$$

$$= |h| \, |\lim_{k \to 0} \{f^{[2]}(x, x+h, x+h+k) + f^{[2]}(x, x+k, x+h+k)\}| \leqq 2M_2 |h|,$$

and therefore $\lim_{h \to 0} [f'(x+h) - f'(x)] = 0$. $f'(x)$ thus exists and is continuous.

By a precisely similar argument making use of the inequality $|f'^{[2]}(x_0 x_1 x_2)| \leqq 3M_3$ it is shown that $f''(x)$ exists and is continuous. The complete theorem follows by induction.

THE UNIVERSITY OF MINNESOTA,
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NOTE ON THE EXISTENCE OF ANALYTIC SOLUTIONS OF NON-HOMOGENEOUS LINEAR q-DIFFERENCE EQUATIONS, ORDINARY AND PARTIAL.*

By C. Raymond Adams.

Introduction

The existence of analytic solutions of the homogeneous linear ordinary q-difference equation was demonstrated by Carmichael† some time ago; recently the existence of analytic solutions of a class of homogeneous linear partial q-difference equations has been shown by the author.‡ In the present note are proved certain theorems concerning the existence of solutions of non-homogeneous equations of the same general types. To obtain solutions of a non-homogeneous difference equation of any type (difference or q-difference, ordinary or partial, etc.) when the solutions of the associated homogeneous equation are known, it is necessary only to discover suitable evaluations for an operator. Such evaluations, effective in an extensive class of cases, are easily found in the present instance. No originality can be claimed for the general methods used here, but it seems to the writer that the results found make a sufficient contribution to the theory of q-difference equations to warrant their being set down.

For the sake of uniformity of notation we find it desirable to recall briefly the main facts concerning homogeneous q-difference equations. In the system

$$g_i(qx) = \sum_{j=1}^{n} a_{ij}(x)\, g_j(x) \qquad (i = 1, 2, \cdots, n), \tag{1}$$

or in matrix notation

$$G(qx) = A(x)\, G(x),$$

let the coefficient functions $a_{ij}(x)$ be expressible as

$$\begin{aligned} a_{ij}(x) &= x^{\kappa}[a_{ij} + a_{ij1}\, x + a_{ij2} x^2 + \cdots] \text{ for } |x| < r, \\ a_{ij}(x) &= x^{\mu}[a'_{ij} + a'_{ij1}/x + a'_{ij2}/x^2 + \cdots] \text{ for } |x| > R \quad (i, j = 1, 2, \cdots, n). \end{aligned} \tag{2}$$

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* Presented to the American Mathematical Society, Dec. 30, 1924.

† "The general theory of linear q-difference equations", Amer. Journ. of Math., vol. 34 (1912), pp. 147–168.

‡ "The general theory of a class of linear partial q-difference equations", Trans. Amer. Math. Soc., vol. 26 (1924), pp. 283–312.

Then if $|q|$ is $\neq 1$ and if certain conditions concerning the roots of the characteristic equations for zero and infinity

$$|a_{ij} - \delta_{ij}\,\sigma| = 0, \qquad |a'_{ij} - \delta_{ij}\,\sigma'| = 0, \qquad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

are satisfied, the system (1) possesses two matrices of solutions

$$(3)\qquad \begin{aligned} (g_{ij}(x)) &= \left(q^{\frac{\varkappa}{2}(\varrho_j - t)}\, x^{\varrho_j}\, \alpha_{ij}(x)\right), \\ (h_{ij}(x)) &= \left(q^{\frac{\mu}{2}(\varrho'_j - t)}\, x^{\varrho'_j}\, \beta_{ij}(x)\right), \qquad t = \frac{\log x}{\log q}. \end{aligned}$$

The functions $\alpha_{ij}(x)$ and $\beta_{ij}(x)$ are analytic at $x = 0$ and $x = \infty$ respectively, and the determinants of the leading coefficients in the expansions of these functions at these points are not zero. Only the case in which such solutions exist is to be considered here. If the coefficient functions in (1) are analytic throughout the plane except for certain singularities, the region of analyticity of the solutions $g_{ij}(x)$ and $h_{ij}(x)$ is extensible by repeated use of (1).

Secondly, let the system

$$4)\qquad g_i(qx, ry) = \sum_{j=1}^{n} a_{ij}(x, y)\, g_j(x, y) \qquad (i = 1, 2, \cdots, n),$$

or in matrix notation

$$G(qx, ry) = A(x, y)\, G(x, y),$$

have coefficient functions expressible in these forms:

$$(5)\qquad \begin{aligned} a_{ij}(x, y) &= x^{\varkappa} y^{\lambda} [a_{ij} + a_{ij10}\, x + a_{ij01}\, y + \cdots] \text{ for } |x| < r_1, \ |y| < r_2; \\ a_{ij}(x, y) &= x^{\mu} y^{\nu} [a'_{ij} + a'_{ij10}/x + a'_{ij01}/y + \cdots] \text{ for } |x| > R_1, \ |y| > R_2; \\ a_{ij}(x, y) &= x^{\xi} y^{\eta} [a''_{ij} + a''_{ij10}\, x + a''_{ij01}/y + \cdots] \text{ for } |x| < r_1, \ |y| > R_2 \\ a_{ij}(x, y) &= x^{\varphi} y^{\psi} [a'''_{ij} + a'''_{ij10}/x + a'''_{ij01}\, y + \cdots] \text{ for } |x| > R_1, \ |y| < r_2. \\ &\qquad (i, j = 1, 2, \cdots, n). \end{aligned}$$

When $|q|$ and $|r|$ are not 1 and certain conditions concerning the roots of the characteristic equations for $(0, 0)$, (∞, ∞), $(0, \infty)$, and $(\infty, 0)$

$$\begin{aligned} |a_{ij} - \delta_{ij}\,\sigma| &= 0, \qquad |a'_{ij} - \delta_{ij}\,\sigma'| = 0, \\ |a''_{ij} - \delta_{ij}\,\sigma''| &= 0, \qquad |a'''_{ij} - \delta_{ij}\,\sigma'''| = 0 \end{aligned} \qquad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

are satisfied, the system (4) possesses four matrices of formal series solutions

$$
\begin{aligned}
(g_{ij}(x,y)) &= \left(q^{\frac{\varkappa}{2}(t^2-t)}\, r^{\frac{\lambda}{2}(\tau^2-\tau)}\, x^{\varrho_j}\ \alpha_{ij}(x,y)\right),\\
(h_{ij}(x,y)) &= \left(q^{\frac{\mu}{2}(t^2-t)}\, r^{\frac{\nu}{2}(\tau^2-\tau)}\, x^{\varrho_j'}\ \beta_{ij}(x,y)\right),\\
(u_{ij}(x,y)) &= \left(q^{\frac{\xi}{2}(t^2-t)}\, r^{\frac{\eta}{2}(\tau^2-\tau)}\, x^{\varrho_j''}\ \gamma_{ij}(x,y)\right),\\
(v_{ij}(x,y)) &= \left(q^{\frac{\varrho}{2}(t^2-t)}\, r^{\frac{\psi}{2}(\tau^2-\tau)}\, x^{\varrho_j'''}\ \delta_{ij}(x,y)\right),\quad t=\frac{\log x}{\log q},\quad \tau=\frac{\log y}{\log r},
\end{aligned}
\tag{6}
$$

where $\alpha_{ij}(x,y)$ is a development in powers of x and y, $\beta_{ij}(x,y)$ one in powers of $1/x$ and $1/y$, $\gamma_{ij}(x,y)$ one in powers of x and $1/y$, and $\delta_{ij}(x,y)$ one in powers of $1/x$ and y. If $|q|$ and $|r|$ are both greater or less than 1, the series $\alpha_{ij}(x,y)$ and $\beta_{ij}(x,y)$ converge in the vicinity of $(0,0)$ and (∞,∞) respectively; if one of the quantities $|q|$, $|r|$ is greater and the other less than 1, the series $\gamma_{ij}(x,y)$ and $\delta_{ij}(x,y)$ converge respectively in the neighborhood of $(0,\infty)$ and $(\infty,0)$. The determinant of the leading coefficients in each matrix of expansions is not zero. If the coefficient functions $a_{ij}(x,y)$ are analytic throughout the x- and y-planes except for certain singularities, the region of analyticity of the solutions $g_{ij}(x,y)$ and $h_{ij}(x,y)$, or $u_{ij}(x,y)$ and $v_{ij}(x,y)$ can be extended by repeated use of (4).

1. **The non-homogeneous ordinary q-difference equation.** Let us consider the system

$$
\begin{aligned}
g_i(qx) &= \sum_{j=1}^{n} a_{ij}(x)\, g_j(x) \qquad (i=1,2,\cdots,n-1),\\
g_n(qx) &= \sum_{j=1}^{n} a_{nj}(x)\, g_j(x) + b(x),
\end{aligned}
\tag{7}
$$

where $|q|$ is $\neq 1$, the $a_{ij}(x)$ are such that the associated homogeneous system obtained by deleting $b(x)$ has solutions (3), and where $b(x)$ is expressible in the forms

$$
\begin{aligned}
b(x) &= x^{\gamma}[b+b_1 x+b_2 x^2+\cdots] \quad \text{for} \quad |x|<r;\\
b(x) &= x^{\delta}[b'+b_1'/x+b_2'/x^2+\cdots] \quad \text{for} \quad |x|>R.
\end{aligned}
$$

It is assumed further that $\varkappa$ and μ are positive or negative integers or zero (an hypothesis not needed heretofore). There is no added generality in making each equation of the system (7) non-homogeneous, for the solution of such a system is obtainable by summing the solutions of the n systems each of which is formed from that one by deleting all but one of the additive terms like $b(x)$. If $\varkappa[\mu]$ is zero, we make also the hypo-

thesis that $q^{\gamma+j}\,[q^{\delta-j}]$ for $j = 0, 1, 2, \cdots$ is not a root of the characteristic equation for $0[\infty]$ of the associated homogeneous system. Under these conditions the system (7) possesses two formal series solutions:*

$$s_i(x) = x^{\gamma-\varkappa}[s_i + s_{i1}\,x + s_{i2}\,x^2 + \cdots], \tag{8}$$

$$s_i'(x) = x^{\delta-\mu}[s_i' + s_{i1}'/x + s_{i2}'/x^2 + \cdots] \qquad (i = 1, 2, \cdots, n); \tag{9}$$

the s's and s''s may be calculated by substituting the series in (7) and reckoning out the coefficients directly. It is our purpose to show that under certain conditions these series converge respectively in the vicinity of $x = 0$ and of $x = \infty$.

Let now $(g_{ij}(x))$ be the matrix solution of the associated homogeneous system which is valid at $x = 0$. In a manner suggested by the method of variation of the constants in the theory of linear differential equations† we put

$$\bar{g}_i(x) = \sum_{k=1}^{n} l_k(x)\, g_{ik}(x) \qquad (i = 1, 2, \cdots, n) \tag{10}$$

and seek to determine the $l_k(x)$ $(k = 1, 2, \cdots, n)$ so that the $\bar{g}_i(x)$ $(i = 1, 2, \cdots, n)$ will be a solution of (7). If we set

$$l_k(qx) - l_k(x) = \Delta l_k(x)$$

and substitute the expressions (10) in (7), the following set of equations for the determination of the $\Delta l_k(x)$ is obtained:

$$\sum_{k=1}^{n} \Delta l_k(x)\, g_{ik}(qx) = 0 \qquad (i = 1, 2, \cdots, n-1),$$

$$\sum_{k=1}^{n} \Delta l_k(x)\, g_{nk}(qx) = b(x).$$

The determinant of the coefficients $|g_{ij}(qx)|$, which we shall denote by $\mathfrak{G}(qx)$, does not vanish identically, since the determinant of the leading coefficients in the expansions of the $g_{ij}(x)$ is different from zero. Therefore

* The forms given in (8) and (9) are used because they hold in all cases. It turns out, however, when $\varkappa > 0$, that the s_i and s_{ij} for $i = 1, 2, \cdots, n-1$; $j = 1, 2, \cdots, 2\varkappa$, and s_n and s_{nj} for $j = 1, 2, \cdots, \varkappa$, are all zero. Similarly, when $\mu < 0$, the s_i' and s_{ij}' for $i = 1, 2, \cdots, n-1$; $j = 1, 2, \cdots, 2\mu$, and s_n' and s_{nj}' for $j = 1, 2, \cdots, \mu$, are zero.

† This device has already been employed in the theory of ordinary difference equations by Birkhoff, "General theory of linear difference equations", Trans. Amer. Math. Soc., vol. 12 (1911), p. 258 and by Williams, "The solutions of non-homogeneous difference equations and their asymptotic forms", ibid., vol. 14 (1913), pp. 213–214.

we can solve for $\Delta l_k(x)$, and if we allow $\mathfrak{G}_k(qx)$ to designate the minor of the element in the last row and kth column of $\mathfrak{G}(qx)$, we find

$$\Delta l_k(x) = (-1)^{n+k} \frac{b(x)\,\mathfrak{G}_k(qx)}{\mathfrak{G}(qx)}$$

or

$$l_k(x) = \Sigma \frac{(-1)^{n+k} b(x)\,\mathfrak{G}_k(qx)}{\mathfrak{G}(qx)} \quad (k = 1, 2, \cdots, n), \tag{11}$$

Σ being used to denote the operator which is the inverse of Δ (i. e., such that $\Sigma\Delta f(x) = \Delta\Sigma f(x) = f(x)$).

The operator Σ admits two simple series evaluations:

$$\Sigma f(x) = f\left(\frac{x}{q}\right) + f\left(\frac{x}{q^2}\right) + f\left(\frac{x}{q^3}\right) + \cdots, \tag{12}$$

$$\Sigma f(x) = -f(x) - f(qx) - f(q^2x) - \cdots. \tag{13}$$

The quotient in (11) may be written as

$$\frac{x^{\gamma-\varkappa-\varrho_k}}{q^{\frac{\varkappa}{2}(t-\ell)}} [\theta + \theta_1 x + \theta_2 x^2 + \cdots]. \tag{14}$$

Let us now suppose $|q| > 1$; then it is clear that of the evaluations (12) and (13) only the first can possibly be used to secure analytic solutions here. Employing (12) we have*

$$\begin{aligned} l_k(x) &= \sum_{m=1}^{\infty} \frac{(-1)^{n+k} b\left(\frac{x}{q^m}\right) \mathfrak{G}_k\left(\frac{x}{q^{m-1}}\right)}{\mathfrak{G}\left(\frac{x}{q^{m-1}}\right)} \\ &= \sum_{m=1}^{\infty} \frac{x^{(m-1)\varkappa}}{q^{m(\gamma-\varkappa-\varrho_k)+(1+2+\cdots+m)\varkappa}} \frac{x^{\gamma-\varrho_k}}{q^{\frac{\varkappa}{2}(t-\ell)}} \left[\theta + \theta_1 \frac{x}{q^m} + \cdots\right]. \end{aligned} \tag{15}$$

Let for the moment x be regarded as fixed, not zero, and within the circle of convergence of the series in (14). Then the ratio of the $(m+1)$th

* The quotient here is in general an infinitely many-valued function because of the presence of $x^{\gamma-\varkappa}$. In order that it may be a single-valued function of position it is therefore necessary to regard x as free to wander over a Riemann surface with infinitely many leaves, like that used for $\log x$. This being the case, we stipulate that the passage from the point x to the point x/q, and similarly from that point to x/q^2, and so on, be made along a logarithmic spiral in a clockwise direction, let us say, through a distance which subtends at the origin an angle that is negative or zero and greater than -2π.

term to the mth for sufficiently large m is clearly seen to be approximately $x^{\varkappa}/q^{\gamma-\varrho_k+m\varkappa}$; as m increases, this ratio tends toward zero if $\varkappa$ is positive. Thus, for $\varkappa>0$, the series converges rapidly, at least beyond a certain point. The $x^{(m-1)\varkappa}$ and the power of q beneath it in (15) can be absorbed into the series and, since the series converges absolutely, it can be rearranged according to powers of x; thus we have for all values of x within some circle about the origin

$$l_k(x) = \frac{x^{\gamma-\varrho_k}}{q^{\frac{\varkappa}{2}(t^2-t)}}[l+l_1x+l_2x^2+\cdots]$$

and hence

$$\bar{g}_i(x) = x^{\gamma}[\bar{g}_i+\bar{g}_{i1}x+\bar{g}_{i2}x^2+\cdots] \quad (i=1,2,\cdots,n). \tag{16}$$

These functions must constitute a solution of (7) and, inasmuch as the equations determining the coefficients in (8) have unique solutions, the series (16) and (8) must be identical.* Hence the formal series (8) converge and represent an analytic solution of (7) in the vicinity of $x=0$.

If $|q|$ is <1, the sum formula (13) may be used; the resulting series converges if $\varkappa$ is <0, and can be identified with (8).

When $\varkappa$ is zero the situation is somewhat different; then the series (14) is simply

$$x^{\gamma-\varrho_k}[\theta+\theta_1x+\theta_2x^2+\cdots]. \tag{17}$$

We shall now make the additional assumption that† $R[\varrho_k-\gamma]$ be $\neq j$ for $k=1,2,\cdots,n$; $j=0,1,2,\cdots$.‡ Let us suppose $|q|>1$. Then if $R[\gamma-\varrho_k]$ is >0, we use evaluation (12) for Σ. If $R[\gamma-\varrho_k]$ is <0, let l be the largest integer less than $R[\varrho_k-\gamma]$ and break up the series (17) into two parts thus:

$$\begin{aligned} &x^{l+\gamma-\varrho_k}\left[\frac{\theta}{x^l}+\frac{\theta_1}{x^{l-1}}+\cdots+\theta_l\right] \\ &+x^{l+1+\gamma-\varrho_k}[\theta_{l+1}+\theta_{l+2}x+\theta_{l+3}x^2+\cdots]. \end{aligned} \tag{18}$$

Now the operator Σ is distributive; thus Σ operating on (17) is equal to the sum of the results of Σ operating on the parts of (18). Σ operating

* Cf. first note, p. 76. It appears that $\bar{g}_i$ and $\bar{g}_{ij}$ for $i=1,2,\cdots,n-1$; $j=1,2,\cdots,\varkappa$ must be zero; this is also shown by carrying out the calculation of these constants.

† We shall use the notation $R[x]$ to denote the *real part* of x.

‡ In case γ and ϱ_k $(k=1,2,\cdots,n)$ are all real, the possibility of $R[\varrho_k-\gamma]$ being a positive integer or zero has already been precluded by the assumption which we found it necessary to make immediately before (8) in order to secure formal solutions.

on the first part of (18) will be evaluated by (13) and Σ operating on the second part by (12). In all these cases the series involved are seen to converge since after a time the terms decrease in approximately geometric ratio. If $|q|$ is <1, the use of the sum formulas (12) and (13) is interchanged.

An entirely parallel discussion can be given to demonstrate, in corresponding cases, the convergence of the series (9). Naturally the matrix solution $(h_{ij}(x))$ of the associated homogeneous system replaces $(g_{ij}(x))$ in the foregoing treatment. If we denote the matrix $(a_{ij}(x))$ by $A(x)$ and its inverse by $A^{-1}(x)$, the results may be summed up as follows.

THEOREM I. *In the following cases*

$\|q\| > 1$, $\varkappa > 0$;	$\|q\| > 1$, $\mu > 0$;
$\|q\| < 1$, $\varkappa < 0$;	$\|q\| < 1$, $\mu < 0$;
$\|q\| \gtrless 1$, $\varkappa = 0$, $R[\varrho_k - \gamma] \neq j$ *for* $k = 1, 2, \cdots, n$; $j = 0, 1, 2, \cdots$;	$\|q\| \gtrless 1$, $\mu = 0$, $R[\varrho'_k - \delta] \neq j$ *for* $k = 1, 2, \cdots, n$; $j = 0, 1, 2, \cdots$;

the formal series solutions (8) *and* (9) *converge respectively for x in the vicinity of* 0 *und* ∞ *and so represent analytic solutions there.* By the aid of the system* (7) *itself the region of validity of these analytic solutions can be extended. The solution which in the vicinity of $x = 0$ is represented by* (8) *is analytic away from that point with the following exceptions*: ∞; *when $|q|$ is >1, the singularities of $A(x)$ and $b(x)$ and points externally congruent† to them; when $|q|$ is <1, the singularities of $A^{-1}(x)$ and $b(x)$ and points externally congruent to them. The solution which in the neighborhood of $x = \infty$ is represented by* (9) *is analytic away from that point with the following exceptions*: 0; *when $|q|$ is >1, the singularities of $A^{-1}(x)$ and $b(x)$ and points congruent internally to them; when $|q|$ is <1, the singularities of $A(x)$ and $b(x)$ and points congruent internally to them.*

It would be of interest to know what is the significance of the formal solutions (8) [(9)] in the cases $|q| < 1$, $\varkappa > 0$ and $|q| > 1$, $\varkappa < 0$ [$|q| < 1$, $\mu > 0$ and $|q| > 1$, $\mu < 0$]. The sum formulas (12) and (13) when applied in these instances yield rapidly divergent series. Are there, however, other evaluations possible for Σ which will yield analytic solutions in these cases—solutions which can be identified with the formal developments (8) and (9) or which are asymptotically represented by them?

2. The relation between the solutions. Frequently the system (7) possesses two solutions; let them be called $\overline{g}_i(x)$ and $\overline{h}_i(x)$ $(i = 1, 2, \cdots, n)$.

* At the point 0 [∞] the solution is analytic or has a pole according as the exponent of x appearing before the series has a real part which is $\geqq 0$ [$\leqq 0$] or <0 [>0].

† Of the points $q^i x$ $(i = 1, 2, \cdots, -1, -2, \cdots)$ those whose moduli are $>|x|$ [$<|x|$] will be said to be *externally* [*internally*] *congruent* to x.

It is clear that the set of functions $\bar{g}_i(x) - \bar{h}_i(x)$ $(i = 1, 2, \cdots, n)$ constitutes a solution of the associated homogeneous system. But any such solution is related to the solutions of the matrix* $(h_{ij}(x))$ in the following way:

$$\bar{g}_i(x) - \bar{h}_i(x) = \sum_{j=1}^{n} \theta_j(x)\, h_{ij}(x) \quad (i = 1, 2, \cdots, n), \tag{19}$$

where the functions $\theta_j(x)$ are periodic, satisfying the relation $f(qx) = f(x)$. The equations (19) may be regarded as a non-homogeneous system for the determination of the $\theta_j(x)$. Since the determinant of the coefficients $|h_{ij}(x)|$ is not identically zero (the determinant of the leading coefficients $|s'_{ij}|$ is $\neq 0$), we can solve for the θ's. We find

$$\theta_j(x) = \frac{\mathfrak{H}_j(x)}{\mathfrak{H}(x)} \qquad (j = 1, 2, \cdots, n),$$

where $\mathfrak{H}(x)$ stands for the determinant $|h_{ij}(x)|$ and $\mathfrak{H}_j(x)$ for the determinant formed from $\mathfrak{H}(x)$ by replacing the jth column by the left-hand members of (19).

In certain instances the nature of the $\theta_j(x)$ can be completely determined. In particular it is seen that if the $a_{ij}(x)$ and $b(x)$ are rational functions, then $\theta_j(x)$ $(j = 1, 2, \cdots, n)$ away from 0 and ∞, is analytic save for poles. Indeed the $\theta_j(x)$ may be subjected to further analysis by the method employed by Birkhoff† in examining the character of the periodic functions of the same type that arise in connection with the homogeneous system (1). The results are stated in

THEOREM II. *When the coefficient functions in* (7) *are rational functions with* $\varkappa \geqq 0$, $\mu \geqq 0$ *and* $|q|$ *is* > 1 *(or* $\varkappa \leqq 0$, $\mu \leqq 0$, *and* $|q| < 1$*), the relation between the solutions of* (7) *may be expressed in the form* (19) *where the* $\theta_j(x)$ $(j = 1, 2, \cdots, n)$ *satisfy the relation* $f(qx) = f(x)$ *and are analytic save for poles over the entire plane away from* 0 *and* ∞. *If we make the transformation* $x = q^t$ *and set* $\theta_j(x) = \omega_j(t)$, *we find*

$$\omega_j(t+1) = \omega_j(t),$$

$$\omega_j\left(t + \frac{2\pi\sqrt{-1}}{\log q}\right) = (-1)^{\mu}\, e^{-2\pi\sqrt{-1}(\mu t + \varrho'_j) + 2\pi\mu^2/\log q}\, \omega_j(t).$$

* The matrix $(g_{ij}(x))$ may equally well be used here.

† "The generalized Riemann problem for ...", Proc. Amer. Acad. of Arts and Sci., vol. 49 (1913), pp. 561–564.

$\omega_j(t)$ is then expressible in the form

$$\omega_j(t) = ce^{at^2+bt}\,\sigma(t-\alpha_1)\,\sigma(t-\alpha_2)\cdots\sigma(t-\alpha_\mu),$$

where $\sigma(t)$ is the Weierstrassian sigma function belonging to the periods 1 *and* $2\pi\sqrt{-1}/\log q$, *and c, a, b, and the α's are suitable constants.*

3. A class of non-homogeneous partial q-difference equations. We next consider the system

$$\begin{aligned} g_i(qx, ry) &= \sum_{j=1}^{n} a_{ij}(x, y)\, g_j(x, y) \qquad (i = 1, 2, \cdots, n-1), \\ g_n(qx, ry) &= \sum_{j=1}^{n} a_{nj}(x, y)\, g_j(x, y) + b(x, y), \end{aligned} \tag{20}$$

where $|q|$ and $|r|$ are $\neq 1$, the $a_{ij}(x, y)$ are such that the associated homogeneous system obtained by deleting $b(x, y)$ has solutions (6), and $b(x, y)$ can be expressed thus:

$$\begin{aligned} b(x, y) &= x^{\alpha}\, y^{\beta}[b + b_{10}\, x + b_{01}\, y + \cdots] \quad \text{for} \quad |x|<r_1, \quad |y|<r_2, \\ b(x, y) &= x^{\gamma}\, y^{\delta}[b' + b'_{10}/x + b'_{01}/y + \cdots] \quad \text{for} \quad |x|>R_1, \quad |y|>R_2, \\ b(x, y) &= x^{\omega}\, y^{\pi}[b'' + b''_{10}\, x + b''_{01}/y + \cdots] \quad \text{for} \quad |x|<r_1, \quad |y|>R_2, \\ b(x, y) &= x^{\sigma}\, y^{\tau}[b''' + b'''_{10}/x + b'''_{01}\, y + \cdots] \quad \text{for} \quad |x|>R_1, \quad |y|<r_2. \end{aligned}$$

It will be assumed that $\varkappa$, λ, μ, ν, ξ, η, φ and ψ are positive or negative integers or zero; we further impose the condition that, if $\varkappa$ and λ [μ and ν] are both different from zero, they are of the same sign, and that, if ξ and η [φ and ψ] are both different from zero, they are opposite in sign. As in § 1 no generality is lost in taking but one equation of (20) to be non-homogeneous. If $\varkappa$ and λ [μ and ν] are both zero, we make the additional hypothesis that $q^{\alpha+i}\,r^{\beta+j}$ [$q^{\gamma-i}\,r^{\delta-j}$] $(i, j = 0, 1, 2, \cdots)$ is not a root of the characteristic equation for $(0, 0)$ [(∞, ∞)] of the associated homogeneous system; corresponding hypotheses are made in the cases $\xi = \eta = 0$ and $\varphi = \psi = 0$. Under these conditions it can be shown by direct substitution and calculation of the coefficients that the system (20) possesses four formal series solutions:

$$s_i(x, y) = x^{\alpha-\varkappa}\, y^{\beta-\lambda}[s_i + s_{i10}\, x + s_{i01}\, y + \cdots], \tag{21}$$

$$s'_i(x, y) = x^{\gamma-\mu}\, y^{\delta-\nu}[s'_i + s'_{i10}/x + s'_{i01}/y + \cdots], \tag{22}$$

$$s''_i(x, y) = x^{\omega-\xi}\, y^{\pi-\eta}[s''_i + s''_{i10}\, x + s''_{i01}/y + \cdots], \tag{23}$$

$$s'''_i(x, y) = x^{\sigma-\varphi}\, y^{\tau-\psi}[s'''_i + s'''_{i10}/x + s'''_{i01}\, y + \cdots] \quad (i = 1, 2, \cdots, n). \tag{24}$$

The convergence of these series under certain conditions can be established by the methods of § 1.* In the discussion there arises a new operator Δ defined by the relation

$$\Delta f(x, y) = f(qx, ry) - f(x, y);$$

we call its inverse Σ and employ for Σ the following evaluations:

$$\Sigma f(x, y) = f\left(\frac{x}{q}, \frac{y}{r}\right) + f\left(\frac{x}{q^2}, \frac{y}{r^2}\right) + f\left(\frac{x}{q^3}, \frac{y}{r^3}\right) + \cdots,$$

$$\Sigma f(x, y) = -f(x, y) - f(qx, ry) - f(q^2x, r^2y) - \cdots.$$

The results may perhaps best be classified according as $|q|$ is $\gtrless 1$ and $|r|$ is $\gtrless 1$ and grouped in four theorems of which the following is typical:

THEOREM III A. ($|q| > 1, |r| > 1$): *In the following cases*

$\varkappa \geqq 0,\ \lambda \geqq 0$, *but not both zero*;	$\mu \geqq 0,\ \nu \geqq 0$, *but not both zero*;
$\varkappa = \lambda = 0,\ R\left[\alpha - \varrho_k + l + (\beta + m)\frac{\log r}{\log q}\right] \neq 0$ *for any* l *and* m *that are positive integers or zero* ($k = 1, 2, \cdots, n$);	$\mu = \nu = 0,\ R\left[\gamma - \varrho'_k - l + (\delta - m)\frac{\log r}{\log q}\right] \neq 0$ *for any* l *and* m *that are positive integers or zero* ($k = 1, 2, \cdots, n$);

the formal series solutions (21) *and* (22) *converge for* (x, y) *in the vicinity of* $(0, 0)$ *and* (∞, ∞) *respectively and so represent analytic solutions there. By the aid of the system* (20) *itself the region of validity of these analytic solutions can be extended. The solution which in the neighborhood of* $(0, 0)$ *is represented by* (21) *is analytic away from that place with the following exceptions:* $x = \infty$; $y = \infty$; *the singularities of*† $A(x, y)$ *and* $b(x, y)$ *and places externally congruent*‡ *to them. The solution which in the neighborhood*

* In the typical case $|q| > 1$, $|r| > 1$, the only point in the treatment essentially different from that of § 1 arises when $\varkappa$ and λ are zero and Σ operates upon a function

$$x^{\alpha - \varrho_k} y^{\beta} [\theta + \theta_{10} x + \theta_{01} y + \cdots].$$

If $R[\alpha - \varrho_k]$ or $R[\beta]$ is negative, or both, complications arise; but it is not difficult to show that if $R\left[\alpha - \varrho_k + l + (\beta + m)\frac{\log r}{\log q}\right]$ is $\neq 0$ for any l and m that are positive integers or zero, this function can be broken up into a finite number of pieces for each of which one or other of the evaluations for Σ given here can be employed.

† $A(x, y)$ is used to denote the matrix $(a_{ij}(x, y))$; $A^{-1}(x, y)$ stands for its inverse.

‡ When $|q|$ and $|r|$ are > 1 the places $(q^i x, r^i y)$ $(i = 1, 2, \cdots)$ $[i = -1, -2, \cdots]$ will be spokon of as *externally* [*internally*] *congruent* to (x, y).

of (∞, ∞) *is represented by* (22) *is analytic with the following exceptions:* $x = 0$; $y = 0$; *the singularities of* $A^{-1}(x, y)$ *and* $b(x, y)$ *and places congruent internally to them.*

4. The relation between the solutions. The system (20) often has two solutions; let us call them $\bar{g}_i(x, y)$ and $\bar{h}_i(x, y)$ $(i = 1, 2, \cdots, n)$ confining ourselves to the typical case $|q| > 1$, $|r| > 1$. The set of functions $\bar{g}_i(x, y) - \bar{h}_i(x, y)$ $(i = 1, 2, \cdots, n)$ constitutes a solution of the associated homogeneous system, and so may be expressed thus in terms of the matrix solution $(h_{ij}(x, y))$ of that system:

$$\bar{g}_i(x, y) - \bar{h}_i(x, y) = \sum_{j=1}^{n} \theta_j(x, y)\, h_{ij}(x, y) \quad (i = 1, 2, \cdots, n); \tag{25}$$

the $\theta_j(x, y)$ are periodic functions, satisfying the equation $f(qx, ry) = f(x, y)$. We may solve for the $\theta_j(x, y)$, obtaining

$$\theta_j(x, y) = \frac{\mathfrak{H}_j(x, y)}{\mathfrak{H}(x, y)},$$

where $\mathfrak{H}(x, y)$ denotes the determinant $|h_{ij}(x, y)|$ and $\mathfrak{H}_j(x, y)$ the determinant formed from $\mathfrak{H}(x, y)$ by replacing its jth column by the set of functions on the left in (25).

When the coefficient functions in (20) are polynomials, we can subject the $\theta_j(x, y)$ to an analysis like that made by the author* in the case of the periodic functions of this same type that arise in connection with the homogeneous system (4). A statement of the results follows.

THEOREM IV A. $(|q| > 1, |r| > 1)$: *When the coefficient functions in* (20) *are polynomials and the system possesses two solutions, the relation between the solutions is expressed by the equation* (25), *where the* $\theta_j(x, y)$ $(j = 1, 2, \cdots, n)$ *satisfy the relation* $f(qx, ry) = f(x, y)$ *and are analytic over the entire* x- *and* y-*planes away from* $x = 0$, $x = \infty$, $y = 0$, *and* $y = \infty$. *Under the transformation* $x = q^t$, $y = r^\tau$, $\theta_j(x, y)$ *becomes a function* $\omega_j(t, \tau)$ *that is found to be triply quasi-periodic and of the sort studied by Cousin.*†

The theorems of this and the preceding section can be extended at once to a system like (20) in which there are any number N of independent variables.

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* Loc. cit., pp. 304–307.

† "Sur les fonctions triplement périodiques de deux variables", Acta Math., vol. 33 (1910), pp. 105–232.

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INCIDENCE AND PARALLELISM IN BIAFFINE GEOMETRY.

By Albert A. Bennett.

Projective geometry of space admits of a duality between point and plane, and has no specialized points and planes. Affine geometry does not admit of such a duality and is distinguished as a subgroup of projective geometry by the invariance of a plane "at infinity", this plane being excluded as "improper", from the subgeometry. In affine geometry any two points are equivalent, the "points at infinity" being excluded entirely from the discussion. In the affine geometry of vectors, a subject of long standing and of increasing importance, there is a unique invariant zero vector. Following Euclidean precedents the zero vector is usually accepted as a vector in full and regular standing without question, although it fails to have the familiarly stipulated property of "direction". However an explicit exclusion of the zero vector from the application of the term, "vector", leads to a self dual geometry, (which we therefore designate as Biaffine Geometry) of considerable interest and simplicity, and which is the direct geometrical interpretation of the algebraic relations among matrices of the third order.

We may imagine a space with a given fixed "center" or "origin". By the term "point" will be meant any finite point other than the origin, while the origin will never be called a point. A complete linear one-dimensional set of points will be called a line if and only if the origin is not included in the set, and so on.

We shall state without proofs the elementary relations of incidence and parallelism holding in biaffine geometry of three dimensions, making use of the duality to define relations among points hitherto only familiar in the case of planes and vice versa. The discussion presents no difficulty and its algebraic interpretation is often trivial. Its novelty and usefulness are sufficient justification for its presentation. The classification of quadrics and the applications of inner and outer products, and other such familiar topics of geometry and vector analysis are capable of equally simple treatment.

For the figures indicated the following are complete systems of non-vacuous, mutually exclusive types of relations.

1. Two distinct planes, (i) on a common line, (ii) parallel.
2. Two distinct points, (i) on a common line, (ii) parallel.

3. A point and a plane, (i) incident, (ii) the point parallel to a unique point of the plane and the plane parallel to a unique plane through the point, (the point and plane are then said to be relatively skew), (iii) parallel.

4. Two distinct lines, (i) doubly incident, that is meet in a common point and lie in a common plane, (ii) point-incident, that is, meet in a common point but do not lie in a common plane, (iii) plane-incident, that is, lie in a common plane but do not meet in a common point, (iv) doubly parallel, that is, have no common point or common plane but for each point of either line there is a point of the other parallel to it and for each plane through either line there is a plane through the other parallel to it, (v) skew-parallel, that is, have no common point or common plane, but for these there is a unique pair of parallel points, one on each line, and a unique pair of parallel planes, one through each line, (vi) doubly skew, that is, have no common point or common plane, and there is no point of either line parallel to a point of the other and no plane through either line parallel to a plane through the other.

5. A point and a line, (i) incident, (ii) point-parallel, that is the point is parallel to a unique point of the line, (iii) doubly parallel, that is, are contained in no common plane and there is no point of the line parallel to the given point, (iv) independent, that is, are contained in one and only one common plane.

6. A plane and a line, (i) incident, (ii) plane-parallel, that is, the plane is parallel to a unique plane through the line, (iii) doubly parallel, that is, contain no common point, and there is no plane through the line parallel to the given plane, (iv) independent, that is, contain one and only one common point.

The following are further relations of incidence and parallelism.

1. If a point and plane are parallel, each point parallel to the given point is parallel to each plane parallel to the given plane.

2. If a point and line are independent, the same is true of any point parallel to the given point with any line, point-incident to the given line.

3. If a point and line are point-parallel, the same is true of any non-incident pair, where the point is parallel to the given point and the line is doubly parallel to the given line.

4. If a point and line are point-parallel, the same is true of any non-incident pair which is not doubly parallel, where the point is parallel to the given point, and the line is point-incident to the given line.

5. If a point and a line are doubly parallel, the same is true of any pair where the point is parallel to the given point and the line is either plane-incident or is doubly parallel to the given line.

6. If a plane and a line are independent, the same is true of any plane parallel to the given plane with any line plane-incident to the given line.

7. If a plane and line are plane-parallel, the same is true of any non-incident pair, where the plane is parallel to the given plane and the line is doubly parallel to the given line.

8. If a plane and line are plane-parallel, the same is true of any non-incident pair which is not doubly parallel, where the plane is parallel to the given plane and the line is plane-incident to the given line.

9. If a plane and a line are doubly parallel, the same is true of any pair where the plane is parallel to the given plane and the line is either point-incident or is doubly parallel to the given line.

10. If two lines are doubly skew, these may be uniquely distinguished, one as the point-bearer, and the other as the plane-bearer. The point-bearer has a unique point doubly parallel to the other line and the plane-bearer has a unique plane doubly parallel to the other line. The point-bearer has no plane either plane-parallel or doubly parallel to the other line, and the point-bearer has no point either point-parallel or doubly parallel to the other line.

11. Given two doubly incident lines, any line doubly parallel to one of the lines is skew-parallel to the other.

12. Given two doubly incident lines. Any line plane-incident to one of the lines is in one and only one of the following relations to the other, (i) doubly incident, (ii) skew-parallel, which occurs when the new line is plane-parallel to the plane of the given two lines, (iii) doubly skew, which occurs when the new line is doubly parallel to the plane of the two given lines. In this case the second of the given lines is the plane-bearer and the plane of the given lines is the plane doubly parallel to the new line.

13. Given two doubly incident lines. Any line point-incident to one of the lines is in one and only one of the following relations to the other, (i) doubly incident, (ii) skew-parallel, which occurs when the new line is point-parallel to the point of intersection of the given two lines, (iii) doubly skew, which occurs when the new line is doubly parallel to the point of intersection of the two given lines. In this case the second of the given lines is the point-bearer and the point of intersection of the given lines is the point doubly parallel to the new line.

NEW PROOFS OF THE SIMPLICITY OF EVERY ALTERNATING GROUP WHOSE DEGREE IS NOT FOUR.

By G. A. Miller.

C. Jordan proved that every alternating group whose degree is not four is simple, *Traité des substitutions*, 1870, page 66. Several years earlier he had proved that equations belonging to this group are simple, Paris *Comptes Rendus*, volume 60 (1865), page 773. In what follows we shall establish an elementary theorem from which the simplicity of every alternating group whose degree exceeds 5 and of certain other groups can readily be deduced. This theorem may be stated in the following form: *If every subgroup composed of all the substitutions which omit a given letter of a substitution group G whose degree exceeds 4 is at least triply transitive then G must be at least fourfold transitive and cannot involve more invariant subgroups than every one of the given triply transitive subgroups contains.* In particular, when one of these triply transitive subgroups is simple G must be simple. The fact that this theorem does not apply to all the groups of degree 4 is illustrated by the symmetric group of this degree.

It is evident that this theorem establishes the simplicity of every alternating group whose degree exceeds 5 provided the simplicity of the alternating group of degree 5 has been established. Hence we shall prove that this group is simple before proving the theorem in question. It may be of interest to note that this special case is also included in a very elementary general theorem relating to the subgroups composed of all the substitutions which omit a given letter. This theorem may be expressed as follows: *If a subgroup composed of all the substitutions which omit a given letter of a transitive group is of class p and of degree $p+1$, p being an odd prime number, then this transitive group is simple and there is only one such group for a given value of p.*

It is obvious that the subgroup in question is of degree 2^α and involves invariantly the abelian regular group of order 2^α and of type $(1, 1, 1, \cdots)$. Since this subgroup is primitive the transitive group G must be of degree $p+2$. The group G cannot involve invariantly a regular group of degree $p+2$ since the subgroup composed of all the substitutions which omit one letter of the holomorph of such a regular group cannot be multiply transitive. For the same reason G cannot involve an invariant subgroup of order $(p+1)(p+2)$ since such a subgroup would be of class $p+1$ and hence would involve a characteristic regular group of order $p+2$. Since G is

generated by its Sylow subgroups of order p it cannot involve an invariant subgroup whose order is divisible by p. Hence G must be simple. For $p = 3$ this theorem establishes the simplicity of the alternating group of order 60. Not more than one group of degree $p+2$ can contain such a given subgroup of degree $p+1$ since the substitutions of degree $p+2$ which transform a subgroup of order p contained in G into itself are completely determined by this subgroup and the letters of G.

When G is assumed to be transitive the word "every" in the theorem stated near the close of the first paragraph may be replaced by "a", since all the subgroups composed of all the substitutions which omit a given letter are then conjugate under G. As this is the case which applies to the alternating groups we shall consider it first, and shall prove later that G must be transitive whenever the conditions of the theorem are satisfied. It is obvious that whenever the subgroup composed of all the substitutions which omit a fixed letter of a given transitive group G is primitive then the degree of this subgroup is exactly one less than the degree of G. Hence it results that when this subgroup G_1 is at least triply transitive then G must be at least fourfold transitive. Moreover, G_1 is a maximal subgroup of G since G is primitive, and every invariant subgroup of G besides the identity must involve more than one of the substitutions of G_1. The latter part of this statement results from the fact that such an invariant subgroup would have to be transitive since G is primitive, and it could not be regular since the subgroup composed of all the substitutions which omit one letter of the holomorph of a regular group cannot be more than doubly transitive whenever the degree of this regular is not 4. An invariant subgroup of G must therefore involve an invariant subgroup of G_1 which cannot be the identity whenever the invariant subgroup of G is not the identity. Hence every invariant subgroup of G is generated by the conjugates of an invariant subgroup of G_1 under G.

The fact that G must be transitive whenever the conditions imposed by the given theorems are satisfied is involved in the following theorem: *At least one of the possible subgroups composed of all the substitutions which omit a given letter of an intransitive group is either intransitive or regular.* When an intransitive group G has at least one transitive constituent which is non-regular a subgroup composed of all the substitutions of G which omit a letter of such a constituent must be intransitive unless G is both simply isomorphic with this constituent and this constituent is not generated by all its substitutions whose degrees are less than the degree of this constituent. In the latter case these substitutions must generate an intransitive invariant subgroup of this constituent. This invariant subgroup must correspond to the identity of the group formed by the remaining

transitive constituents of G unless a subgroup composed of all the substitutions of G which omit a letter of the non-regular transitive constituent in question is intransitive. When this intransitive invariant subgroup corresponds to the identity of the group formed by the remaining transitive constituents of G then the subgroup composed of all the substitutions of G which omit a letter of any one of these remaining constituents is obviously intransitive. That is, if an intransitive group involves at least one non-regular transitive constituent then at least one of the subgroups composed of all the substitutions which omit a fixed letter of this intransitive group must itself be intransitive.

It remains therefore only to consider the case when every transitive constituent of G is regular and also every subgroup composed of all the substitutions of G which omit a fixed letter is transitive. It is obvious that both of these conditions are satisfied whenever G is the direct product of any two regular groups and only then. In fact, to the identity of one of these regular transitive constituents there must correspond a transitive subgroup of G. This must be regular since all the transitive constituents of G were assumed to be regular. This completes a proof of the theorem in question, and hence it establishes the fact that every substitution group which satisfies the conditions imposed on G in the theorem noted towards the close of the first paragraph must be transitive.

From the given theorem it results directly that if the alternating group of degree 5 is simple and triply transitive then the alternating group of every larger degree must be simple. The fact that the alternating group of degree 5 is simple results also directly from the well known theorems that an invariant subgroup of a primitive group must be transitive whenever its order exceeds the identity and that the order of a transitive group is a multiple of its degree. Hence it results that if the alternating group of degree 5 were not simple its invariant subgroup whose order exceeds the identity would contain six Sylow subgroups of order 5. Its order would therefore be 30 and it would involve four sets of conjugate cycles of order 5. As it could not have more than four sets of conjugate cycles* it could not contain any cycles of order 2 or of order 3. From Sylow's theorem it follows that cycles of both of these orders must be present in every substitution group of order 30. That is, the alternating group of degree 5 must be simple. The theorem of the first paragraph includes also a proof of the fact that the two fivefold transitive groups discovered by E. Mathieu are simple since their respective maximal subgroups of one lower degree are known to be simple. In fact, in the case of the fivefold

* G. A. Miller, Proc. Nat. Acad., vol. 9 (1923), p. 52.

transitive group of degree 24 its simplicity as well as the simplicity of its maximal subgroup of degree 23 are established by the theorem in question if it is noted that a maximal subgroup of degree 22 contained in the latter is simple.

When the degree of G is not of the form 2^α the theorem stated at the close of the first paragraph remains true if the term "triply transitive" is replaced by the term "doubly transitive", since the subgroup composed of all the substitutions which omit a given letter of the holomorph of a regular group cannot be doubly transitive unless the order of this regular group is of the form 2^α. It may also be added that for the sake of proving the simplicity of every alternating group whose degree exceeds 4 the theorem in question might be replaced by the following: *If a subgroup composed of all the substitutions which omit a given letter of a multiply transitive substitution group of degree n is simple and involves a substitution besides the identity whose degree is less than $n/2$ then this transitive group must be simple.* A proof of this theorem follows directly from the proof given in the second paragraph and the fact that each substitution of a subgroup composed of all the substitutions which omit one letter of a holomorph of a regular group omits exactly as many letters of this holomorph as the number of the substitutions of this regular group which it transforms into itself. This number must be a divisor of the degree of this regular group, and hence it results that if the alternating group of degree 6 is simple then the alternating group of every higher degree must be simple.

SYMMETRIC TENSORS OF THE SECOND ORDER WHOSE COVARIANT DERIVATIVES VANISH.

By Harry Levy.*

In a paper appearing in the *Transactions of the American Mathematical Society*, vol. 25, p. 303, Professor Eisenhart obtained, in analytic form, necessary and sufficient conditions that a Riemann space admit a symmetric tensor of the second order whose covariant derivatives are zero. Here we consider this problem from another point of view, and obtain a new set of necessary and sufficient conditions for the existence of such tensors.

1. Let us begin by considering a covariant tensor of the second order, the determinant of whose components vanishes. If we denote the components by b_{ij}, the equations†

$$b_{ij} X^j = 0 \qquad i = 1, 2, \cdots, n, \tag{1.1}$$

admit a non-zero solution in the X's as functions of the coördinates $x^1, x^2, \cdots, x^n$ of the space. If we define functions $\overline{X}$ of variables $\overline{x}$, when the $\overline{x}$'s are the space coördinates, by means of the tensor equations†

$$\overline{X}^i = X^j \frac{\partial \overline{x}^i}{\partial x^j}, \qquad i = 1, 2, \cdots, n, \tag{1.2}$$

it follows that the left hand side of equations (1.1) are the components of a vector, and hence equations (1.1) hold in every coördinate system. On the other hand, from equations (1.2) X^i are the components of a contravariant vector, and as such defines, in the space, a congruence of curves. If we introduce new coördinates, taking these curves as the curves of parameter x^1 it follows that in this coördinate system

$$X^1 \neq 0, \qquad X^r = 0, \qquad r = 2, 3, \cdots, n, \tag{1.3}$$

and from equations (1.1) we have that

$$b_{i1} = 0, \qquad i = 1, 2, \cdots, n. \tag{1.4}$$

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† We employ the usual convention that, unless otherwise stated, an index appearing twice, once as a subscript and once as a superscript, is to be summed over all values from 1 to n.

Hence we have the theorem:

If the determinant of the components of a covariant tensor of the second order is zero, there exists a coördinate system in which the components $b_{i1}\,(i = 1, 2, \cdots, n)$ *are zero.*

If the determinant $b = |b_{ij}|$ is of rank less than $n-1$, then the determinant of b_{rs} for $r, s = 2, 3, \cdots, n$ vanishes, and we may make in any hypersurface $x^1 =$ constant, a transformation of the coördinates $x^2, x^3, \cdots, x^n$ similar to the preceding one. Hence we have the theorem:

If the rank of the determinant of the components, b_{ij}, *of a covariant tensor of the second order is* p, *we may make the components* $b_{i\alpha} = 0$ $(i = 1, 2, 3, \cdots, n;\ \alpha = 1, 2, 3, \cdots. n-p)$ *in a suitably chosen variety of* $p+1$ *dimensions.*

We observe that in general it is not impossible to make the above reduction throughout the space; for in order that $b'_{i\alpha} = 0$ for $i = 1, 2, \cdots, n$ and $\alpha = 1, 2, \cdots, n-p$ it is necessary and sufficient that the equations

$$b_{jk}\frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial x^k}{\partial \bar{x}^\alpha} = 0, \tag{1.5}$$

be satisfied for $i = 1, 2, \cdots, n$ and $\alpha = 1, 2, \cdots, n-p$; since the Jacobian $\left|\frac{\partial x^j}{\partial \bar{x}^i}\right|$ cannot vanish, we are led to a Pfaff system of equations

$$b_{jk}\frac{\partial x^k}{\partial \bar{x}^\alpha} = 0, \quad j = 1, 2, \cdots, n;\ \alpha = 1, 2, \cdots, n-p. \tag{1.6}$$

Equations of this type have been studied extensively and it has been shown that, for (1.6) to admit n independent solutions, certain restrictions must be imposed upon the b's.

2. Let us consider the case when b_{ij} is a symmetric tensor $(b_{ij} = b_{ji})$ whose covariant derivatives with respect to a fundamental quadratic differential form are zero. We take as the fundamental form

$$\varphi = g_{ij}\,dx^i\,dx^j, \tag{2.1}$$

where $g_{ij} = g_{ji}$, and the determinant $g = |g_{ij}| \neq 0$, and we denote by $\left\{\begin{matrix} h \\ ij \end{matrix}\right\}$ the Christoffel symbols of the second kind formed with respect to the form (2.1); the covariant derivative, $b_{ij,k}$, of b_{ij} with respect to x^k and the form (2.1) is, by definition,

$$b_{ij,k} = \frac{\partial b_{ij}}{\partial x^k} - b_{hj}\left\{\begin{matrix} h \\ ik \end{matrix}\right\} - b_{ih}\left\{\begin{matrix} h \\ jk \end{matrix}\right\}. \tag{2.2}$$

If we choose a coördinate system in which equations (1.4) hold, the conditions that $b_{ij,k} = 0$ reduce upon substitution of (1.4) in (2.2) to the set of equations

$$
\begin{gathered}
\sum_{r=2}^{n} b_{rs} \left\{ \begin{matrix} r \\ 1\,k \end{matrix} \right\} = 0, \qquad \frac{\partial b_{rs}}{\partial x^1} = 0, \\
\frac{\partial b_{rs}}{\partial x^t} - \sum_{q=2}^{n} b_{qs} \left\{ \begin{matrix} q \\ r\,t \end{matrix} \right\} - \sum_{q=2}^{n} b_{rq} \left\{ \begin{matrix} q \\ s\,t \end{matrix} \right\} = 0, \\
k = 1, 2, \cdots, n; \; r, s, t = 2, 3, \cdots, n.
\end{gathered} \tag{2.3}
$$

From the middle of these three sets of equations combined with equations (1.4) we observe that the b's are independent of x^1. If the rank of $b = |b_{ij}|$ is less than $n-1$, the equations (1.1) have at least two independent solutions, one of which is given by equations (1.3), which solution we denote by $X_{1|}{}^{i}$, and the second is a solution of the equations

$$
\sum_{s=2}^{n} b_{rs} X^s = 0, \qquad r = 2, 3, \cdots, n. \tag{2.4}
$$

Denoting this solution by $X_{2|}{}^{i}$, we observe that $X_{2|}{}^{1}$ is arbitrary and that $X_{2|}{}^{s}$ $(s = 2, 3, \cdots, n)$ are functions only of $x^2, x^3, \cdots, x^n$, and consequently by reasoning similar to that employed in the preceding section, we may make a transformation of the coördinates $x^2, x^3, \cdots, x^n$ which makes

$$
X_{2|}{}^{2} \neq 0, \qquad X_{2|}{}^{r} = 0, \qquad r = 3, 4, \cdots, n, \tag{2.5}
$$

and since x^1 is left unchanged by this transformation of coördinates, equations (1.4) still hold, and the solution $X_{1|}{}^{i}$ satisfies (1.3). Substituting equations (2.5) in equations (2.4) we obtain that

$$
b_{r2} = 0, \qquad r = 2, 3, 4, \cdots, n. \tag{2.6}
$$

If we substitute this result first in the equation from the last set of equations (2.3), for $r = 2$, $s \neq 2$, and then for $t = 2$, $r, s \neq 2$, we obtain that

$$
\begin{gathered}
\frac{\partial b_{rs}}{\partial x^2} = 0, \\
\sum_{r=3}^{n} b_{rs} \left\{ \begin{matrix} r \\ 2\,k \end{matrix} \right\} = 0, \qquad k = 1, 2, \cdots, n; \; r, s = 3, 4, \cdots, n.
\end{gathered}
$$

If the rank of b is less than $n-2$, the determinant $|b_{rs}|$ for $r, s = 3, 4, \cdots, n$ vanishes, and the b's are functions of $x^3, \cdots, x^n$ only; we are able to continue the reduction in the above fashion. If we let the rank of b be p, equations (1.1) admit $n-p$ independent solutions $X_{\alpha|}{}^{i}$ $(\alpha = 1, 2, \cdots, n-p; \; i = 1, 2, \cdots, n)$ and a coördinate system may be determined such that

$$
X_{\alpha|}{}^{i} = 0, \qquad \alpha = 1, 2, \cdots, n-p; \; i = \alpha + 1, \cdots, n. \tag{2.7}
$$

It then follows that in this coördinate system

$$b_{i\alpha} = 0, \qquad \alpha = 1, 2, \cdots, n-p;\ i = 1, 2, \cdots, n. \tag{2.8}$$

The conditions (2.3) that the covariant derivatives of b_{ij} be zero, reduce by virtue of (2.8) to the equations

$$\begin{Bmatrix} r \\ \alpha\, k \end{Bmatrix} = 0, \qquad \frac{\partial b_{rs}}{\partial x^{\alpha}} = 0,$$

$$\frac{\partial b_{rs}}{\partial x^{t}} - \sum_{q=n-p+1}^{n} b_{qs} \begin{Bmatrix} q \\ r\, t \end{Bmatrix} - \sum_{q=n-p+1}^{n} b_{rq} \begin{Bmatrix} q \\ s\, t \end{Bmatrix} = 0, \tag{2.9}$$

$$\alpha = 1, 2, \cdots, n-p;\ r, s, t = n-p+1, \cdots, n;$$
$$k = 1, 2, \cdots, n.$$

3. In order to interpret the first set of these equations, that $\begin{Bmatrix} r \\ \alpha\, k \end{Bmatrix}$ is zero, which we observe is a condition only on the space, we introduce an *orthogonal ennuple*, that is n contravariant vectors, $\lambda_{h|}{}^{i}$, subject to the conditions that

$$g_{ij}\, \lambda_{h|}{}^{i}\, \lambda_{k|}{}^{j} = 0,$$
$$g_{ij}\, \lambda_{h|}{}^{i}\, \lambda_{h|}{}^{j} = e_h, \quad h, k = 1, 2, \cdots, n;\ h \neq k, \tag{3.1}$$

where e_h is plus or minus one. For our purpose we take the first $n-p$ of these congruences linearly dependent upon the X's, that is

$$\lambda_{\alpha|}{}^{i} = \sum_{\beta=1}^{n-p} A_{\alpha\beta}\, X_{\beta|}{}^{i}, \quad \alpha = 1, 2, \cdots, n-p;\ i = 1, 2, \cdots, n. \tag{3.2}$$

The problem of finding functions $A_{\alpha\beta}$ such that (3.1) hold, is equivalent to the problem of finding a generalized tetrahedron self conjugate with respect to a quadric hypersurface, for

$$g_{ij}\, \lambda_{\alpha|}{}^{i}\, \lambda_{\beta|}{}^{j} = \sum_{\alpha,\beta=1}^{n-p} (g_{ij}\, X_{\alpha|}{}^{i}\, X_{\beta|}{}^{j})\, A_{\alpha\gamma}\, A_{\beta\delta},$$

and this is possible.

From equations (2.7) it follows that

$$\lambda_{\alpha|}{}^{r} = 0, \qquad \alpha = 1, 2, \cdots, n-p;\ r = n-p+1, \cdots, n. \tag{3.3}$$

The remaining congruences are chosen arbitrarily, except that (3.1) hold; if we denote the covariant components of a congruence by $\lambda_{h|i}$ so that $\lambda_{h|i} = g_{ij}\, \lambda_{h|}{}^{j}$ we must have from equations (3.1) by virtue of equations (3.3) that

$$\sum_{\beta=1}^{n-p} \lambda_{\alpha|}{}^{\beta}\, \lambda_{r|\beta} = 0, \qquad \alpha = 1, 2, \cdots, n-p;\ r = n-p+1, \cdots, n,$$

from which it follows that

$$\lambda_{r|\beta} = 0, \qquad \beta = 1, 2, \cdots, n-p;\ r = n-p+1, \cdots, n. \tag{3.4}$$

The invariants*, γ_{ijk}, of an orthogonal $n$ple are defined by the equations

$$\gamma_{ijk} = \lambda_{i|h,l}\, \lambda_{j|}{}^{h}\, \lambda_{k|}{}^{l}, \tag{3.5}$$

where $\lambda_{i|h,l}$ is the covariant derivative of $\lambda_{i|h}$ with respect to x^l and the fundamental form (2.1). In particular $\lambda_{r|\alpha,i}$, for $\alpha = 1, 2, \cdots, n-p$; $r = n-p+1, \cdots, n$; $i = 1, 2, \cdots, n$, vanishes by virtue of equations (3.4) and the first set of (2.9). From (3.3) it then follows that

$$\gamma_{r\beta j} = 0 \qquad \begin{matrix} \beta = 1, 2, \cdots, n-p; \\ r = n-p+1, \cdots, n; \\ j = 1, 2, \cdots, n. \end{matrix} \tag{3.6}$$

Since $\gamma_{ijk} + \gamma_{jik} \equiv 0$†, from (3.6) it follows that $\gamma_{\beta rs} = 0$ and $\gamma_{r\alpha\beta} = 0$. These, the author has shown,‡ are sufficient conditions that congruences $\lambda_{1|}, \lambda_{2|}, \cdots, \lambda_{n-p|}$ be the normals to a family of varieties, V_p, of dimensionality p, and that congruences $\lambda_{n-p+1|}, \cdots, \lambda_{n|}$ be the normals to a family of varieties, V_{n-p}, of dimensionality $n-p$. We let the family of V_p's be defined by the system of equations

$$f_{\alpha|}(x^1, x^2, \cdots, x^n) = c_\alpha, \qquad \alpha = 1, 2, \cdots, n-p, \tag{3.7}$$

where the c's are parameters, and similarly we take the equations

$$f_{r|}(x^1, x^2, \cdots, x^n) = c_r, \qquad r = n-p+1, \cdots, n, \tag{3.8}$$

to be the equations defining the family of V_{n-p}'s. From equations (3.7) $\partial f_{\alpha|}/\partial x^i$ for any $\alpha = 1, 2, \cdots, n-p$ and for $i = 1, 2, \cdots, n$ are the components of a vector orthogonal to V_p, and hence $\partial f_{\alpha|}/\partial x^i$ must be a linear combination of $\lambda_{1|i}, \lambda_{2|i}, \cdots, \lambda_{n-p|i}$. Similarly $\partial f_{r|}/\partial x^i$ is a linear combination of $\lambda_{n-p+1|i}, \lambda_{n-p+2|i}, \cdots, \lambda_{n|i}$. From (3.1) written in the form

* Ricci and Levi-Civita, Méthodes de calcul différential absolu, Math. Annalen, vol. 54 (1901), p. 148.

† Ricci and Levi-Civita, loc. cit., p. 148.

‡ "Normal congruences of curves in Riemann space", Bull. Amer. Math. Soc., vol. 31, (1925), p. 41.

$g^{ij}\,\lambda_{h|i}\,\lambda_{k|j} = 0$ $(h \neq k)$ it follows that any one of the hypersurfaces $f_{\alpha|} = \text{constant}$ $(\alpha = 1, 2, \cdots, n-p)$ is orthogonal to every one of the hypersurfaces $f_{r|} = \text{constant}$ $(r = n-p+1, \cdots, n)$. Consequently if we effect a transformation of coördinates, taking the hypersurface $f_{i|} = \text{constant}$ for the hypersurface $x^i = \text{constant}$, in the new coördinate system

$$g_{\alpha r} = 0, \quad \alpha = 1, 2, \cdots, n-p;\ r = n-p+1, \cdots, n. \tag{3.9}$$

The varieties V_p are now given by the set of equations $x^1 = \text{constant}$, $x^2 = \text{constant}, \cdots, x^{n-p} = \text{constant}$, and since $\lambda_{\alpha|i}$ for $i = 1, 2, \cdots, n$ are the components of a vector normal to V_p, $\lambda_{\alpha|r} = 0$ for $\alpha = 1, 2, \cdots, n-p$; $r = n-p+1, \cdots, n$. From (3.9) it follows that $\lambda_{\alpha|}{}^r = 0$, and from (3.2) that $X_{\alpha|}{}^r = 0$. But $X_{\alpha|}{}^i$ for $\alpha = 1, 2, \cdots, n-p$ are independent solutions of equations (1.1), from which we obtain that equations (2.8) hold. Hence equations (2.9) and (3.9) hold simultaneously.

If we denote the Christoffel symbol of the first kind by $[ij, k]$ so that $[ij, k] = g_{kh}\left\{ \begin{matrix} h \\ i\ j \end{matrix} \right\}$, it follows from the first of equations (2.9) and (3.9) that $[\alpha k, r] = 0$ for $\alpha = 1, 2, \cdots, n-p$; $r = n-p+1, \cdots, n$; $k = 1, 2, \cdots, n$. From this it follows that $g_{\alpha\beta}$ is independent of x^r and that g_{rs} is independent of x^α for $\alpha, \beta = 1, 2, \cdots, n-p$, and $r, s = n-p+1, \cdots, n$. The fundamental form (2.1) is then expressible as the sum of two forms

$$\varphi = \sum_{\alpha,\beta=1}^{n-p} g_{\alpha\beta}\,dx^\alpha\,dx^\beta + \sum_{r,s=n-p+1}^{n} g_{rs}\,dx^r\,dx^s$$

where the g's are functions only of the x's of its form. We may sum up these results in the form of a theorem:

*If the covariant derivatives of a symmetric tensor of the second order are zero, and if the determinant of the components of the tensor is of rank p, the fundamental form of the space is expressible as the sum of two forms, one in $n-p$ variables, the other in p variables, the coefficients in each form being functions only of the x's of that form. The tensor is reducible to a tensor in the space of the p variables whose components are functions only of the x's of that space, and whose covariant derivatives with respect to the fundamental form of the space of the p variables are equal to zero.**

* This result is in accordance with that obtained by Eisenhart, loc. cit., p. 303, under the assumption of a positive definite form. Quadratic forms of the above type, that is forms which are expressible as the sum of several forms, were obtained by Eisenhart in the above mentioned paper, and have also received the attention of Kasner in his paper "Separable quadratic differential forms", Proc. Nat. Acad. Sci., vol. 11 (1925), p. 95.

4. We consider finally the case of a tensor whose determinant is not zero, and whose covariant derivatives are zero. Take as the fundamental form of the space the general quadratic form

$$\varphi = g_{ij}\,dx^i\,dx^j, \tag{4.1}$$

where $g = |g_{ij}| \neq 0$; we then have

$$b_{ij,k} \equiv \frac{\partial b_{ij}}{\partial x^k} - b_{hj}\left\{\begin{matrix} h \\ i\ k \end{matrix}\right\} - b_{ih}\left\{\begin{matrix} h \\ j\ k \end{matrix}\right\} = 0. \tag{4.2}$$

If we denote the Christoffel symbols of the second kind with respect to the tensor b_{ij} by $\left\{\begin{matrix} h \\ i\ k \end{matrix}\right\}_b$, the equations

$$\frac{\partial b_{ij}}{\partial x^k} - b_{hj}\left\{\begin{matrix} h \\ i\ k \end{matrix}\right\}_b - b_{ih}\left\{\begin{matrix} h \\ j\ k \end{matrix}\right\}_b = 0 \tag{4.3}$$

are always true. If we subtract one of this set of equations from the corresponding equations of the set (4.2), and if therein we interchange i, j, k, we obtain the equations

$$\left\{\begin{matrix} h \\ ik \end{matrix}\right\} - \left\{\begin{matrix} h \\ ik \end{matrix}\right\}_b = 0. \tag{4.4}$$

It has been shown* that a necessary and sufficient condition that two Riemann spaces whose fundamental forms are (4.1) and

$$\varphi' = b_{ij}\,dx^i\,dx^j, \tag{4.5}$$

respectively, be in geodesic correspondence is that

$$\left\{\begin{matrix} h \\ ik \end{matrix}\right\} - \left\{\begin{matrix} h \\ ik \end{matrix}\right\}_b = \frac{1}{4}\left(\delta_i^h \frac{b_{kk,k}}{b_{kk}} + \delta_k^h \frac{b_{ii,i}}{b_{ii}}\right), \tag{4.6}$$

where δ_i^h is a Kronecker delta, that is

$$\delta_i^h = \begin{cases} 0, & h \neq i, \\ 1, & h = i. \end{cases}$$

Hence by virtue of equations (4.2), (4.6) are satisfied. A further necessary condition that (4.2) be true is that

$$b_{ij,k} + b_{jk,i} + b_{ki,j} = 0. \tag{4.7}$$

* "Geodesic representation between Riemann spaces", Levy and Bramley, these Annals (2), vol. 25 (1923), p. 55.

But these equations are the conditions that b_{ij} be a quadratic first integral of the equations of the geodesics of (4.1),* that is, that $b_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$ be constant along the geodesics of (4.1). Conversely these two conditions are sufficient to make $b_{ij,k}$ vanish. For if (4.7) hold, we obtain that $b_{ii,i}$ vanishes, and from (4.6) it follows that $\left\{\begin{matrix} h \\ ik \end{matrix}\right\}_b = \left\{\begin{matrix} h \\ ik \end{matrix}\right\}$; substituting in (4.3), it follows that (4.2) hold. Hence we have the theorem:

A necessary and sufficient condition that a symmetric tensor, b_{ij}, $(b = |b_{ij}| \neq 0)$ of the second order have covariant derivatives equal to zero is first, that the tensor be a first integral of the geodesics of the given space, and second, that the Riemann space defined by taking b_{ij} as the fundamental tensor be capable of geodesic representation upon the given space.

5. Suppose (4.1) represents a space of constant curvature, K_0. Then we have the relation†

$$R^i{}_{jkl} = K_0(\delta^i_k g_{jl} - \delta^i_l g_{jk}) \tag{5.1}$$

where $R^i{}_{jkl}$ is the Riemann tensor of the second kind referred to the fundamental form (4.1). Beltrami has shown‡ that the spaces which admit of geodesic representation upon a space of constant curvature are themselves spaces of constant curvature. Hence from the hypothesis that equations (4.2) and (5.1) hold, it must follow that

$$R'^i{}_{jkl} = K_1(\delta^i_k b_{jl} - \delta^i_l b_{jk}) \tag{5.2}$$

where $R'^i{}_{jkl}$ refers to the fundamental form (4.5) and K_1 is the constant curvature of that form. But from (4.4) it follows that $R'^i{}_{jkl} = R^i{}_{jkl}$; equating from (5.1) and (5.2) for $i = k \neq l$ we find that

$$K_0 g_{jl} = K_1 b_{jl}.$$

Hence we have the theorem:

If, in a space of constant curvature, a symmetric tensor of the second order with non-vanishing determinant has first covariant derivatives equal to zero, it must be a constant multiple of the fundamental tensor of the space.

* Ricci and Levi-Civita, loc. cit., p. 183.

† Schouten, Der Ricci-Kalkül, p. 171.

‡ Schouten, loc. cit., p. 204.

GENERALIZATIONS OF THE EIGHT SQUARE AND SIMILAR IDENTITIES.

By E. T. Bell.

1. **Introduction and summary.** In generalizing Euler's four square identity (all references are in § 5), Lagrange[1] merely writes down the result without stating what led him to it. Seeking to reconstruct his computation we find an extension of his formula, and an extremely simple method which enables us to generalize the eight square identity in a similar manner. The same process can also be applied at once to yield like generalizations of the norm theorems, first given by Lagrange[2], for algebraic integers of any degree, and to the norm theorems for linear algebras.

The gist of the method is as follows. First, the usual proof of Euler's identity by means of norms of quaternions (Tait[3], Dickson[4]), may be modified by taking quaternions over a given algebraic number field. One of the simplest examples leads in § 2 to the above mentioned generalization of Lagrange's identity, viz., the form

$$x^2+bcy^2+cau^2+abv^2$$

repeats under multiplication. The field here is of the sixth degree, and is defined by

$$\xi^6-(ab+bc+ca)\,\xi^4+abc(a+b+c)\,\xi^2-a^2b^2c^2=0.$$

For its bearing on the next application we note that the above generalization is included with that of Lagrange in the important theorem of Hermite[5], that the quaternary quadratic form $x^2+\varphi(u,v,w)$ where $\varphi(u,v,w)$ is the adjoint of the (general) ternary quadratic form, repeats under multiplication. It would be interesting to know whether Hermite's theorem can be obtained from Euler's by taking quaternions over the appropriate field. If so, the like could be tried in the next instance.

Extending the norm theorem for the algebras of order 8 of Cayley[6] or Dickson[7], we are led in the same way, as the simplest example, to the repetition under multiplication of the octenary quadratic form

$$p^2+abct^2+gh(au^2+bcq^2)+hf(bv^2+car^2)+fg(cx^2+abs^2);$$

that is, if this form be denoted by

$$F(p,q,r,s,t,u,v,x),$$

then (cf. § 3, end),

$$F(p,\cdots,x)\cdot F(p_1,\cdots,x_1)=F(p_2,\cdots,x_2),$$

where $p_2, \cdots, x_2$ are *rational and integral in* $p_1, \cdots, x_1, p_2, \cdots, x_2, a, b, c, f, g, h,$ *with rational integer coefficients.* The field here is of degree 14; it is unnecessary to write down its defining equation.

In all such theorems, unless otherwise noted, it is postulated that the new variables $p_2, q_2, \cdots$ occurring in the compound form shall be rational and integral in *all* of the letters appearing in the component forms, and further that the numerical coefficients shall be rational integers.

These two theorems together suggest the existence of a theorem for octenary quadratic forms analogous to Hermite's for quaternary, but I have not found this. They further hint that the complete theory of composition for octenary quadratic will be amenable to treatment similar to the recent discussion by Brandt[8] for quaternary quadratics. Other lines of evidence point in the same direction; e. g., the fact that 8 is the maximum order of quadratic forms for which the principal genus is represented by a single class. There can be no true analogue of Hermite's identity for quadratic forms in 16, 32, 64, $\cdots$ indeterminates, since, if there were, we should have multiplication theorems for these numbers of squares, and such are known not to exist.

There are similar extentions of the norm theorems for algebraic number fields of any degree. The simplest example for fields of degree >2, given in § 4, shows that the ternary cubic

$$x^3+ab^2y^3+a^2bz^3-3abxyz$$

repeats under multiplication. This is obtained from its special case $a=b=1$, the case $a=1$ has been discussed by several writers in connection with Diophantine analysis.

Finally, the whole question of repetition under multiplication of forms whose coefficients are integers of a given algebraic number field has barely been opened. The only instances in the literature appear to be the ternary cubic of Eisenstein[9] and the quinary quintic of the writer,[10] belonging respectively to the theories of trisection and quinquesection in cyclotomy.

2. **Extension of Lagrange's identity.** A full treatment of this case will enable us to omit several longer formulas for the next. In Euler's identity

$$(x^2+y^2+u^2+v^2)\,(x_1^2+y_1^2+u_1^2+v_1^2) = x_2^2+y_2^2+u_2^2+v_2^2,$$

where

$$\begin{aligned}
x_2 &= xx_1-yy_1-uu_1-vv_1,\\
y_2 &= (x'y_1'+y'x_1')+(u'v_1'-v'u_1'),\\
u_2 &= (x'u_1'+u'x_1')-(y'v_1'-v'y_1'),\\
v_2 &= (x'v_1'+v'x_1')+(y'u_1'-u'y_1'),
\end{aligned}$$

take

$$(x, y, u, v) = (x', y'\sqrt{\alpha}, u'\sqrt{\beta}, v'\sqrt{\gamma}),$$

with a similar substitution for the letters with suffixes, and impose the conditions that the resulting form

$$x^2 + \alpha y^2 + \beta u^2 + \gamma v^2$$

shall repeat under multiplication. We have

$$\begin{aligned}
x_2 &= x'x_1' - \alpha y'y_1' - \beta u'u_1' - \gamma v_0'v_1', \\
y_2 &= \sqrt{\alpha}(x'y_1' + y'x_1') + \sqrt{\beta\gamma}(u'v_1' - v'u_1'), \\
u_2 &= \sqrt{\beta}(x'u_1' + u'x_1') - \sqrt{\gamma\alpha}(y'v_1' - v'y_1'), \\
v_2 &= \sqrt{\gamma}(x'v_1' + v'x_1') + \sqrt{\alpha\beta}(y'u_1' - u'y_1').
\end{aligned}$$

The conditions will be satisfied provided

$$x_2 = x_2', \quad y_2 = y_2'\sqrt{\alpha}, \quad u_2 = u'\sqrt{\beta}, \quad v_2 = v'\sqrt{\gamma},$$

where x', y', u', v' satisfy the conditions of rationality and integrality as stated in § 1. These require that

$$\sqrt{\beta\gamma} = \alpha'\sqrt{\alpha}, \quad \sqrt{\gamma\alpha} = \beta'\sqrt{\beta}, \quad \sqrt{\alpha\beta} = \gamma'\sqrt{\gamma},$$

where α', β', γ', are new parameters. From these, attending only to the positive roots, we get

$$\alpha = \beta'\gamma', \quad \beta = \gamma'\alpha', \quad \gamma = \alpha'\beta'.$$

Finally, writing $(\alpha', \beta', \gamma') \equiv (a, b, c)$, and dropping accents, we have

$$(x^2 + bcy^2 + cau^2 + abv^2)(x_1^2 + bcy_1^2 + cau_1^2 + abv_1^2) = x_2^2 + bcy_2^2 + cau_2^2 + abv_2^2,$$

where

$$\begin{aligned}
x_2 &= xx_1 - bcyy_1 - cauu_1 - abvv_1, \\
y_2 &= xy_1 + yx_1 + a(uv_1 - vu_1), \\
u_2 &= xu_1 + ux_1 - b(yv_1 - vy_1), \\
v_2 &= xv_1 + vx_1 + c(yu_1 - uy_1).
\end{aligned}$$

Other forms of this are obtained by changing the signs of some of the variables. For $c = -1$ it becomes Lagrange's identity.

In this example the process leads to a determinate system of equations for finding α, β, γ, but this is the exception. Thus, in the next, the corre-

sponding set contains 21 equations between 7 parameters. As the main discussion is precisely similar to the above we need consider only the equations and the final result.

3. **Extension of the eight square theorem.** The eight square identity is given in full by Dickson[4]. If desired it can be written down from our final formula by taking therein each of a, b, c, f, g, h equal to unity. Proceeding as before we make the substitution

$$(p, q, r, s, t, u, v, x) = (p', q'\sqrt{\alpha}, r'\sqrt{\beta}, s'\sqrt{\gamma}, t'\sqrt{\delta}, u'\sqrt{\varepsilon}, v'\sqrt{\zeta}, x'\sqrt{\theta}),$$

and impose the conditions (upon the eight square identity as transformed by the substitution) that the form

$$p'^2+\alpha q'^2+\beta r'^2+\gamma s'^2+\delta t'^2+\varepsilon u'^2+\zeta v'^2+\theta x'^2$$

shall repeat under multiplication. The integrality conditions are met as they arise in the course of the solution; the rationality conditions are satisfied provided the following system of equations has a rational solution for $\alpha, \beta, \cdots, \theta$ in terms of parameters,

$$\begin{aligned}
\beta\gamma &= \alpha_1^2\alpha, & \delta\varepsilon &= \alpha_2^2\alpha, & \zeta\theta &= \alpha_3^2\alpha,\\
\alpha\gamma &= \beta_1^2\beta, & \delta\zeta &= \beta_2^2\beta, & \varepsilon\theta &= \beta_3^2\beta,\\
\alpha\beta &= \gamma_1^2\gamma, & \delta\theta &= \gamma_2^2\gamma, & \varepsilon\zeta &= \gamma_3^2\gamma,\\
\alpha\varepsilon &= \delta_1^2\delta, & \beta\zeta &= \delta_2^2\delta, & \gamma\theta &= \delta_3^2\delta,\\
\alpha\delta &= \varepsilon_1^2\varepsilon, & \beta\theta &= \varepsilon_2^2\varepsilon, & \gamma\zeta &= \varepsilon_3^2\varepsilon,\\
\alpha\theta &= \zeta_1^2\zeta, & \beta\delta &= \zeta_2^2\zeta, & \gamma\varepsilon &= \zeta_3^2\zeta,\\
\alpha\zeta &= \theta_1^2\theta, & \beta\varepsilon &= \theta_2^2\theta, & \gamma\delta &= \theta_3^2\theta,
\end{aligned}$$

in which the letters with suffixes denote rational integral functions of parameters to be determined.

From the first three in the first column we find

$$\alpha = \beta_1\gamma_1, \qquad \beta = \gamma_1\alpha_1, \qquad \gamma = \alpha_1\beta_1;$$

from the first three in the third column,

$$\varepsilon = \alpha_1\beta_3\gamma_3/\alpha_3, \qquad \zeta = \beta_1\gamma_3\alpha_3/\beta_3, \qquad \theta = \gamma_1\alpha_3\beta_3/\gamma_3.$$

To get integral solutions in terms of new parameters take

$$\alpha_1 = \alpha_1'\alpha_3, \qquad \beta_1 = \beta_1'\beta_3, \qquad \gamma_1 = \gamma_1'\gamma_3.$$

From the first three in the second column we find now

$$\delta^3 = \alpha_2^2\beta_2^2\gamma_2^2\alpha_1'\beta_1'\gamma_1',$$

which will give a rational δ provided

$$\alpha_2 = k\alpha_1', \qquad \beta_2 = k\beta_1', \qquad \gamma_2 = k\gamma_1',$$

where k is a new parameter, giving $\delta = k\alpha_1'\beta_1'\gamma_1'$.

To satisfy the fourth row we must have

$$\delta_1^2 = \beta_3^2\gamma_3^2/k, \qquad \delta_2^2 = \gamma_3^2\alpha_3^2/k, \qquad \delta_3^2 = \alpha_3^2\beta_3^2/k,$$

and to make these integral we must take

$$\alpha_3 = k\alpha_3', \qquad \beta_3 = k\beta_3', \qquad \gamma_3 = k\gamma_3',$$

and therefore, for rationality, $k = 1$. Hence $\delta = \alpha_1'\beta_1'\gamma_1'$, and $(\alpha_3', \beta_3', \gamma_3') = (\alpha_3, \beta_3, \gamma_3)$.

We have therefore in summary

$$\alpha = \beta_1'\gamma_1'\beta_3\gamma_3, \quad \beta = \gamma_1'\alpha_1'\gamma_3\alpha_3, \quad \gamma = \alpha_1'\beta_1'\alpha_3\beta_3, \quad \delta = \alpha_1'\beta_1'\gamma_1',$$
$$\epsilon = \alpha_1'\beta_3\gamma_3, \quad \zeta = \beta_1'\gamma_3\alpha_3, \quad \theta = \gamma_1'\alpha_3\beta_3$$

as a rational integral solution in terms of the parameters α_1', β_1', γ_1', α_3, β_3, γ_3 of the 12 equations in the first 4 rows. It remains to choose $\alpha_1', \cdots, \gamma_3$ so that the last three rows are satisfied in the same way. Direct substitution shows that the above values give the required solution, and we have also the following subsidiary results, needed in writing out the final formula:

$$\begin{array}{lll}
\alpha_1 = \alpha_1'\alpha_3, & \beta_1 = \beta_1'\beta_3, & \gamma_1 = \gamma_1'\gamma_3, \\
\alpha_2 = \alpha_1', & \beta_2 = \beta_1', & \gamma_2 = \gamma_1', \\
\delta_1 = \beta_3\gamma_3, & \delta_2 = \gamma_3\alpha_3, & \delta_3 = \alpha_3\beta_3, \\
\epsilon_1 = \beta_1'\gamma_1', & \epsilon_2 = \gamma_1'\alpha_3, & \epsilon_3 = \beta_1'\alpha_3, \\
\zeta_1 = \gamma_1'\beta_3, & \zeta_2 = \alpha_1'\gamma_1', & \zeta_3 = \alpha_1'\beta_3, \\
\theta_1 = \beta_1'\gamma_3, & \theta_2 = \alpha_1'\gamma_3, & \theta_3 = \alpha_1'\beta_1'.
\end{array}$$

By means of these it is easily verified that the stated values of $\alpha, \cdots, \theta$ are a solution of the original 21 equations having the desired properties. Write

$$(\alpha_1', \beta_1', \gamma_1', \alpha_3, \beta_3, \gamma_3) \equiv (a, b, c, f, g, h).$$

Then the required values of $p_2, \cdots, x_2$ in § 1 (1) are

$$\begin{aligned}
p_2 &= pp_1+abctt_1+gh(auu_1+bcqq_1)+hf(bvv_1+carr_1)+fg(cxx_1+abss_1),\\
q_2 &= pq_1-qp_1 \quad +af(rs_1 \quad -sr_1) \quad +a(tu_1 \quad -ut_1) \quad +f(vx_1 \quad -xv_1),\\
r_2 &= pr_1-rp_1 \quad -bg(qs_1 \quad -sq_1) \quad \mp b(tv_1 \quad -vt_1) \quad \pm g(ux_1 \quad -xu_1),\\
s_2 &= ps_1-sp_1 \quad +ch(qr_1 \quad -rq_1) \quad \pm c(tx_1 \quad -xt_1) \quad \pm h(uv_1 \quad -vu_1),\\
t_2 &= pt_1-tp_1 \quad -gh(qu_1 \quad -uq_1) \quad \pm hf(rv_1 \quad -vr_1) \quad \mp fg(sx_1 \quad -xs_1),\\
u_2 &= pu_1-up_1 \quad +bc(qt_1 \quad -tq_1) \quad \mp cf(rx_1 \quad -xr_1) \quad \mp bf(sv_1 \quad -vs_1),\\
v_2 &= pv_1-vp_1 \quad -cg(qx_1 \quad -xq_1) \quad \mp ca(rt_1 \quad -tr_1) \quad \pm ag(su_1 \quad -us_1),\\
x_2 &= px_1-xp_1 \quad +bh(qv_1 \quad -vq_1) \quad \pm ah(ru_1-ur_1) \quad \pm ab(st_1 \quad -ts_1).
\end{aligned}$$

The result of taking any one of a, b, c, f, g, h equal to zero in this is readily identified with the extension in § 2 of Lagrange's identity.

4. **Extensions of algebraic norm theorems.** The simplest norm theorem for a field of degree >2 is

$$x^3+y^3+z^3-3xyz = (x+y+z)\,(x+wy+w^2z)\,(x+w^2y+wz),$$

w being a complex cube root of unity. Replacing (x, y, z) by $(x\sqrt[3]{\alpha}, y\sqrt[3]{\beta}, z\sqrt[3]{\gamma})$, and requiring the resulting form to repeat under multiplication, we get easily the following final result. Write

$$\varphi(x, y, z) \equiv x^3+ab^2y^3+a^2bz^3-3abxyz.$$

Then $\varphi(x, y, z)\cdot\varphi(x_1, y_1, z_1) = \varphi(x_2, y_2, z_2)$, where

$$\begin{aligned}
x_2 &= xx_1+aby z_1+abzy_1,\\
y_2 &= xy_1+yx_1 \quad +azz_1,\\
z_2 &= xz_1+byy_1 \quad +zx_1.
\end{aligned}$$

This example is sufficient to show the character of such extensions.

5. **References.** [1] Lagrange, *Œuvres*, vol. 3, p. 201. [2] ibid., vol. 7, pp. 164–179. [3] Tait, *Quaternions*, 3d. edit., pp. 63, 147. [4] Dickson, *Annals of Mathematics*, vol. 20, p. 157. [5] Hermite, *Œuvres*, vol. 1, p. 213; cf. also Bachmann, *Zahlentheorie*, vol. 4, pp. 13, 409 (the last for a point of departure for the suggested extension to octenary forms). [6] Cayley, *Coll. Papers*, vol. 11, pp. 368–371; vol. 1, p. 127. [7] Dickson, *Annals of Mathematics*, vol. 20, pp. 157–159, 164. [8] Brandt, *Mathematische Annalen*, vol. 91, pp. 300–316. [9] Eisenstein, *Crelle's Journal*, vol. 28, pp. 289–274; vol. 29, pp. 19–53. [10] *Cyclotomic Quinary Quintic* (New Era Press, 1912), pp. 1–5. Papers bearing on the discussion are analyzed in Dickson's *History*, vol. 2, pp. 277, 281, 283; vol. 3, pp. 215, 226, 232, 259.

ON A GEOMETRICAL THEORY OF CONTINUOUS GROUPS.

I. FAMILIES OF PATH-CURVES OF CONTINUOUS ONE-PARAMETER GROUPS OF THE PLANE.*

By B. de Kerékjártó.

1. The purpose of the present paper is to investigate the continuous one-parameter groups of topological (i. e. continuous one-to-one) transformations of the plane into itself from a purely geometrical point of view. The analytical theory of continuous groups as developed by Lie gives for this problem actually only the answer that every one-parameter group of the plane can be generated by one single infinitesimal transformation which can be represented, for properly chosen coördinates (x, y) and parameter t, by the infinitesimal operator

$$U = \frac{\partial}{\partial y};$$

this is equivalent to saying that all the transformations in the neighborhood of the identity can be expressed in the form:

$$x' = x, \qquad y' = y + t.$$

Our purpose is to determine all the different geometrical types of one-parameter groups of the plane. The problem is similar to that treated by Poincaré concerning families of curves defined by differential equations. Indeed, if we assume that the group is defined by functions

$$x' = f(x, y; t), \qquad y' = g(x, y; t)$$

which are analytic in the variables (x, y, t), the question of considering the path-curves of the group is essentially the same as that of considering the integral curves of the differential equation

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}$$

where

$$\xi(x, y) = \left(\frac{\partial f(x, y, t)}{\partial t}\right)_{t=0}, \qquad \eta(x, y) = \left(\frac{\partial g(x, y, t)}{\partial t}\right)_{t=0};$$

* Presented to the Hungarian Academy of Sciences in the meeting of June 8, 1925 (except § 9 and § 10).

$t = 0$ denotes that value of the parameter t to which corresponds the identity and we assume that the group is represented by functions such that $\xi(x, y)$ and $\eta(x, y)$ do not vanish both identically and none of them is infinite.

However, we do not assume the analyticity of the functions defining the transformations of the group but only their continuity; and so we shall first investigate the structure of the family of path-curves of a continuous one-parameter group of the plane and then construct for a given family of curves a continuous one-parameter group of continuous one-to-one transformations of the plane into itself.

2. We understand by a *one-parameter group* a group formed by transformations S_t whose set is a continuous one-to-one image of the real parameter values t $(-\infty < t < +\infty)$.

Given a continuous one-parameter group of topological transformations of the plane into itself, we consider a point P and the set of all the points into which P is carried by the different transformations of the group. These points form together a continuous curve (i. e. continuous uniform image of the infinite straight line) which is called the *path-curve* passing through the point P.

If P is not invariant for any transformation of the group except the identity $(t = 0)$ for two different values t and t' of the parameter, the point P is transformed into two different images; indeed, if the transformations S_t and $S_{t'}$ carried the point P into the same point Q, the transformation $S_t S_{t'}^{-1}$ would carry the point P into itself contrary to our assumption. Hence the path-curve passing through a point which is not invariant for any transformation $S_t (t \neq 0)$ is a *simple continuous curve* (continuous one-to-one image of an infinite straight line). On this line transformations other than the identity have no invariant points; consequently *the group on such a path-curve is an ordinary translation group.**

If a point P is invariant for all the transformations S_t corresponding to a certain interval $t_0 \leqq t \leqq t_1$, then P is invariant for all the transformations of the group. Indeed, if $\tau\,(>0)$ is sufficiently small, the transformation $S_{t_1+\tau}$ is the same as the transformation $S_{t_1} S_{\tau'}$ where $\tau'\,(>0)$ is arbitrarily small if τ is sufficiently small. Hence the effect of the transformation $S_{t_1+\tau}$ is the same with regard to the point P as that of the transformation $S_{t_0} S_{\tau'} = S_{t_0+\tau''}$ where again $\tau''\,(>0)$ is arbitrarily small. If we remark that the parameter values t for which P is invariant form a closed set, it follows that P is invariant for all the values of t. Such points are called *fixed invariant points* or invariant points of the group. For a fixed invariant point the path curve reduces to one single point.

* Brouwer, Math. Annalen 67 (1909), p. 256.

If P is not a fixed invariant point but only invariant for certain transformations of the group, then the parameter values t for which P is carried into itself form a closed set. Consider one of the complementary intervals $t_0 \leqq t \leqq t_1$ of this set; as the effect of the transformations

$$S_{t_1+\tau} \quad (\tau > 0, \text{ sufficiently small})$$

with respect to the point P is the same as that of the transformations

$$S_{t_0+\tau'} \quad (\tau' > 0),$$

it follows that the closed continuous curve corresponding to the set of images of P for the transformations $S_t\,(t_0 \leqq t \leqq t_1)$ is transformed into itself by all the transformations of the group. From this it follows that this must be a *simple* closed curve; otherwise there would be two values t_0' and t_1' of the parameter t such that $t_0 < t_0' < t_1' < t_1$ and

$$S_{t_0'}(P) = S_{t_1'}(P);$$

the closed continuous curve $S_t(P)$ $(t_0' \leqq t \leqq t_1')$ would then be transformed into itself by all the transformations of the group; but as P does not belong to this curve this is a contradiction. Hence the path-curve passing through a point which is invariant for certain transformations of the group without being invariant for all the transformations of the group, is a simple closed curve on which the group is an *ordinary rotation group*.*

As every path-curve is invariant under the group, two different path-curves cannot have any points of intersection. Hence the family of path-curves has its only singularities at the invariant points of the group.

We call a path-curve also a *characteristic* and we understand by a *semi-characteristic* of an open path-curve any one of the two parts into which it is divided by one of its points. A semicharacteristic can either converge to infinity, or to an invariant point of the group, or to a certain point set (§ 6).

3. So far we have considered the different path-curves making use chiefly of the continuity of the group. But if we consider that the image $S_t(P)$ of a point P for the transformation S_t depends continuously on the simultaneous variables (P, t) we can easily see that for the family of path-curves the following condition is satisfied:

Around every non-singular point P there is a neighborhood V_P such that, for any sequence of arcs of the family $\widehat{P_1Q_1}$, $\widehat{P_2Q_2}$, $\cdots$ which lie in this neighborhood and whose extremities converge to one and the same

* Brouwer, l. c., p. 257.

point Q of V_P, the arcs $\widehat{P_iQ_i}$ also converge to the single point Q. (That is, if R_i is an arbitrary point on $\widehat{P_iQ_i}$, the points $R_1, R_2, \cdots$ converge to Q.)

Let V_P be a neighborhood of P which does not contain any singular points nor a closed path-curve of the family, and let $\widehat{P_iQ_i}$ $(i = 1, 2, \cdots)$ be a sequence of arcs lying in V_P whose extremities converge to a point Q of V_P. If we denote by t_i the smallest absolute value of the parameter t for which

$$S_t(P_i) = Q_i$$

the values $t_1, t_2, \cdots$ converge to zero. Indeed, since the transformation $S_t(P)$ is continuous in (P, t) and consequently uniformly continuous for any finite region of (P, t), there is a certain positive number ε such that for any two points Q and Q' of V_P the smallest absolute value of the parameter t for which $S_t(Q) = Q'$ is less than ε. Thus, $R_1, R_2, \cdots$ being any sequence of points chosen on the arcs $\widehat{P_1Q_1}, \widehat{P_2Q_2}, \cdots$, the smallest values t_i' for which

$$S_{t_i'}(P_i) = R_i$$

converge to zero and, since the points $P_1, P_2, \cdots$ converge to Q, consequently the points $R_1, R_2, \cdots$ also converge to Q. So we have shown that the above condition is satisfied for the considered family of path-curves in the neighborhood of each point different from the fixed invariant points.

Hence it follows* that the family is equivalent to a family of parallel lines in the neighborhood of every non-singular point. This means that we can map the neighborhood of a non-singular point on a region of the (x, y)-plane by a topological transformation so that the arcs of the family lying in this neighborhood are transformed into parallel lines $x = \text{const.}$

We understand by a *cross-arc* a simple arc a neighborhood of which can be mapped on a plane region so that the arcs of path curves lying in this neighborhood are carried into parallel lines and the cross-arc into a segment orthogonal to these parallel lines.

4. Consider now an arbitrary family of curves in the plane which we assume has a finite number of singular points and is regular in the sense given in § 3 in the neighborhood of every non-singular point.

In order to define the *index of a singular point* in the way due to Hamburger† we surround the singular point by a small polygon which is composed of a finite number of arcs of the family and of cross-arcs and which does not contain in its interior any other singular points. A vertex of this polygon is called *convex* or *concave* according as the continuation

* Kerékjártó, Vorlesungen über Topologie, I, Berlin, 1923, p. 249.

† Sitzungsberichte der Preuß. Akademie der Wiss., Berlin, 21 (1922), p. 258.

of the arc of the family beyond this vertex goes into the exterior or into the interior of the polygon. The index of the singular point is then

$$\varrho = 1 - \frac{c - c'}{4},$$

where c and c' denote the number of convex respectively concave vertices of the polygon.

Another way to define the index of the singular points, due to Poincaré, is as follows. Assume first that the curves have continuously turning tangents and consider the variation of the direction of the tangent while a variable point describes a small circuit arround the singular point; this variation will then be $2\varrho\pi$, where ϱ is the index of the singular point. If we do not assume the existence of a continuously turning tangent, we consider two continuously variable points P and P' which lie at every moment on the same arc of the family sufficiently near to each other but not coinciding. While the point P describes a small circuit around the singular point, the direction of the vector $\overrightarrow{PP'}$ will change by $2\varrho\pi$, ϱ being the index of the singular point. The equivalence of these two definitions is obvious.

For the indices of the singular points the following relation holds:

$$\sum \varrho_i = 2,$$

where the sum is taken over all the singular points. This relation, also, is due to Poincaré.

By definition the index of a singular point is either an integer or the half of an integer.

If the family in question is that of the path-curves of a continuous one-parameter group, we have on every curve a positive sense of orientation corresponding to increasing values of the parameter; this orientation changes continuously with the curves. It follows that for this case the variation of the direction of the vector $\overrightarrow{PP'}$ (where P' is the image of P for a fixed sufficiently small value of the parameter) while P describes a small circuit around the singular point, must be a multiple of 2π; hence *for the family of path-curves of a continuous one-parameter group every singular point has an integer as index.*

5. For the singular points the same propositions are true which Poincaré and Bendixson have found for the family of integral curves of a linear differential equation of the first order and which the writer has developped

for the general topological case in "Vorlesungen über Topologie, I." I repeat here these results in a few words.

A singular point is called a *center* if there is no semicharacteristic whose points converge to this singular point. In any neighborhood of a center there is a simple closed curve of the family.

A singular point to which one or more semi-characteristics converge is called a *multiple point.* A neighborhood of a multiple point which is bounded by a polygon Π formed by path- and cross-arcs falls into a finite number of regions of the following three types:

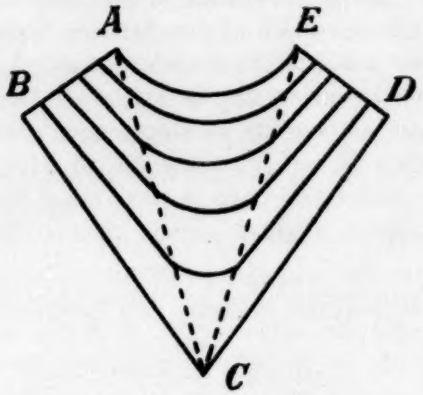

Fig. 1.

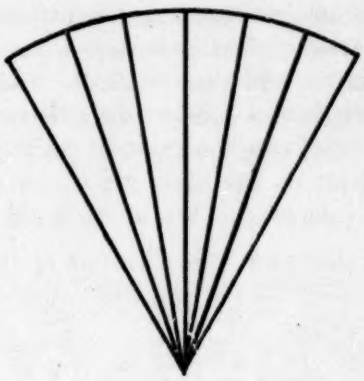

Fig. 2.

Fig. 3.

1) *col*; bounded by two semi-characteristics converging to the singular point, by two cross-arcs and by an arc of a path-curve on the polygon Π; every path-arc in this region goes from a point of one of the bounding cross-arcs to a point of the other cross-arc. (Fig. 1.)

2) *open nodal region*; bounded by two semi-characteristics and by a cross-arc; every path-arc in this region is a semi-characteristic which converges to the singular point. (Fig. 2.)

3) *closed nodal region*; its boundary consists of the two semi-characteristics of one and the same path-curve which converge to the same point (together with this point the curve forms a simple closed curve). Every curve of the family in this region is of the same type, that is to say, both of its semi-characteristics converge to the same singular point. (Fig. 3.)

Regarding the singular points of families of path-curves of a continuous one-parameter group there are no other restrictions than that of § 4 which, using the last observations, is equivalent saying that at every multiple point *the total number of cols and closed nodal regions must be even.*

6. Concerning the *ends* of the semi-characteristics we know from the results of Poincaré and Bendixson that for families of integral curves an end consisting of more than one point is a simple closed curve of the family around which the characteristic turns like a spiral. We know also that, for the analytic case, in the neighborhood of a closed integral curve either all the other curves are closed or all of them are open.*

For the topological case there might be in any neighborhood of a closed curve of the family a closed and also an open curve of the family. If between two closed curves there are no singular points and no other closed curves, the family between these two curves is one of the following two types:

$$\left.\begin{aligned} \varphi &= t + tg\,(r-2)\frac{\pi}{2} \\ \varphi &= t + tg\,|r-2|\frac{\pi}{2} \end{aligned}\right\} 1 \leqq r \leqq 3,\ -\infty < t < +\infty,$$

where r, φ are polar coördinates in the ring-shaped surface bounded by the circles $r = 1$ and $r = 3$†.

If closed curves of the family converge to a closed curve, there are, in general, in any neighborhood of the latter ring-shaped regions consisting only of closed curves and regions of the first type, while the second type cannot appear in every neighborhood of the limit curve.

For the case of families of path-curves of a continuous group we may make the following additional remark with regard to the periods. We understand by the period of a closed path-curve the smallest positive value of the parameter t for which every point on this curve is transformed into itself. From the continuity of the transformations $S_t(P)$ with regard to (P, t) it follows that *the periods corresponding to a sequence of closed path-curves converge to the period of the limit curve of the sequence.*

7. *Given a regular family of curves in the plane with a finite number of singular points of integral index, we want to construct a continuous one-parameter group of topological transformations of the plane into itself whose path-curves are the curves of the given family and whose invariant points are the singular points of the family.*

Our first step is to determine a positive sense of orientation on every

* See, for instance, Bieberbach, Differenzialgleichungen, Berlin, 1923, p. 59–60.

† Kerékjártó, Vorlesungen über Topologie, I, p. 255.

curve corresponding to the increasing values of the parameter which will be defined. This orientation should change continuously with the curves of the family in the following sense: if we take two cross-arcs in the neighborhood of a non-singular point which do not intersect each other and which intersect the very same arcs of the family in this neighborhood, then the point of intersection of an arc of the family with the first cross-arc should lie in the positive direction from the point of intersection of the same arc with the second cross-arc, for all the arcs of the family in consideration, where a positive or negative direction means that direction corresponding to the given positive or negative sense on the path-curve.

For this purpose we order a positive direction to a certain curve c of the family; then we take two points P and P' on c, P' lying in the positive direction from the point P. If P describes any continuous curve, we let P' describe a continuous curve so that at every moment P and P' lie on the same curve of the family but never coincide. While P describes a simple closed curve, the point P' returns to c, into a point P'' which lies from P in the same direction as P'. Indeed this last proposition is obvious if we restrict the variation of P and P' to a sufficiently small neighborhood of a non-singular point; furthermore, we have assumed that the singular points have integral indices so that this holds also for the neighborhoods of the singular points; from this and from the simply-connectedness of the plane it follows that the same is true without any restriction upon the variation of P and P'. Consequently the continuation of the positive direction on the curve c leads to a positive sense on each curve which changes continuously with these curves.

Our second step is to determine a *scale of addition* for every curve of the family which changes continuously with the curves. By the scale of addition we shall mean that there will be ordered to every couple of points P, Q, lying on one and the same curve of the family, a real value $(P, Q) = -(Q, P)$ which is zero if and only if $P = Q$ so that for any three points P, Q, R on the same curve of the family we have $(P, Q)+(Q, R)+(R, P) = 0$.

For this purpose it is convenient to represent the plane by a stereographic projection on a sphere and to consider the point at infinity as an ordinary point of the plane.

We exclude first the singular points by constructing around each of them a small polygon consisting of arcs of the family and of cross-arcs. Each of these polygons may be assumed to be a closed curve of the family in the neighborhood of a center and to have the normal forms given in § 5 at a multiple point in every region of the three different types discussed in § 5. The region R outside of these polygons can be devided into a finite

number of "squares" that is to say into regions each of which is bounded by two arcs of the family and two cross-arcs and in which the arcs of the family are situated like lines parallel to an edge in a square. In each of these squares we take a diagonal, that is to say, a cross-arc which joins two opposite corners of the square. In this way the whole region R is divided into a finite number of "triangles"; each triangle is bounded by two cross-arcs and by one arc of the family which we shall call the base of the triangle.

We take an arc of the family which is an edge of the triangulation on R; its extremities should be corners of the triangulation but besides these it should not contain any other corners of the triangulation. We determine on this arc a system of scale-values $0 \leqq t \leqq 1$ by a continuous one-to-one correspondence such that the corner corresponding to the value 1 should lie (according to the positive sense on this path-curve) in the positive direction from the corner corresponding to the value 0. In the same way we determine the scale on every arc of the family belonging to the edges of the triangulation.

We take a triangle Δ and map it by a continuous one-to-one transformation into a regular triangle whose edges have the length n, where $n+1$ denotes the number of the corners of the division lying on the base of Δ; we determine this transformation of Δ into the representing regular triangle in such a way that the corners on the base of Δ should be transformed into equidistant points on the base of the representing triangle and that two arbitrary points on this last one should have a distance equal to the scale-value of the corresponding points; finally the arcs of the family lying in Δ should be transformed into parallel lines. Then we understand by scale-value of any two points of Δ lying on the same arc of the family the distance of the corresponding points in the representing triangle supplied with the proper sign.

In order to extend the scale for the whole plane, we consider first a *center*. Around this center we take a sequence of closed curves of the family $c_1, c_2, \cdots$ converging to this point and we determine successively the scale for the region lying between c_i and c_{i+1} which should be the same on c_i as the scale already determined on this curve.

In a similar way we take for an *open nodal region* a sequence of cross-arcs which converge to the singular point; we order to two consecutive ones the scale-value 1 and extend it for the region between the two cross-arcs.

For a *col* we take two cross-arcs AC and EC (fig. 1) joining the two corners of the polygon Π with the singular point C. We assign to the points of the cross-arc AB the values of t from 0 to $+\infty$; let P be a point of AB, t the assigned value, Q the point of intersection of the arc passing

through P with the cross-arc AC, then we order to P, Q the scale-value $(P, Q) = t$. Mapping the triangle ABC into a regular triangle as above, we extend the scale for the whole triangle. (In the same way for CDE). For the triangle ACE we extend the scale already determined for the arc AE by the above method.

Finally for a *closed nodal region* we preceed as follows. We replace the neighborhood in question by a smaller one. By such a reduction of the neighborhood the closed nodal region will be replaced by a finite number of closed and open nodal regions and cols. For that part of the first neighborhood which does not belong to the second one, and for the cols and open nodal regions of the second neighborhood, we determine the scale by the above method; hence the only part for which the scale is not yet determined will consist of a finite number of closed nodal regions whose diameters are smaller than that of the first nodal region. Consequently, indefinite repetition of this process (replacing neighborhoods by smaller ones which finally converge to the singular point) leads to a determination of the scale for the whole region.

In this way we determine the scale of addition in every triangle in such a way that the determination will be unique on the common part of the basis of neighboring triangles. In order to determine the scale for the whole plane we define the scale-value for any two points P and Q lying on the same curve of the family as the sum:

$$(P, Q) = (P, P_1) + (P_1, P_1) + \cdots + (P_n, Q),$$

where $P_1, P_2, \cdots, P_n$ are the consecutive points of intersection of the arc PQ with the edges of the triangulation (whose number is necessarily finite).

The scale defined by the above considerations satisfies all the imposed conditions. By the following prescription we order to every value t of the parameter a topological transformation of the plane into itself which carries every curve of the given family into itself: take an arbitrary point P, denote by Q that point lying on the same path-curve for which the scale-value $(P, Q) = t$. The inverse of this transformation corresponds to the parameter value $-t$. The product of the transformations corresponding to the parameter values t and t' has the parameter value $t + t'$. Finally it follows that the transformations depend continuously on the parameter t.

8. Regarding the case of groups with infinitely many fixed invariant points, the above methods can be extended if the set of fixed invariant points is a discontinuous point set. If the set of fixed invariant points contains continua, the same reductions may be used as those that the writer has

developped for the analogous problem concerning the parametric representations of continuous surfaces.*

9. In view of an application of the above results we want to determine the conditions under which a continuous one-parameter group of the plane is equivalent with the group of *translations of one and the same direction*:

$$y' = y, \qquad x' = x + t.$$

The path-curves of a group of this type form a family with the only singular point at infinity. However this property of the family of path-curves is not sufficient for their characterisation, as is shown by the example of the following group, which is not a translation group:

$$\text{for } y \leqq -1\colon \quad x' = x - t, \quad y' = y,$$

$$\text{for } y \geqq +1\colon \quad x' = x + t, \quad y' = y,$$

$$\text{for } -1 < y < +1 \begin{cases} x' = x + \left| t + \operatorname{tg}\frac{\pi y}{2} \right| - \left| \operatorname{tg}\frac{\pi y}{2} \right|, \\ y' = \frac{2}{\pi} \operatorname{arctg}\left(t + \operatorname{tg}\frac{\pi y}{2} \right), \end{cases} \quad \begin{matrix} (t = \text{parameter} \\ \text{of the group}). \end{matrix}$$

We shall understand by the *spherical distance* of two points in the plane the spherical distance of the points on the tangent sphere of radius unity corresponding to them by a stereographic projection. Thus we see that the family of path-curves of a one-parameter translation group has the following property: if the points $P_1, P_2, \cdots$ converge to the point P, the curves of the family $c_1, c_2, \cdots$ passing through $P_1, P_2, \cdots$ converge *uniformly* to the path-curve c passing through P. We understand by uniform convergence of the curves $c_1, c_2, \cdots$ to the curve c that the spherical parameter-distances (or écart's)† of the curves c_n and c converge to zero if $n \to \infty$.

On the other hand we want to show that *a one-parameter continuous group without fixed invariant points* (other than infinity) *whose path-curves satisfy the last condition is equivalent with the group of translations*:

$$y' = y, \qquad x' = x + t.$$

First we want to prove that the family of path-curves is equivalent with the family of parallel lines $y = \text{const}$. It is clear that the family

* Proc. Nat. Acad. Sc., Washington, *10* (1924), p. 268–271.

† Fréchet, Rend. d. Circ. Mat. d. Palermo, 22 (1906), p. 53.

cannot have any closed path-curves; otherwise the interior of such a curve would contain a singular point of the family* which would be a fixed invariant point, against our assumption. Hence it follows that every path-curve converges to infinity in both directions.†

For our next purpose it is convenient to represent the plane on a sphere by a stereographic projection. The family of path-curves will then appear as a family of simple closed curves each of them passing through the pole O of the sphere. We show next: if c_1, c_2, c_3 are three arbitrary curves of the family there is one of them, say c_2, which separates the other two from each other in the sense that every arc joining a point of c_1 to a point of c_3, without passing through the point O, intersects the curve c_2.—Denote by (c_1, c_3) the region on the sphere bounded by the curves c_1 and c_3. Take a point P of c_3 different from O; if P' is a point of (c_1, c_3) sufficiently near to P, the path-curve c' passing through P' has a very small parameter-distance from the curve c_3; then it is easy to see that c' separates c_1 and c_3 from one another. In the very same way we can see that if a curve c_2 separates c_1 from c_3 all the curves of the family in a sufficiently small neighborhood of c_2 have the same property. On the other hand, if $c^1, c^2, \cdots$ are curves of the family which separate c_1 and c_3 from each other, any limiting curve of this sequence different from c_1 and c_3 has the same property, according to the remark just made. Hence it follows that all the curves of the family lying in the region (c_1, c_3) separate the curves c_1 and c_3 from one another. In this way we have obtained a linear order for the curves of the family. After this the homoeomorphism of the family with the family of the circles which are the images of the parallel lines $y = \text{const.}$ by a stereographic projection, can be proved in the same way as in the analogous problem concerning families of concentric circles.‡

We map then the path-curves of the group into the family of parallel lines $y = \text{const.}$ by a topological transformation of the plane into itself. We order to a point P lying on a path-curve c as coördinate x that value of the group-parameter t for which the point of intersection of the line c with the y-axis is carried into the point P. In this way we obtain a system of coördinates in which the given group is expressed by the formulas:

$$y' = y, \qquad x' = x + t.$$

* See Kerékjártó, Vorlesungen über Topologie, I, p. 253.

† Hamburger, Sitzungsberichte der Preuß. Akademie der Wiss., Berlin, 21 (1922), p. 258–262.

‡ Kerékjártó, Vorlesungen über Topologie, I, p. 241–246. A simpler proof is given by the writer in the paper: Sur les familles de surfaces et de courbes, I (under press at the Acta Lit. ac Scient. of the University of Szeged).

By similar reasonings it can be proved that *every periodic continuous one-parameter group of topological transformations of the plane into itself is equivalent with the group of rotations around the origin.*

10. From the translation theorem of Brouwer* combined with the result of the preceding paragraph we obtain the following *characterisation of plane translations*:

A topological transformation of the plane into itself with invariant indicatrix is homoeomorphic with a plane translation if and only if it has no invariant point and its powers form a uniformly continuous set of transformations.

The uniform continuity of the powers T^n $(n = \pm 1, \pm 2, \cdots)$ of the given transformation T should mean that for any finite region of the plane and for any positive number ε there is a positive number δ such that two arbitrary points of the region whose distance is less than δ are carried by every transformation T^n into points P_n, Q_n whose spherical distance is less than ε.

By the translation theorem of Brouwer we can construct to any point of the plane a translation field.† Hence it follows by the uniform continuity of the transformations T^n that it is possible to determine for any positive number ε and for any path-curve‡ passing through a point P a positive number δ such that through an arbitrary point Q whose distance from P is less than δ a path-curve can be constructed whose spherical parameter-distance from the path-curve passing through P is less than ε. We thus construct for a number ε (>0) a finite sequence of non-intersecting path-curves such that two consecutive ones always have a parameter-distance less than ε; then we interpolate between any two consecutive curves of the first sequence a finite sequence of path-curves such that two consecutive ones always have a parameter-distance less than $\varepsilon/2$; and so forth. Finally we obtain a regular family of path-curves covering the whole plane for which moreover the condition of uniform convergence also is satisfied. There is a continuous one-parameter group whose path-curves are the curves of this family (§ 7); this group is equivalent with a translation group (§ 9). Furthermore we can construct such a group in a way that it gives for the value $t = 1$ of the parameter the transformation T given before. Consequently the transformation T is equivalent with the plane translation:

$$x' = x+1, \quad y' = y.$$

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* Brouwer, Math. Annalen 72 (1912), p. 37–54.

† Brouwer, l. c., p. 46 et seq.

‡ A *path-curve* is a simple open line invariant for the transformation T; see Brouwer l. c., p. 38 et seq.

PRINCETON UNIVERSITY,
June 1925.

GROUP VELOCITY AND THE PROPAGATION OF DISTURBANCES IN DISPERSIVE MEDIA.

By Fredrick Wood.

1. **Introduction.** The relation between the group velocity U and the wave velocity V of a disturbance propagated in a dispersive medium has been studied by Stokes, Rayleigh, Kelvin* and others. A paper by Green† and a book by Havelock‡ form the basis for much of the discussion in this paper. After discussing the method of Kelvin these writers have given certain special examples in which the group formula leads to very satisfactory results. However, it is apparent from the very fact that confirmation of the results is sought in special cases, that there should be further study of the mode of derivation of the group formula and of the limits within which it may be expected to apply in the case of a given initial disturbance. In this paper examples are given in which only a very limited part of the disturbance is given by the group method. Furthermore, some of the integrals§ to which the group method has been applied in the past are divergent. These points have been discussed and illustrated by special examples of both finite and infinite displacements at the origin and the discussion of these examples throws further light upon the group method.

2. **Kelvin's group formula for an infinite displacement at the origin.** In 1887 Kelvin derived a formula‖ for the approximate displacement resulting from an initial infinite displacement at the origin. This formula is the one used by Havelock, Green and others. With Havelock consider a displacement y, a function of x and t, and let $y = f(x)$, $\frac{\partial y}{\partial t} = 0$, when $t = 0$. The disturbance at any position and time is given by

$$(1) \quad y = \frac{1}{2\pi}\int_0^\infty \varphi(k)\cos k(x - Vt)\,dk + \frac{1}{2\pi}\int_0^\infty \varphi(k)\cos k(x + Vt)\,dk,$$

* Lord Kelvin, On the waves produced by a single impulse in water of any depth, or in a dispersive medium, Proc. Roy. Soc., 42, 1887, p. 80.

† G. Green, On group velocity and the propagation of disturbances in a dispersive medium, Proc. Roy. Soc. Edin., 29, 1909, p. 445.

‡ T. H. Havelock, The propagation of disturbances in dispersive media, Cambridge Tract No. 17, 1914. Bibliography, pp. 85–87.

§ See e. g., Havelock, loc. cit., p. 32, equation 37.

‖ Kelvin, loc. cit., p. 80.

where

$$\varphi(k) = \int_{-\infty}^{\infty} f(\omega) \cos k\omega \, d\omega, \tag{2}$$

and V is a function of k.

In the special case* where $f(x)$ is zero everywhere except in the range $-\epsilon < x < \epsilon$, and equals $\frac{1}{2\epsilon}$ in that range, $\varphi(k)$ approaches 1 as ϵ approaches zero. The initial disturbance is then an infinitely intense displacement at the origin, and at any subsequent time

$$y = \frac{1}{2\pi}\int_0^{\infty} \cos k(x - Vt)\, dk + \frac{1}{2\pi}\int_0^{\infty} \cos k(x + Vt)\, dk, \tag{3}$$

provided that V is such a function of k that the integrals which occur are convergent.

The usual procedure is to assume that the principal part of the disturbance in the region where x is positive is given by the first integral. It is then assumed that positions and times can be found at which an exceptionally large number of elements of the first integral in (3) have the same phase; if this were the case these elements would reinforce each other and would presumably produce the predominant part of the total effect, all the other elements mutually interfering because of differences of phase. Mutual reinforcement of waves of nearly the same wave-length will occur when the phase $k(x - Vt)$ is stationary with respect to k, i. e., when

$$\frac{d}{dk}\{k(x - Vt)\} = 0,$$

or

$$x - Ut = 0, \tag{4}$$

where U is the group velocity.† Corresponding to this value of U there will be one or more particular values k_0 satisfying this relation. If k_0 is fixed, a definite relation between x and t is determined, or if particular values of x and t are considered, k_0 is determined.

The evaluation of the predominant part of the disturbance is accomplished by assuming that sufficient time has elapsed for it to be given by the predominant group consisting of an infinite number of waves for which k lies in the vicinity of the value k_0 given by equation (4). Assuming that the phase is given with sufficient accuracy by three terms of a Taylor series, and by making the substitution

* Havelock, loc. cit., p. 19.

† Havelock, loc. cit., p. 20.

$$\sigma^2 = \frac{1}{2} t \left| \left(\frac{dU}{dk} \right)_0 \right| (k - k_0)^2 \tag{5}$$

and taking t large, the limits for σ can be taken as $\pm \infty$ and the value* of the predominant group is given by

$$y = \frac{1}{2\pi} \left\{ \frac{2\pi}{t \left| \left(\frac{dU}{dk} \right)_0 \right|} \right\}^{\frac{1}{2}} \cos \left\{ k_0 (x - V_0 t) \mp \frac{\pi}{4} \right\}, \tag{6}$$

k_0 being given in terms of x and t by (4). Equation (6) is the result reached by Kelvin; the notation and details are due to Havelock, and a similar treatment is given by Green.

For an arbitrary initial disturbance symmetrical with respect to the origin, if the amplitude factor $\varphi(k)$ is sensibly constant, the group formula is

$$y = \frac{1}{2\pi} \left\{ \frac{2\pi}{t \left| \left(\frac{dU}{dk} \right)_0 \right|} \right\}^{\frac{1}{2}} \varphi(k_0) \cos \left\{ k_0 (x - V_0 t) \mp \frac{\pi}{4} \right\}, \tag{7}$$

This, according to Havelock, appears to limit the application of the formula to places and times associated with values of k_0 in the vicinity of values of k for which $\varphi(k)$ is a maximum.

3. **The integral** $\int_0^\infty \cos k(x + Vt)\, dk$. The group formula (6) was derived from a consideration of the first integral in (3) for large values of t. Since the initial disturbance has been resolved into an infinite number of components each one being propagated equally to the right and to the left with its appropriate velocity, the second integral in (3) will of course represent an appreciable part of the disturbance for smaller values of t. Since the group method assumes the contribution of this second integral to be negligible it is possible to place certain restrictions on the range of values within which the method is applicable by studying the effect of this integral. So far as I am aware no attempt has been made to connect the restriction that t shall be sufficiently large with the effect of the second integral in (3) nor to indicate definitely a method of determining the range of applicability of the group formula in a specific case.

4. **Infinite displacement at the origin.** $V = k$. Consider first the case of flexural vibrations of an elastic rod. Here†

$$V = k, \qquad U = 2k, \qquad k_0 = \frac{x}{2t}.$$

* Havelock, loc. cit., p. 22.

† Havelock, loc. cit., p. 30.

In the case of the infinite displacement at the origin of section 2, equation (3) gives the displacement at any subsequent time and the group solution (6) takes the form

$$y = \frac{1}{2}(\pi t)^{-\frac{1}{2}} \cos\left(\frac{x^2}{4t} - \frac{\pi}{4}\right). \tag{8}$$

Now the integrals in (3) can be evaluated by exact methods. After putting $V = k$, completing the squares, and changing the variable, we get finally equation (8).

Havelock concludes that, since we get equation (8) in each case and also since the Taylor series for the argument of the cosine is exact, that this is a case for which the group method gives a result exact for all times. It is true that the Taylor series is an exact expression for the argument of the cosine but to say that the group method is exact for all times is to contradict the whole method, it being expressly stated that the formula (6) was derived on the supposition that t was large. It was supposed that sufficient time had elapsed so that after the change of variable the limits of integration could be taken to be $-\infty$ and ∞. This is the sole justification for this choice of limits.

As a matter of fact, using the limits of integration $-\infty$ and ∞ amounts in this particular case to including the integral involving $x + Vt$ and therefore to the exact calculation of the sum of the two integrals in (3) and not to an approximate calculation of the first, as it might seem at first sight. We conclude in the case $V = k$, for an infinite displacement at the origin, that although the group method leaves out of consideration the second integral, it is included by integration from $-\infty$ to ∞, and that instead of being the real group solution, which for $k_0\sqrt{t}$ very small would give but one-half the result, it is really the exact solution by ordinary integration. Accordingly we cannot permit the agreement in this case of the result of the application of the group formula with the exact result, to increase too greatly our confidence in the group method. As the foregoing calculation has shown the agreement for all values of t is due to the unusual circumstances of this particular case in which $V = k$ and $\varphi(k)$ is constant, corresponding to an infinite displacement at the origin of the type assumed. If $\varphi(k)$ is not constant, and this is the usual case when the initial displacement is arbitrary, entire agreement of the group and exact solutions will not be found even for the case in which $V = k$.

The second integral can be expressed in terms of Fresnel integrals. In order that these integrals may be neglected, the lower limit must be moderately large, of the order of magnitude given by

$$\sqrt{\frac{2}{\pi}}\, k_0 \sqrt{t} > 8 \cdot 5,$$

which means approximately that $x > 22\sqrt{t}$, a restriction upon the range of applicability of the group method. A weaker restriction can be used if we allow for a moderate error, e. g., 10 percent.

In other examples similar restrictions on x and t will be shown, which enter as in this case through the method of derivation of the formula from the first integral alone.

5. **Relation between $\varphi(k)$ and $f(x)$.** In the case of an arbitrary initial displacement $\varphi(k)$ plays an important role in the group formula (See section 2), the usual statement being that the formula holds only for $\varphi(k)$ sensibly constant.* In the particular infinite displacement at the origin just considered $\varphi(k) = 1$ for all values of k. We will now consider both finite and infinite displacements at the origin, determine $\varphi(k)$, and seek to find, by means of special examples and by use of equation (2) the limits within which the group method may be expected to hold for a given initial disturbance.

Consider the case where

$$f(x) = \frac{\mu}{\sqrt{\pi}} e^{-\mu^2 x^2}, \qquad V = k, \qquad \varphi(k) = e^{-\frac{k^2}{4\mu^2}}.$$

The group solution is

$$y = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{16\mu^2 t^2}} \cos\left(\frac{x^2}{4t} - \frac{\pi}{4}\right). \tag{9}$$

To obtain the exact solution substitute in (1) and obtain

$$y = \frac{1}{\pi}\int_0^\infty e^{-\frac{k^2}{4\mu^2}} \cos kx \cos k^2 t \, dk,$$

which when evaluated becomes

$$y = \frac{1}{2\sqrt{2\pi}\sqrt{\frac{1}{16\mu^4}+t^2}}\, e^{-\frac{x^2}{16\mu^2\left(\frac{1}{16\mu^4}+t^2\right)}} \left\{ \sqrt{\frac{1}{4\mu^2}+\sqrt{\frac{1}{16\mu^4}+t^2}}\cos\frac{tx^2}{4\left(\frac{1}{16\mu^4}+t^2\right)} \right.$$
$$\left. + \sqrt{-\frac{1}{4\mu^2}+\sqrt{\frac{1}{16\mu^4}+t^2}}\sin\frac{tx^2}{4\left(\frac{1}{16\mu^4}+t^2\right)} \right\}. \tag{10}$$

* Havelock, loc. cit., p. 26.

If in equation (10) we assume that t is large compared with $1/4\mu^2$, we get equation (9). We then conclude that the group method holds for t large compared with $1/4\mu^2$. If μ is very small, t must be very large; $f(x)$ has a broad maximum at the origin, and $\varphi(k)$ has a sharp maximum. As μ increases to unity, t must be large compared with $\frac{1}{4}$, and as μ approaches infinity, the restriction on t is gradually removed. The maximum of $f(x)$ becomes sharper and sharper and approaches an infinite displacement at the origin, while $\varphi(k)$ broadens out, approaching a constant value over a wide range. The group and exact solutions then agree as in section 4 because the calculation by the group formula really amounts in this case to the exact calculation of the two integrals containing $x-Vt$ and $x+Vt$. This situation is approached more and more nearly the sharper the maximum of $f(x)$ at the origin.

This single example furnishes a good illustration of the way in which the group method works. The group formula holds in this case when $f(x)$ has a sharp maximum at the origin. This will be seen in other examples which will be presented. The failure of the group method for small values of μ is to be traced primarily to the rapid variation of $\varphi(k)$ in the vicinity of k_0 with the consequence that $\varphi(k)$ cannot be treated as a constant and removed from under the integral sign as in section 2.

It is possible to perform the integration and get exact results for

$$f(x) = \frac{\mu^3}{\sqrt{\pi}} x^2 e^{-\mu^2 x^2} \text{ and } f(x) = \frac{\mu}{\sqrt{\pi}} e^{-\mu^2 x^2} \cos \mu x$$

when $V = k$. It is found that the group and exact methods agree better and better as μ increases, i. e., as the maxima of $f(x)$ become sharper.

As another example indicating a similar relation between $\varphi(k)$ and $f(x)$ consider the initial displacement

$$f(x) = \frac{\sin nx}{x}$$

and let $V = k$. For subsequent values of t,

$$(11) \quad y = \frac{1}{2\pi}\int_0^\infty \varphi(k) \cos(kx - k^2 t)\, dk + \frac{1}{2\pi}\int_0^\infty \varphi(k) \cos(kx + k^2 t)\, dk,$$

where

$$(12) \quad \varphi(k) = \pi, \quad k < n; \qquad \varphi(k) = \frac{\pi}{2}, \quad k = n; \qquad \varphi(k) = 0, \quad k > n.$$

The group solution is

$$(13) \quad y = \frac{\sqrt{\pi}}{2\sqrt{t}} \cos\left(\frac{x^2}{4t} - \frac{\pi}{4}\right), \quad k_0 < n; \qquad y = 0, \quad k_0 > n.$$

Equation (11) becomes, making use of (12) and changing variables

$$(14)\quad y = \frac{1}{2\sqrt{t}} \int_{-k_0\sqrt{t}}^{\sqrt{t}\left(n-\frac{x}{2t}\right)} \cos\left(\sigma^2 - \frac{x^2}{4t}\right) d\sigma + \frac{1}{2\sqrt{t}} \int_{k_0\sqrt{t}}^{\sqrt{t}\left(n+\frac{x}{2t}\right)} \cos\left(\sigma^2 - \frac{x^2}{4t}\right) d\sigma.$$

The group method leads to the first integral with limits $-\varepsilon\sqrt{t}$, $\varepsilon\sqrt{t}$, if $0 < (k_0 - \varepsilon) < (k_0 + \varepsilon) < n$; with limits $-k_0\sqrt{t}$, $\varepsilon\sqrt{t}$ if k_0 is very small; and with limits $-\varepsilon\sqrt{t}$, $(n - k_0)\sqrt{t}$ if k_0 is very nearly equal to n.

The last two cases indicate the restrictions

$$\sqrt{\frac{2}{\pi}}\, k_0\sqrt{t} = \frac{x}{2\sqrt{t}}\sqrt{\frac{2}{\pi}} > 8.5,$$

$$\sqrt{\frac{2}{\pi}}\,(n - k_0)\sqrt{t} = \left(n - \frac{x}{2t}\right)\sqrt{\frac{2t}{\pi}} > 8.5,$$

if to a high degree of approximation the limits of the integral corresponding to the one from which (6) is obtained are to be taken to be $-\infty$ and ∞.

For particular values of n, x, and t, the values of y can be calculated by using Fresnel integrals, and the group and exact solutions can then be compared graphically.

This is an example of a constant $\varphi(k)$ in the range $0 < k < n$ which should insure the holding of the group formula. However, it will be subject to the following restrictions:

$$x > 22\sqrt{t}; \qquad x < 2tn, \text{ since } \frac{x}{2t} = k_0 < n; \text{ and } \sqrt{\frac{2t}{\pi}}\left(n - \frac{x}{2t}\right) > 8.5,$$

in which case, the exact solution reduces to the group solution (13). For exact agreement the conditions then are, the second being included by the third,

$$x > 22\sqrt{t}, \qquad x < 2tn, \qquad x < (2tn - 22\sqrt{t}).$$

The following table shows the limits for x corresponding to given values of n and t, $a < x < b$.

From this table it follows that the group formula holds only for large values of t if n is small. If n is increased the group formula holds for smaller values of x and t. If attention is fixed on any particular value of t, the range on x is increased as n increases. For a large n $f(x)$ has a sharp

maximum at the origin as in the previous examples. As $f(x) = \frac{\sin nx}{x}$ tends toward an infinite displacement at the origin, $\varphi(k)$ is a constant, different from zero, over a wider and wider range.

n	t	a	b	n	t	a	b
1	484	484	484	10	16	88	232
1	529	506	552	10	25	110	390
1	576	528	624	20	2	31	49
1	625	550	700	20	3	38	84
10	5	49	51	20	4	44	116
10	9	66	114	20	5	49	151

It is clear that because of the restriction $x > 22\sqrt{t}$ which results because the group method neglects the integral $\int_0^\infty \cos k(x+Vt)\,dk$, and also from the fact that $\varphi(k) = 0$, $k_0 > n$, we have here an example of a rather limited range of applicability of the group method. In fact, the table shows that for $n = 1$ the method has little value. That the exact solution gives a displacement when $\varphi(k) = 0$ will be shown later.

In case of the same initial displacement $y = \frac{\sin nx}{x}$ and $V = \sqrt{\frac{g}{k}}$, the group solution can be written down at once and the exact solution treated as in the preceding examples. The conditions resulting are similar to the ones given above. As no new points arise in the problem, the conditions on x will be merely stated here. $x > \frac{3t}{\sqrt{n}}$, $x > \left[\frac{2}{.35n} + \left(\frac{t}{.35n} + \frac{4}{.12n^2}\right)^{\frac{1}{2}}\right]^2$, $x < \frac{t^2}{16}$, $\sqrt{x}$ must be large.

The following are further examples of displacements which have sharp maxima at the origin and which can be made to become infinite displacements at the origin resulting in a constant $\varphi(k)$ and the consequent holding of the group formula when $V = k$.

(a) $$f(x) = \frac{an}{\pi[a^2 + (xn)^2]} = \frac{a}{\pi n\left[\left(\frac{a}{n}\right)^2 + x^2\right]},$$

$$\varphi(k) = \int_{-\infty}^{\infty} f(\omega)\cos k\omega\, d\omega = e^{-\frac{ka}{n}}, \quad k \geqq 0.$$

(b) $$f(x) = ne^{nx}, \quad -\infty < x < 0, \quad f(x) = ne^{-nx}, \quad 0 < x < \infty,$$
$$\varphi(k) = \frac{2}{1+\left(\frac{k}{n}\right)^2}.$$

(c) $$f(x) = \frac{b}{c}\cos x, \quad -c < x < c; \quad f(x) = 0 \text{ elsewhere},$$
$$\varphi(k) = \frac{b}{c}\left[\frac{\sin c(k+1)}{k+1} + \frac{\sin c(k-1)}{k-1}\right].$$

In each of the first two cases as n becomes infinite, and in the third as c approaches zero, $f(x)$ represents an infinite displacement at the origin, and $\varphi(k)$ is constant. Such examples are plentiful and others will be given later.

In fact, if we take an even function $f(x)$, which is different from zero in the interval $-\varepsilon < x < \varepsilon$, and essentially zero elsewhere, and which has a maximum at the origin, we get for small values of ε and k,

$$\varphi(k) = 2\int_0^{\varepsilon} f(\omega)\cos k\omega \, d\omega = 2\int_0^{\varepsilon} f(\omega)\, d\omega = \text{a constant},$$

since $\cos k\omega$ is practically equal to unity if ε is sufficiently small and k has moderately large values which can be taken larger as ε is taken smaller. This then establishes the fundamental relation between $f(x)$ and $\varphi(k)$ which the examples indicated, i. e., a sharp maximum at the origin of $f(x)$ gives a broad slowly-changing $\varphi(k)$ for which the group method will hold.

6. **Discussion of $\varphi(k) = 0$.** In the example in which $f(x) = \frac{A\sin nx}{x}$ the solution given by the group method was steadily zero for $k_0 > n$ since $\varphi(k)$ was zero. That the exact solution gives an appreciable displacement for $k_0 > n$ can be shown by changing the integrals in (14) to Fresnel integrals and substituting particular values of n, x, and t. For example,

y	n	t	x
$-.02\ A$	1	121	250
$-.014A$	1	144	400
$-.01\ A$	1	196	400

Calculation by the group formula shows that, within the range within which it can be applied, the maximum value of y is approximately $0.08A$ when $t = 121$. The values of y for the region in which $\varphi(k) = 0$ are

of this order of magnitude while the group method would give the displacement zero.

In other examples $\varphi(k)$ is zero at isolated points. Thus if $f(x) = \frac{A}{c}$, $-c<x<c$; $f(x)=0$ elsewhere, $\varphi(k)=2A\frac{\sin kc}{kc}$; if $f(x)=\frac{bx}{c^2}+\frac{b}{c}$, $-c<x<0$; $f(x)=-\frac{bx}{c^2}+\frac{b}{c}$, $0<x<c$; $f(x)=0$ elsewhere, $\varphi(k)=\frac{2b}{c^2k^2}(1-\cos kc)$. In the first example with $V=k$ it is found that the exact solution gives a displacement that is negligible for values of k which make $\varphi(k)=0$. In the second example the displacement is not negligible. In both cases as the initial displacement approximates an infinite displacement at the origin better agreement of the exact and group solutions is found.

Another example of the same type is found if we let

$$f(x) = n\cos n^2x^2$$

where $V=k$.

The examples in this section show the possibility that the group method will fail for $\varphi(k)=0$, but that the error becomes of less importance as the maximum of $f(x)$ at the origin becomes sharper and sharper and disappears in the case of an infinite displacement at the origin.

7. **Divergent integrals.** The integrals in equation (1), which represent the displacement at any time and place, may not be convergent in certain cases. In fact, we get an interesting example if we take the infinite displacement at the origin discussed in section 2 and take $V=\sqrt{\frac{g}{k}}$. Then the integral given by Havelock* is obtained, viz.,

$$(15)\quad y=\frac{1}{2\pi}\int_0^\infty \cos k\left(x-\sqrt{\frac{g}{k}}\,t\right)dk+\frac{1}{2\pi}\int_0^\infty \cos k\left(x+\sqrt{\frac{g}{k}}\,t\right)dk.$$

These integrals are not convergent and it is doubtful if results obtained by discussing by the group method have any value. In particular the integral given later by Havelock† is not equivalent to (15).

Let

$$f(x)=\frac{1}{\sqrt{x}},\quad x>0;\qquad f(x)=\frac{1}{\sqrt{-x}},\quad x<0,$$

* Havelock, loc. cit., p. 32.
† Havelock, loc. cit., p. 33.

where $V = \sqrt{\frac{g}{k}}$. We get $\varphi(k) = 2\sqrt{\frac{\pi}{2k}}$ and the group solution is

$$y = \frac{2}{\sqrt{2x}} \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right). \tag{16}$$

The exact solution is given by

$$y = \frac{2}{\sqrt{2\pi}} \int_0^\infty \cos \sqrt{kg}\, t \cos kx \frac{dk}{\sqrt{k}}.$$

This gives, after replacing k by k^2 and integrating, a result identical with (16). The presence of the factor $\frac{1}{\sqrt{k}}$ in the integrand insures the convergence of the integral. In this case there is complete agreement between the group formula and the exact result.

ON AN EXPRESSION OF A MINOR OF ORDER TWO OF THE *M*th COMPOUND OF A DETERMINANT *A* IN TERMS OF MINORS OF *A* OF ORDER HIGHER THAN *M*.

By W. H. Metzler.

In 1893 the writer* gave an expression for the most general minor of the second order of the second compound of a determinant of the fourth order in terms involving minors of order greater than two which are either coaxial or pairs of conjugates. Thus using inclusive notation

$$(A_1)\quad \left(\begin{vmatrix}12\\12\end{vmatrix}\begin{vmatrix}34\\34\end{vmatrix}\right) = 1\begin{vmatrix}234\\234\end{vmatrix} + 2\begin{vmatrix}134\\134\end{vmatrix} + 1\begin{vmatrix}234\\134\end{vmatrix} + 2\begin{vmatrix}134\\234\end{vmatrix} - \begin{vmatrix}1234\\1234\end{vmatrix}.$$

Where $1\begin{vmatrix}234\\234\end{vmatrix}$ stands for $a_{11}A^{234}_{234}$ and $\begin{vmatrix}1234\\1234\end{vmatrix}$ is the determinant of order 4.

The next case where we take a minor of order two of the third compound of a determinant of order six or greater is

$$\begin{aligned}(A_2)\quad &\left(\begin{vmatrix}123\\123\end{vmatrix}\begin{vmatrix}456\\456\end{vmatrix}\right)\\
&\equiv \begin{vmatrix}123\\123\end{vmatrix}\cdot\begin{vmatrix}456\\456\end{vmatrix} - \begin{vmatrix}123\\456\end{vmatrix}\cdot\begin{vmatrix}456\\123\end{vmatrix}\\
&= \Big[\begin{vmatrix}1234\\1234\end{vmatrix}\cdot\begin{vmatrix}56\\56\end{vmatrix} + \begin{vmatrix}1235\\1235\end{vmatrix}\cdot\begin{vmatrix}46\\46\end{vmatrix} + \begin{vmatrix}1236\\1236\end{vmatrix}\cdot\begin{vmatrix}45\\45\end{vmatrix}\\
&\quad - \begin{vmatrix}1236\\1235\end{vmatrix}\cdot\begin{vmatrix}45\\46\end{vmatrix} - \begin{vmatrix}1235\\1236\end{vmatrix}\cdot\begin{vmatrix}46\\45\end{vmatrix} + \begin{vmatrix}1236\\1234\end{vmatrix}\cdot\begin{vmatrix}45\\56\end{vmatrix} + \begin{vmatrix}1234\\1236\end{vmatrix}\cdot\begin{vmatrix}56\\45\end{vmatrix}\\
&\quad - \begin{vmatrix}1235\\1234\end{vmatrix}\cdot\begin{vmatrix}46\\56\end{vmatrix} - \begin{vmatrix}1234\\1235\end{vmatrix}\cdot\begin{vmatrix}56\\46\end{vmatrix}\Big]\\
&\quad - \Big[\begin{vmatrix}12345\\12345\end{vmatrix}\begin{matrix}6\\6\end{matrix} + \begin{vmatrix}12346\\12346\end{vmatrix}\begin{matrix}5\\5\end{matrix} + \begin{vmatrix}12356\\12356\end{vmatrix}\begin{matrix}4\\4\end{matrix}\\
&\quad - \begin{vmatrix}12356\\12346\end{vmatrix}\begin{matrix}4\\5\end{matrix} - \begin{vmatrix}12346\\12356\end{vmatrix}\begin{matrix}5\\4\end{matrix} + \begin{vmatrix}12356\\12345\end{vmatrix}\begin{matrix}4\\6\end{matrix} + \begin{vmatrix}12345\\12356\end{vmatrix}\begin{matrix}6\\4\end{matrix}\\
&\quad - \begin{vmatrix}12346\\12345\end{vmatrix}\begin{matrix}5\\6\end{matrix} - \begin{vmatrix}12345\\12346\end{vmatrix}\begin{matrix}6\\5\end{matrix}\Big]\\
&\quad + \begin{vmatrix}123456\\123456\end{vmatrix}\\
&= S_1 - S_2 + S_3,\end{aligned}$$

* Amer. Journ. Math., vol. 16, pp. 131–150.

where S_1 represents the terms in the first square brackets, S_2 represents those in the second square brackets, and S_3 represents $\left|\begin{matrix}123456\\123456\end{matrix}\right|$. If we represent the left-hand side of (A_2) by S_0, we may write

$$(A_2) \qquad -S_0+S_1-S_2+S_3=0.$$

To prove the truth of (A_2) we have but to show that any minor of order three together with its cofactor has a total coefficient zero.

For instance the first term in S_1 contains $\left|\begin{matrix}123\\123\end{matrix}\right|\cdot\left|\begin{matrix}56&4\\56&4\end{matrix}\right|$ the last terms contains $-\left|\begin{matrix}123\\123\end{matrix}\right|\cdot\left|\begin{matrix}56&4\\46&5\end{matrix}\right|$, and the 7th term contains $\left|\begin{matrix}123\\123\end{matrix}\right|\cdot\left|\begin{matrix}56&4\\46&6\end{matrix}\right|$, which three together gives $\left|\begin{matrix}123\\123\end{matrix}\right|\cdot\left|\begin{matrix}456\\456\end{matrix}\right|$. There are two other sets of three terms in S_1 which give $\left|\begin{matrix}123\\123\end{matrix}\right|\cdot\left|\begin{matrix}456\\456\end{matrix}\right|$ making three in all. Similarly for S_2 and S_3.

It is therefore readily seen that the total coefficient

$$\text{of } \left|\begin{matrix}123\\123\end{matrix}\right|\cdot\left|\begin{matrix}456\\456\end{matrix}\right| \text{ is } [-1+3-3+1]=0,$$

$$\text{of } \left|\begin{matrix}123\\124\end{matrix}\right|\cdot\left|\begin{matrix}456\\356\end{matrix}\right| \text{ is } [\ \ 0-1+2-1]=0,$$

$$\text{of } \left|\begin{matrix}123\\145\end{matrix}\right|\cdot\left|\begin{matrix}456\\236\end{matrix}\right| \text{ is } [\ \ 0+0-1+1]=0,$$

$$\text{of } \left|\begin{matrix}123\\456\end{matrix}\right|\cdot\left|\begin{matrix}456\\123\end{matrix}\right| \text{ is } [+1+0+0-1]=0.$$

These are the type products and any others give the same result.

To indicate the next case briefly we have

$$(A_3)\quad \left(\left|\begin{matrix}1234\\1234\end{matrix}\right|\ \left|\begin{matrix}5678\\5678\end{matrix}\right|\right)$$

$$\equiv \left|\begin{matrix}1234\\1234\end{matrix}\right|\cdot\left|\begin{matrix}5678\\5678\end{matrix}\right|-\left|\begin{matrix}1234\\5678\end{matrix}\right|\cdot\left|\begin{matrix}5678\\1234\end{matrix}\right|$$

$$=\Big[\left|\begin{matrix}12345\\12345\end{matrix}\right|\cdot\left|\begin{matrix}678\\678\end{matrix}\right|+\left|\begin{matrix}12346\\12346\end{matrix}\right|\cdot\left|\begin{matrix}578\\578\end{matrix}\right|+\left|\begin{matrix}12347\\12347\end{matrix}\right|\cdot\left|\begin{matrix}568\\568\end{matrix}\right|+\left|\begin{matrix}12348\\12348\end{matrix}\right|\cdot\left|\begin{matrix}567\\567\end{matrix}\right|$$

$$-\left|\begin{matrix}12345\\12346\end{matrix}\right|\cdot\left|\begin{matrix}678\\578\end{matrix}\right|-\left|\begin{matrix}12346\\12345\end{matrix}\right|\cdot\left|\begin{matrix}578\\678\end{matrix}\right|+\left|\begin{matrix}12345\\12347\end{matrix}\right|\cdot\left|\begin{matrix}678\\568\end{matrix}\right|+\left|\begin{matrix}12347\\12345\end{matrix}\right|\cdot\left|\begin{matrix}568\\678\end{matrix}\right|$$

$$-\left|\begin{matrix}12345\\12348\end{matrix}\right|\cdot\left|\begin{matrix}678\\567\end{matrix}\right|-\left|\begin{matrix}12348\\12345\end{matrix}\right|\cdot\left|\begin{matrix}567\\678\end{matrix}\right|-\left|\begin{matrix}12346\\12347\end{matrix}\right|\cdot\left|\begin{matrix}578\\568\end{matrix}\right|-\left|\begin{matrix}12347\\12346\end{matrix}\right|\cdot\left|\begin{matrix}568\\578\end{matrix}\right|$$

$$+\left|\begin{matrix}12346\\12348\end{matrix}\right|\cdot\left|\begin{matrix}578\\567\end{matrix}\right|+\left|\begin{matrix}12348\\12346\end{matrix}\right|\cdot\left|\begin{matrix}567\\578\end{matrix}\right|-\left|\begin{matrix}12347\\12348\end{matrix}\right|\cdot\left|\begin{matrix}568\\567\end{matrix}\right|-\left|\begin{matrix}12348\\12347\end{matrix}\right|\cdot\left|\begin{matrix}567\\568\end{matrix}\right|\Big]$$

$$-\left[\begin{vmatrix}123456\\123456\end{vmatrix}\cdot\begin{vmatrix}78\\78\end{vmatrix}+\begin{vmatrix}123457\\123457\end{vmatrix}\cdot\begin{vmatrix}68\\68\end{vmatrix}+\begin{vmatrix}123458\\123458\end{vmatrix}\cdot\begin{vmatrix}67\\67\end{vmatrix}+\begin{vmatrix}123467\\123467\end{vmatrix}\cdot\begin{vmatrix}58\\58\end{vmatrix}\right.$$

$$+\begin{vmatrix}123468\\123468\end{vmatrix}\cdot\begin{vmatrix}57\\57\end{vmatrix}+\begin{vmatrix}123478\\123478\end{vmatrix}\cdot\begin{vmatrix}56\\56\end{vmatrix}-\begin{vmatrix}123456\\123457\end{vmatrix}\cdot\begin{vmatrix}78\\68\end{vmatrix}-\begin{vmatrix}123457\\123456\end{vmatrix}\cdot\begin{vmatrix}68\\78\end{vmatrix}$$

$$\left.+\text{ etc. 36 terms}\right]$$

$$+\left[\begin{vmatrix}1234567\\1234567\end{vmatrix}\begin{vmatrix}8\\8\end{vmatrix}+\begin{vmatrix}1234568\\1234568\end{vmatrix}\begin{vmatrix}7\\7\end{vmatrix}+\begin{vmatrix}1234578\\1234578\end{vmatrix}\begin{vmatrix}6\\6\end{vmatrix}+\begin{vmatrix}1234678\\1234678\end{vmatrix}\begin{vmatrix}5\\5\end{vmatrix}\right.$$

$$-\begin{vmatrix}1234567\\1234568\end{vmatrix}\begin{vmatrix}8\\7\end{vmatrix}-\begin{vmatrix}1234568\\1234568\end{vmatrix}\begin{vmatrix}7\\8\end{vmatrix}+\text{etc. 16 terms}\left.\right]$$

$$-\begin{vmatrix}12345678\\12345678\end{vmatrix}$$

or

$$(A_3)\qquad -S_0+S_1-S_2+S_3-S_4=0.$$

It is just as readily seen here that the total coefficient

$$\text{of }\begin{vmatrix}1234\\1234\end{vmatrix}\cdot\begin{vmatrix}5678\\5678\end{vmatrix}\text{ is }[-1+4-6+4-1]=0,$$

$$\text{of }\begin{vmatrix}1234\\1235\end{vmatrix}\cdot\begin{vmatrix}5678\\4678\end{vmatrix}\text{ is }[\ \ 0-1+3-3+1]=0,$$

$$\text{of }\begin{vmatrix}1234\\1256\end{vmatrix}\cdot\begin{vmatrix}5678\\3478\end{vmatrix}\text{ is }[\ \ 0+0-1+2-1]=0,$$

$$\text{of }\begin{vmatrix}1234\\1567\end{vmatrix}\cdot\begin{vmatrix}5678\\2348\end{vmatrix}\text{ is }[\ \ 0+0+0-1+1]=0,$$

$$\text{of }\begin{vmatrix}1234\\5678\end{vmatrix}\cdot\begin{vmatrix}5678\\1234\end{vmatrix}\text{ is }[+1+0+0+0-1]=0.$$

The general case may be stated as follows

$$(A)\quad \left(\begin{vmatrix}(n\,|\,\underset{\alpha}{m})\\(n\,|\,\underset{\alpha}{m})\end{vmatrix}\,\middle|\,\begin{vmatrix}(n\,|\,\underset{\beta}{m})\\(n\,|\,\underset{\beta}{m})\end{vmatrix}\right)$$

$$=\sum_1^m{}_k(-1)^{k+1}\sum_1^{(m)_k}{}_i\sum_1^{(m)_k}{}_j(-1)^{v_k}\begin{vmatrix}(n\,|\,\underset{\alpha}{m})\,(n\,|\,\underset{\beta}{m}\,|\,\underset{i}{k})\\(n\,|\,\underset{\alpha}{m})\,(n\,|\,\underset{\beta}{m}\,|\,\underset{j}{k})\end{vmatrix}\cdot\begin{vmatrix}(n\,|\,\underset{\beta}{\overline{m}}\,|\,\underset{i}{k})\\(n\,|\,\underset{\beta}{\overline{m}}\,|\,\underset{j}{k})\end{vmatrix},$$

where $\begin{vmatrix}(n\,|\,\underset{\alpha}{m})\\(n\,|\,\underset{\alpha}{m})\end{vmatrix}$ represents a coaxial minor of order m of a determinant of order n, where $(n\,|\,\underset{\beta}{\overline{m}}\,|\,\underset{i}{k})$ is the complementary of the combination $(n\,|\,\underset{\beta}{m}\,|\,\underset{i}{k})$

with respect to m, i. e. $(n|\underset{\beta}{m}|\underset{i}{k})$ and $(n|\underset{\beta}{\overline{m}}|\underset{i}{k})$ are complementary combinations of the numbers $(n|\underset{\beta}{m})$ and where v_k is the sum of the number of inversions in $(n|\underset{\beta}{m}|\underset{i}{k})(n|\underset{\beta}{\overline{m}}|\underset{i}{k})$ plus the number of inversions in $(n|\underset{\beta}{m}|\underset{j}{k})$ $(n|\underset{\beta}{\overline{m}}|\underset{j}{k})$.

Here the m values of k give the m groups of terms $S_1, S_2, \cdots, S_m$. If we write as before

$$(A) \qquad -S_0+S_1-S_2+\cdots+(-1)^{m+1}S_m=0,$$

then the proof follows precisely as in (A_2) or (A_3).

It is to be observed that the number of terms in S_h is $\{(m)_h\}^2$ and that the terms involve either coaxial minors or pairs of terms having the same sign in which the minors in one of the two are the conjugates of those in the other.

By properly adjusting the numbers indicating the rows and columns we may, from this relation, get that giving the value of any minor of the second order of the mth compound.

Thus in (A_1) if we put 2 for the 4 in the columns we get

$$\left(\begin{vmatrix}12\\12\end{vmatrix}\begin{vmatrix}34\\32\end{vmatrix}\right)=0+\begin{matrix}2\\2\end{matrix}\begin{vmatrix}134\\132\end{vmatrix}+\begin{matrix}1\\2\end{matrix}\begin{vmatrix}234\\132\end{vmatrix}+0-0$$

or

$$\left(\begin{vmatrix}12\\12\end{vmatrix}\begin{vmatrix}34\\23\end{vmatrix}\right)=\begin{matrix}2\\2\end{matrix}\begin{vmatrix}134\\123\end{vmatrix}+\begin{matrix}1\\2\end{matrix}\begin{vmatrix}234\\123\end{vmatrix},$$

and if we put 2 for 3 and 3 for 4 in the rows and columns we get

$$\left(\begin{vmatrix}12\\12\end{vmatrix}\begin{vmatrix}23\\23\end{vmatrix}\right)=\begin{matrix}2\\2\end{matrix}\begin{vmatrix}123\\123\end{vmatrix}.$$

It is also seen from the foregoing relation (A) that if all minors of order $m+1$ of the original determinant vanish them, since all minors of higher order vanish as a consequence, all minors of order two of the mth compound vanish, and if in addition the coaxial minors of order m vanish them not more than $(n)_m-1$ of the minors of order m do not vanish.

NEW YORK STATE COLLEGE FOR TEACHERS,
May 1925.

THE NULLITY OF A MATRIX RELATIVE TO A FIELD.*

By C. C. MacDuffee.

1. Let x be an n-rowed square matrix whose elements x_{ij} are elements of a field F. The *rank* r of x is the order of a non-vanishing minor determinant of x of maximum order. Following Sylvester, we call $s = n - r$ the *nullity* of x. It is a well known theorem† that a necessary and sufficient condition that x be of nullity s is that the system of homogeneous equations

$$\sum_{i=1}^{i=n} c_i x_{ij} = 0 \qquad (j = 1, 2, \cdots, n)$$

have exactly s linearly independent solutions

$$(1) \qquad (c_1^i, c_2^i, \cdots, c_n^i) \qquad (i = 1, 2, \cdots, s)$$

the c_j^i being elements of F. An equivalent condition is that

$$\sum_{i=1}^{i=n} c_i x_{ji} = 0 \qquad (j = 1, 2, \cdots, n)$$

have exactly s linearly independent solutions. Hence s might have been defined as the number of linearly independent linear relations among the columns of x, or as the number of linearly independent linear relations among the rows of x.

This latter definition of nullity admits of generalization. Let the x_{ij} now be elements of a semi-field S, i. e., a set which has all the properties of a field except that division by elements not zero need not always be possible. Let F_0 be any field all of whose elements are in S. We define $\varkappa(x)$, the *column-nullity of* x *with respect to* F_0, as the number of linearly independent linear relations among the columns of x with coefficients in F_0; i. e., the number of linearly independent sets (1) all of whose elements c_j^i are in F_0. Similarly we define $\varrho(x)$, the *row-nullity of* x *with respect to* F_0, as the number of linearly independent linear relations among the rows of x with coefficients in F_0.

If $S = F_0$, evidently $\varrho = \varkappa = s$. Usually, however, ϱ, $\varkappa$ and s are all distinct.

2. If a and b are two non-singular n-rowed square matrices with elements in F_0, and if

$$u \equiv axb,$$

* Presented to the American Mathematical Society, April 10, 1925.

† Cf. Bôcher, *Introduction to Higher Algebra*, p. 50.

we can show that the column- and row-nullities of u are equal respectively to the column- and row-nullities of x.

Since

$$x = a^{-1} u b^{-1},$$

the relation is a reciprocal one, and we may choose our notation for the two matrices so that the row-nullity $\varrho(x)$ of x is greater than or equal to the row-nullity $\varrho(u)$ of u. We shall show that these nullities must then be equal.

In case $\varrho(x) = 0$, then $\varrho(u) = 0$ and our theorem is true. In the remaining case there exists at least one set of numbers k_i not all zero in F_0 such that

$$\sum_{i=1}^{i=n} k_i x_{ij} = 0 \qquad (j = 1, 2, \cdots, n). \tag{2}$$

Since a is non-singular with elements a_{ij} in F_0, the system of equations

$$k_i = \sum_{j=1}^{j=n} a_{ji} c_j \qquad (i = 1, 2, \cdots, n) \tag{3}$$

has a unique solution $(c_1, c_2, \cdots, c_n)$ where the c_j are numbers of F_0 not all of which are zero, namely

$$c_i = \sum_{j=1}^{j=n} \frac{A_{ij}}{a} k_j \qquad (i = 1, 2, \cdots, n)$$

where A_{ij} is the cofactor of a_{ij} in a, the determinant of a.

Then

$$\begin{aligned}
(4) \qquad \sum_{h=1}^{n} c_h u_{hi} &= \sum_{h,j=1}^{n} \frac{A_{hj}}{a} k_j u_{hi} \\
&= \sum_{h,j=1}^{n} \frac{A_{hj}}{a} k_j \sum_{l,m=1}^{n} a_{hl} x_{lm} b_{mi} \\
&= \sum_{j,l,m=1}^{n} k_j \left[\sum_{h=1}^{n} \frac{A_{hj}}{a} a_{hl} \right] x_{lm} b_{mi} \\
&= \sum_{j,m=1}^{n} k_j x_{jm} b_{mi} \\
&= \sum_{m=1}^{n} \left[\sum_{j=1}^{n} k_j x_{jm} \right] b_{mi} \\
&= 0 \qquad (i = 1, 2, \cdots, n)
\end{aligned}$$

Hence to every linear relation in F_0 among the rows of x there corresponds by (3) a linear relation in F_0 among the rows of u.

Since there are exactly $\varrho(x)$ linearly independent linear relations

$$(k_1^i,\ k_2^i,\ \cdots,\ k_n^i) \qquad (i = 1, 2, \cdots, \varrho(x))$$

satisfying (2), there are at least $\varrho(x)$ linear relations

$$(c_1^i,\ c_2^i,\ \cdots,\ c_n^i) \qquad (i = 1, 2, \cdots, \varrho(x))$$

satisfying (4). We must show that these last $\varrho(x)$ linear relations are linearly independent. Let us denote by k the matrix (k_j^i) and by c the matrix (c_j^i). Then from (3) we have

$$k = c\,a$$

and inasmuch as a is non-singular, the rank of c is equal to the rank of k, namely $\varrho(x)$.

We have now proved that $\varrho(u) \geqq \varrho(x)$, which together with our assumption that $\varrho(u) \leqq \varrho(x)$ gives $\varrho(u) = \varrho(x)$.

We may show that the column-nullities $\varkappa(x)$ and $\varkappa(u)$ of x and u with regard to F_0 are equal by considering the relation

$$\breve{u} = \breve{b}\,\breve{x}\,\breve{a}$$

where $\breve{u}$ denotes the transpose of u, i. e., the matrix obtained from u by changing rows into columns. Then

$$\varkappa(x) = \varrho(\breve{x}) = \varrho(\breve{u}) = \varkappa(u).$$

3. The arithmetic invariants $\varkappa(x)$ and $\varrho(x)$ of the matrix x under the transformation $u = axb$ lead directly to projective invariants of analytic functions and algebraic forms. For instance, let $f_1, f_2, \cdots f_n$ be n analytic functions of n independent variables $x_1, x_2, \cdots, x_n$ with coefficients in a field F_0. Let f_i become $f_i'(x_1', x_2', \cdots, x_n')$ under the projective transformation

$$x_i = \sum_{j=1}^{j=n} a_{ij}\,x_j', \qquad |a_{ij}| \neq 0 \qquad (i = 1, 2, \cdots, n) \tag{5}$$

where the a_{ij} are in F_0. Then

$$\frac{\partial f_i'}{\partial x_j'} = \sum_{k=1}^{k=n} \frac{\partial f_i}{\partial x_k}\,\frac{\partial x_k}{\partial x_j'} = \sum_{k=1}^{k=n} \frac{\partial f_i}{\partial x_k}\,a_{kj} \tag{6}$$

and therefore the jacobian matrix

$$j' = \left(\frac{\partial f_i'}{\partial x_j'}\right) = \left(\frac{\partial f_i}{\partial x_j}\right)(a_{ij}) = ja$$

is an invariant matrix. Hence $\varrho(j)$, its row-nullity over F_0, is an arithmetic invariant of the n functions $f_1, f_2, \cdots, f_n$.

Let us suppose that $\varrho(j) > 0$ so that there exist linearly independent sets of elements in F_0

$$(c_1^i, c_2^i, \cdots, c_n^i) \qquad (i = 1, 2, \cdots, \varrho(j)) \tag{7}$$

such that

$$\sum_{k=1}^{k=n} c_k^i \frac{\partial f_k}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_{k=1}^{k=n} c_k^i f_k = 0 \quad (j = 1, 2, \cdots, n).$$

Then

$$\sum_{k=1}^{k=n} c_k^i f_k = c \qquad (i = 1, 2, \cdots, \varrho(j)) \tag{8}$$

where c is in F_0. Evidently every relation (8) leads to a set of constants (7), so that $\varrho(j)$ is the number of linearly independent linear relations with coefficients in F_0 among the functions $f_1, f_2, \cdots, f_n$.

This result is an interesting analog of the theorem* that the ordinary nullity $s(j)$ of j gives the number of functionally independent functional relations among the functions $f_1, f_2, \cdots, f_n$.

4. By means of the invariant $\varkappa(h)$ of the hessian matrix of an algebraic form, the complete solution of Hesse's problem† may be very simply stated. This problem is to determine necessary and sufficient conditions that an algebraic form in n variables be projectively equivalent to a form in $n-1$ variables. Hesse stated that the vanishing of the determinant afterwards named for him was such a necessary and sufficient condition, and gave two unsatisfactory proofs.‡ The theorem was finally shown to be false by Gordan and Nöther,§ who proved the existence of quinary forms whose hessians vanish but which cannot be written in terms of $n-1$ variables.

Let $f(x_1, x_2, \cdots, x_n)$ be an algebraic form of degree greater than 1 in n variables with coefficients in F_0 which becomes $f'(x_1', x_2', \cdots, x_n')$ by transformations (5). Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j'} = \sum_{k=1}^{k=n} \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial x_k}{\partial x_j'} = \sum_{k=1}^{k=n} \frac{\partial^2 f'}{\partial x_k' \partial x_j'} \frac{\partial x_k'}{\partial x_i}.$$

* P. Gordan, *Invariantentheorie*, Teubner, Leipzig, 1885, pp. 128–131.

† For a quite complete history of Hesse's problem, see Sir Thomas Muir, *The Theory of Determinants*, Macmillan and Co., Vol. 2, Chap. XIII.

‡ *Crelle's Journal*, vol. XLII, pp. 117–128, and vol. LVI, pp. 263–269.

§ *Mathematische Annalen*, vol. X (1876), pp. 547–569.

In the notation of matrices, this is

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\left(\frac{\partial x_i}{\partial x'_j}\right) = \left(\frac{\partial x'_j}{\partial x_i}\right)\left(\frac{\partial^2 f'}{\partial x'_i \partial x'_j}\right).$$

Since $\sum_{k=1}^{k=n} \frac{\partial x_i}{\partial x'_k}\frac{\partial x'_k}{\partial x_i} = 1$, we have

$$\left(\frac{\partial^2 f'}{\partial x'_i \partial x'_j}\right) = \left(\frac{\partial x_j}{\partial x'_i}\right)\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\left(\frac{\partial x_i}{\partial x'_j}\right)$$

or

$$h' = \breve{a}\, h\, a \tag{9}$$

where h is the hessian matrix of f.

Since h is symmetric, $\varrho(h) = \varkappa(h)$. From (9) we see that $\varkappa(h)$ is an arithmetic invariant of h and hence also of f.

Let us denote by r the minimum number of variables in terms of which f can be written. Then we can so transform f that h will have $n-r$ rows of zeros. Then $\varkappa(h) \geqq n-r$.

Since h is the jacobian of the first partial derivatives of f, there are exactly $\varkappa(h)$ linearly independent linear relations (8) with coefficients in F_0 among these partials. Since f is a form, of degree greater than 1, the right-hand members are zero. That is, we have

$$\sum_{i=1}^{i=n} c_i^j f_i = 0 \qquad (j = 1, 2, \cdots, \varkappa(h))$$

whose matrix (c_i^j) is of rank $\varkappa(h)$.

Let us set

$$a_{ij} = c_i^j \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, \varkappa(h))$$

and let a_{ij} for $j = \varkappa+1, \cdots, n$ be any numbers of F_0 such that

$$|a_{ij}| \neq 0.$$

This can be done unless $\varkappa = n$, i. e., unless f is a constant. We see from (6) that if we transform by this matrix $a = (a_{ij})$, we make $f'_j = 0$ for $j = 1, 2, \cdots, \varkappa$ so that f' is independent of $x'_1, x'_2, \cdots, x'_\varkappa$. Thus

$$r \leqq n-\varkappa, \qquad \varkappa \leqq n-r,$$

which, together with the inequality proved above, gives

$$\varkappa = n-r, \qquad r = n-\varkappa.$$

Thus *a necessary and sufficient condition that the algebraic form* $f(x_1, x_2, \cdots, x_n)$ *of degree* >1 *be projectively equivalent to a form in* r *variables, and equivalent to no form in fewer than* r *variables, is that* $\varkappa(h) = n - r$.

5. The theorem which Hesse stated but did not succeed in proving was equivalent to the statement that the nullity over F_0 of the hessian of every algebraic form is equal to its nullity s in the sense of Sylvester. For quadratic forms and for binary forms this is obviously true. It has been proved true for ternary and quaternary forms,* and not always true for quinary forms.

We can now show that in the special case where $s(h) = n-1$, we also have $\varkappa(h) = n-1$ for forms of arbitrary degree and number of variables. When $s = n-1$, the rank of the hessian is 1, and when $\varkappa = n-1$ the form can be written in terms of one variable only, i. e., as the product of a constant by a power of a single variable. The theorem which we shall prove, then, is the following:

A necessary and sufficient condition that the algebraic form f *of degree* $p>1$ *in* n *variables be the product of a constant by the* p*th power of a linear function, is that the rank of its hessian matrix be* 1.

The necessity of the condition is obvious if we transform f so that it takes the form cx^p.

To prove the sufficiency, we set $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = f_{ij}$ and assume that the matrix

$$h \equiv \begin{Vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{Vmatrix}$$

is of rank 1. By Euler's relation

$$\sum_{i=1}^{i=n} x_i f_{ji} = (p-1) f_j \qquad (j = 1, 2, \cdots, n)$$

We multiply the kth column of h by x_k and add to it x_l times the lth column for every $l \neq k$. We divide the new kth column by $p-1$ and obtain

$$\begin{Vmatrix} f_{11} & f_{12} & \cdots & f_{1,k-1} & f_1 & f_{1,k+1} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2,k-1} & f_2 & f_{2,k+1} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{n,k-1} & f_n & f_{n,k+1} & \cdots & f_{nn} \end{Vmatrix} \qquad (k = 1, 2, \cdots, n)$$

* Gordan and Nöther, loc. cit.

These matrices are all of rank 1, since the above elementary transformations leave the rank invariant. That is,

$$\begin{vmatrix} f_{ij} & f_i \\ f_{kj} & f_k \end{vmatrix} = 0 \qquad (i, j, k = 1, 2, \cdots, n).$$

Let us first consider the case when $f_i f_k \neq 0$. The above equation gives us

$$\frac{\partial \log f_i}{\partial x_j} = \frac{\partial \log f_k}{\partial x_j}$$

and hence

$$\log f_i = \log f_k + \varphi(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n).$$

Since this holds for every j, it follows that φ is a constant, and we may write

$$\log f_i = \log f_k + \log c.$$

Hence there exist constants c_i, c_k not both zero (and evidently in F_0) such that

$$c_i f_i + c_k f_k = 0 \qquad (i, k = 1, 2, \cdots, n;\ i \neq k). \tag{10}$$

In case $f_i f_k \equiv 0$, a relation (10) holds obviously.

From (10) it follows that for every i and k we have a relation

$$c_i f_{ji} + c_k f_{jk} \equiv 0 \qquad (j = 1, 2, \cdots, n). \tag{11}$$

Since the rank of h is 1, there is at least one second partial which is not zero. For convenience of notation we shall assume that for some j, $f_{j1} \neq 0$. From (11) we have

$$c_1^k f_{j1} + c_k^k f_{jk} \equiv 0 \qquad (j = 1, 2, \cdots, n;\ k = 2, 3, \cdots, n). \tag{12}$$

Every c_k^k in (12) is different from zero, for $c_k^k = 0$ implies $c_1^k \neq 0$, and hence $f_{j1} \equiv 0$ for $j = 1, 2, \cdots, n$ contrary to our assumption. Then the matrix of the coefficients of the partials in (12) with the first column deleted is for every j the same $(n-1)$-rowed diagonal matrix none of whose diagonal elements is zero. It is therefore non-singular, and (12) gives us $n-1$ linearly independent linear relations with coefficients in F_0 among the columns of h. Thus $\varkappa(h) \geq n-1$. Since the rank of h is 1, $\varkappa(h) = n-1$. This proves our theorem.

THE OHIO STATE UNIVERSITY,
May, 1925.

DETERMINATION OF THE TERNARY COLLINEATION GROUPS WHOSE COEFFICIENTS LIE IN THE $GF(2^n)$.

By R. W. Hartley.

Introduction.

The object of this paper is to determine the finite groups of collineations in three homogeneous variables, whose coefficients lie in the $GF(2^n)$. Such a collineation is of the form

$$\varrho x_i' = \sum_{j=1}^{3} a_{ij} x_j \qquad (i = 1, 2, 3),$$

wherein the coefficients a_{ij} are in the $GF(2^n)$ and the variables x_i are regarded as homogeneous coördinates in the modular plane $PG(2, 2^n)$.*

The finite groups of collineations in the ordinary plane were completely determined for the first time by Blichfeldt.† They were determined by H. H. Mitchell by using a geometrical method almost entirely; he also determined completely the finite groups of collineations in the modular plane $PG(2, p^n)$ where p is an odd prime.‡

The case of the even prime was considered to some extent and the finite groups in the modular plane $PG(2, 2^2)$ determined by U. G. Mitchell.§

The binary modular groups have been fully determined. The general case was solved by E. H. Moore,|| and Wiman.¶ Dickson gave a solution based on these two.** A still different determination of binary groups on the modular line was given by H. H. Mitchell.††

It was determined that when the coefficients lie in the $GF(2^n)$ the order of the collineation groups, which leave no points fixed, is of the form

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* O. Veblen and W. H. Bussey, Finite projective geometries, Trans. Amer. Math. Soc., vol. 7 (1906), pp. 241–259.

† Blichfeldt, On the order of linear homogeneous groups, Trans. Amer. Math. Soc., vol. 4 (1903), pp. 387–397, and vol. 5 (1904), pp. 310–325; also The finite discontinuous, primitive groups of collineations in three variables, Math. Annalen, vol. 63 (1907), pp. 552–572.

‡ H. H. Mitchell, Determination of the ordinary and modular ternary linear groups, Trans. Amer. Math. Soc., vol. 12 (1911), pp. 207–242.

§ U. G. Mitchell, Geometry and collineation groups of the finite projective plane $PG(2, 2^2)$, Diss. Princeton 1910.

|| E. H. Moore, The subgroups of the generalized finite modular group, Dicennial Publications of the University of Chicago, vol. 9 (1904), pp. 141–190.

¶ Wiman, Bestimmung aller Untergruppen einer doppelt unendlichen Reihe von einfachen Gruppen, Bihang till K. Svenska Vet.-Akad. Handlingar, vol. 25, part 1, no. 2.

** Dickson, Linear Groups, Chap. XII.

†† H. H. Mitchell, loc. cit.

$(2^k+1)2^k(2^k-1)$, where k is a factor of n. A group of order $(2^k+1)2^k(2^k-1)$ can be represented by collineations with coefficients in the $GF(2^k)$. Such a group permutes the 2^k+1 points on the line with coördinates in that field. There are $(2^k+1)(2^k-1)$ transformations of period 2, each leaving fixed one point only on the line. There are transformations of period 2^k-1, each leaving fixed two points and permuting cyclicly the other 2^k-1 points, and transformations of period 2^k+1, the coördinates of whose two fixed points are in a higher field, that permute cyclicly the 2^k+1 points. Metacyclic sub-groups of order $2^k(2^k-1)$ exist each leaving one point fixed. There are also present dihedral groups of order $2(2^k-1)$ and $2(2^k+1)$. The octahedral group does not occur.

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Types of transformations.*

The following five types of transformations are present.

Type I. A transformation that leaves fixed the three vertices of a triangle and its three sides and no other points or lines. By choosing as its fixed triangle the triangle of reference it takes the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{where } \alpha, \beta, \gamma \neq 0.$$

Its order depends on α, β and γ but is odd and a divisor of 2^k-1 if the coefficients are in the $GF(2^k)$.

Type II. A transformation that leaves fixed two points and two lines only. It may be written in the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } \alpha \neq 0 \text{ or } 1.$$

Its order is of the form $2d$ where d is odd. Its square is an homology. Any odd power m of it such that $\alpha^m = 1$ is an elation.

Type III. A transformation that leaves fixed only one point and one line through the point. It may be written in the form

$$\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha^{-1} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } \alpha \neq 0.$$

Its order is four and its square is an elation.

* Cf. Veblen and Young, Projective Geometry; Dickson, Linear Groups, Chap. X, XI.

Type IV. Homology. This leaves fixed all the points on a line called the axis and all the lines through a point not on the axis called the center. It may be written in the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{where } \alpha \neq 0 \text{ or } 1.$$

Its order, which is odd, depends on α.

Type V. Elation. This leaves fixed all the points on a line called the axis and all lines through a point on the axis called the center. It may be written in the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Its order is two.

Groups containing subgroups of elations with a common center and axis and of order greater than two.

We shall first consider such groups as do not leave invariant a point, line or triangle, and leave such groups as do to be enumerated with the sub-groups at the end of the paper. *We shall denote by condition A the condition that no point, line or triangle is to be left invariant.*

THEOREM 1. *The only groups that satisfy condition A, and contain sub-groups of elations with a common center and axis and order greater than two, are the* $HO(3, 2^{2k})$; *groups containing the* $HO(3, 2^{2k})$ *as self-conjugate sub-groups of index* 3, *if* k *is odd*; *the* $LF(3, 2^k)$; *groups containing the* $LF(3, 2^k)$ *as self-conjugate sub-groups of index* 3, *if* k *is even.*

We take as center and axis of a group of elations (010) and $x_3 = 0$. Any elation of this group will be of the form

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda' \\ 0 & 0 & 1 \end{pmatrix}$$

where λ' is any mark. Any two such elations are commutative; hence the order of the group of elations is of the form 2^k. We assume that 2^k is the order of the largest sub-group of this kind present and that $k \geqq 2$.

Since no point or line is to be left invariant there must exist another sub-group of elations of the same order with center not on $x_3 = 0$ and axis not passing through (010). We let the center and axis of such a sub-group be (001) and $x_2 = 0$. This may be represented by

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda'' & 1 \end{pmatrix}.$$

The group generated by E_1 and E_2 leaves fixed (100) and $x_1 = 0$. There can be no homology present with this point and line for center and axis, for such an homology would be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

whereas if it were in the group generated by E_1 and E_2, $\mu^2 = 1$, and hence $\mu = 1$. Therefore this group is simply isomorphic with the binary group on $x_1 = 0$ and the order is $(2^k+1)2^k(2^k-1)$. Then from the theory of binary groups we may, without loss of generality, assume that λ' and λ'' run over all the marks in the $GF(2^k)$.

Since no point or line is to be left invariant there must exist a sub-group of elations of order 2^k with center not on $x_1 = 0$ and axis not passing through (100). We may, without loss of generality, assume that the common axis of this additional group does not pass through (001), (010) or (011). For if it did, some elation of the group generated by E_1 and E_2 would transform it into a group whose axis did not. We can, furthermore, so choose this elation that the center of the transformed group does not lie on $x_2 = 0$, $x_3 = 0$ or $x_2 + x_3 = 0$. If $(1\,\nu\,\mu)$ were taken as such a center, the homology

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu^{-1} & 0 \\ 0 & 0 & \nu^{-1} \end{pmatrix}$$

would send this point to $(1\,1\,\mu\nu^{-1})$ and at the same time leave E_1 and E_2 invariant, hence we may without loss of generality take $(1\,1\,\mu)$ as the new center. We take $x_1 + (1+\sigma\mu)x_2 + \sigma x_3 = 0$ as the axis, where $\sigma \neq 0$, $1+\sigma\mu \neq 0$, and $1+\sigma\mu+\sigma \neq 0$.

A transformation of such a group may be written in the form

$$E_3 = \begin{pmatrix} 1+\lambda & \lambda(1+\sigma\mu) & \lambda\sigma \\ \lambda & 1+\lambda+\lambda\sigma\mu & \lambda\sigma \\ \lambda\mu & \lambda\mu(1+\sigma\mu) & 1+\lambda\sigma\mu \end{pmatrix},$$

where λ is undetermined but runs over 2^k values.

The group generated by E_1 and E_3 will be of the same character as the one generated by E_1 and E_2. It will be simply isomorphic with a binary group and must be represented on the line joining their centers, namely $\mu x_1 + x_3 = 0$, by this binary group of order $(2^k+1)2^k(2^k-1)$. 2^k+1 points including the centers of E_1 and E_3 will be permuted. Under E_1, (11μ) is conjugate with all these points except (010); and under E_3, (010) is conjugate with all except (11μ). A particular transformation F_3 of E_3, wherein λ has a particular value, sends (010) to some one of these 2^k+1 points on $\mu x_1 + x_3 = 0$ and there must be in E_1 a transformation F_1 which sends (11μ) to this same point.

F_1 sends (11μ) into $(1, 1+\lambda'\mu, \mu)$.

F_3 sends (010) into $[\lambda(1+\sigma\mu), \quad 1+\lambda+\lambda\sigma\mu, \quad \lambda\mu(1+\sigma\mu)]$.

In order that these points be identical, we have, since $1+\sigma\mu \neq 0$,

$$\sigma\lambda\lambda'\mu^2 + \lambda\lambda'\mu + 1 = 0. \tag{1}$$

In the same way we consider the particular transformation F_2 of E_2 whose λ'' has such a particular value that it sends (11μ) into the same point as that into which F_3 sends (001).

F_2 sends (11μ) into $(1, 1, \quad \lambda''+\mu)$.

F_3 sends (001) into $(\lambda\sigma, \quad \lambda\sigma, \quad 1+\lambda\sigma\mu)$.

From these we get, since $\sigma \neq 0$,

$$\sigma\lambda\lambda'' = 1. \tag{2}$$

In the group generated by E_1 and E_2 there will be a sub-group of elations with (011) as center and $x_2 + x_3 = 0$ as axis. We designate this group by

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\lambda''' & \lambda''' \\ 0 & \lambda''' & 1+\lambda''' \end{pmatrix}$$

where λ''' runs over all the marks of the $GF(2^k)$.

By cousidering F_3 and the particular transformation F_4 of E_4 determined as in the other two cases, we find that

F_4 sends (11μ) into $(1, \quad \lambda'''+1+\lambda'''\mu, \quad \lambda'''+\lambda'''\mu+\mu)$

F_3 sends (011) into $(\lambda+\lambda\sigma\mu+\lambda\sigma, 1+\lambda+\lambda\sigma+\lambda\sigma\mu, \lambda\mu+\lambda\sigma\mu^2+1+\lambda\sigma\mu)$

From this and (2) we get, since $1+\sigma\mu+\sigma \neq 0$,

$$\lambda'''\mu^2 + \lambda\lambda''\lambda'''\mu + \lambda'' + \lambda''' + \lambda\lambda''\lambda''' = 0. \tag{3}$$

From (1), (2) and (3) we get

$$\lambda = 1/\lambda' + 1/\lambda'' + 1/\lambda''', \tag{4}$$

which shows that λ is in the $GF(2^k)$. Also from (2) σ is in this field.

From (1) and (2) we have

$$\mu^2 + \lambda\lambda''\mu + \lambda''/\lambda' = 0. \tag{5}$$

Hence μ lies in the $GF(2^{2k})$.

We consider two cases:

(i) When μ is not in the $GF(2^k)$.

(ii) When μ is in the $GF(2^k)$.

In the first case, since μ satisfies (5), the other root of (5) will be $\bar{\mu}$. the conjugate of μ, and is determined when μ is given. Hence for a given value of μ from (5) we have that $\lambda\lambda''$ is determined, and hence σ is determined. That is, when a center of a group of elations is taken, the axis is uniquely determined.

The line joining (100) to $(1\,1\,\mu)$ meets $x_1 = 0$ in $(0\,1\,\mu)$. The axis of E_3, $x_1 + (1 + \sigma\mu)\,x_2 + \sigma x_3 = 0$, meets $x_1 = 0$ at the point $(0, 1, \lambda\lambda'' + \mu)$. But from (5)

$$\bar{\mu} = \lambda\lambda'' + \mu.$$

Hence this point is $(0\,1\,\bar{\mu})$.

E_3 with $\lambda = 1$ leaves $(0\,1\,\bar{\mu})$ fixed and interchanges (100) and $(0\,1\,\mu)$. E_1 with $\lambda' = (\mu + \bar{\mu})/\mu\bar{\mu}$ leaves (100) fixed and interchanges $(0\,1\,\mu)$ and $(0\,1\,\bar{\mu})$. Hence all the permutations are made on the vertices of the triangle (100), $(0\,1\,\mu)$ and $(0\,1\,\bar{\mu})$.

There is a binary cyclic group of order $2^k + 1$ present on each side of this triangle, hence there is an homology present with center (100) and axis $x_1 = 0$ of order $(2^k + 1)/3$ or $2^k + 1$ according as $2^k + 1$ is or is not divisible by 3, that is, according as k is odd or even.*

We denote this value by $(2^k + 1)/\nu$, where $\nu = 3$ or 1 according as k is odd or even. If $\nu = 3$ there will not be an homology present of order $2^k + 1$ in the group generated by the elations, since its determinant would not be a cube in the $GF(2^{2k})$.

There can be only one center of elations on each axis of elations. For suppose there was a center other than $(1\,1\,\mu)$ on the axis of E_3. This elation must have the axis of E_3 for its axis or the square of its product and an elation of E_3 would be another elation with the same center and a different axis. All elations with centers on the line joining (100) and

* H. H. Mitchell, loc. cit., pp. 213–214.

(11μ) must have axes that pass through $(01\bar{\mu})$, a point uniquely determined for each line of centers through (100). The axis of any other elation with center on the axis of E_3 other than at (11μ), and not on an axis through (100), must meet $x_1 = 0$ at some point other than $(01\bar{\mu})$, hence would have a different axis than E_3. If the new center were on an axis through (100), and had the axis of E_3 for axis, the square of its product and one of the elations with the axis through (100) for axis would be another elation with the same center but different axis.

Under the homology mentioned above (11μ) is conjugate with other points on the line joining (100) and (11μ), hence there is more than one center on this line and therefore 2^k+1 centers, and the same number on each of the lines through (100) and points with coördinates in the $GF(2^{2k})$ other than those through centers of elations on $x_1 = 0$. There are $2^{2k}-2^k$ such lines, hence there are $(2^{2k}-2^k)(2^k+1)$ centers with coördinates not in the $GF(2^{2k})$, which together with those on $x_1 = 0$, make $2^{3k}+1$ in all.

Since the order of the group on $x_1 = 0$ is $(2^k+1)\,2^k\,(2^k-1)$ and the order of the homology that has (100) for center and $x_1 = 0$ for axis is $(2^k+1)/\nu$, the order of the group that leaves fixed (100) and $x_1 = 0$ is $(2^k+1)^2\,2^k\,(2^k-1)/\nu$.

Furthermore any transformation in the group that leaves a line of centers fixed must leave fixed the point where the corresponding axes meet, hence this is the order of the group leaving $x_1 = 0$ fixed.

Any line on which two centers of elations lie will have on it 2^k+1 centers and since there are $2^{3k}+1$ centers of elations there will be

$$2^{3k}\,(2^{3k}+1)/2^k\,(2^k+1) = 2^{2k}\,(2^{2k}-2^k+1)$$

lines on each of which 2^k+1 centers of elations lie.

Under the cyclic group of order 2^k-1 leaving fixed (100), (010) and (001), the line of centers of E_1 and E_3 is conjugate with 2^k-1 lines of centers through (010); and under the homology of order $(2^k+1)/\nu$ with center (100) and axis $x_1 = 0$, it is conjugate with $(2^k+1)/\nu$ lines of centers through (010). Since 2^k-1 and $(2^k+1)/\nu$ are relatively prime, this line of centers is conjugate, under the group leaving (010) fixed, to $(2^{2k}-1)/\nu$ lines through (010). Since $x_1 = 0$ is conjugate with some of these lines through (010), it will be conjugate, $\nu = 1$, with all the 2^{2k} lines of centers through (010). If $\nu = 3$, $x_1 = 0$ will be conjugate with $(2^{2k}-1)/3+1$ of these lines, but by considering a sub-group leaving fixed a line of centers not among them, the lines of centers other than the one left fixed fall into three conjugate sets in one of which $x_1 = 0$ lies. Hence $x_1 = 0$ must be conjugate with $2\,(2^{2k}-1)/3$ of the lines of centers through (010). Similarly

every line of centers through (010) is conjugate with more than half the lines of centers through that point, and hence they are all conjugate. Since (010) is conjugate with all the centers of elations, all the lines of centers will be conjugate with $x_1 = 0$. Therefore the order of the entire group must be

$$(2^{2k} - 2^k + 1)(2^k + 1)^2 2^{3k}(2^k - 1)/\nu.$$

This group exists and may be identified with the hyperorthogonal group $HO(3, 2^{2k})$* by noting that E_1, E_2 and E_3, and hence the group they generate, leave invariant the function

$$(\mu^2 + \bar{\mu}^2) x_1^{2^k+1} + (\mu x_2 + x_3)^{2^k+1} + (\bar{\mu} x_2 + x_3)^{2^k+1}.$$

The coördinates of the $2^{3k} + 1$ centers of elations satisfy this function equated to zero.

If $\nu = 3$ there exists a group having the $HO(3, 2^{2k})$ as a selfconjugate sub-group of index 3. It may be generated by the $HO(3, 2^{2k})$ and an homology of order $2^k + 1$, having (100) for center and $x_1 = 0$ for axis.

The $HO(3, 2^{2k})$ cannot be further extended. For an additional center of elations must lie on a line in the $GF(2^{2k})$ and all such lines under the $HO(3, 2^{2k})$ fall into two conjugate sets, axes of elations and lines on each of which lie $2^k + 1$ centers, and no further centers can lie on lines of either of these types.

We next consider case (ii), where μ is in the $GF(2^k)$. From (2) we have that σ is alsc in this field.

Since (11μ) now lies on one of the $2^k + 1$ lines in the $GF(2^k)$ through (100), which are all axes of elations with centers on $x_1 = 0$, we may, without loss of generality, assume that the center of the additional elations is on $x_2 = 0$, at (101) say.

As the axis of this additional group of elations E_3 passes through one of the $2^k + 1$ centers on $x_1 = 0$, there will be an axis of elations with center (101) passing through each of the $2^k + 1$ points on $x_1 = 0$, since the transformations of E_2 leave (101) fixed and move (010) into 2^k positions on $x_1 = 0$ and the squares of the products of E_2 and E_3 are elations with (101) for center and $x_2 = 0$ for axis.

The squares of the products of E_1 and transformations of the group with center (101) and axis through (010) generate a group of elations with (010) for center and $x_1 + x_3 = 0$ for axis. Under this group (100) is conjugate with 2^k points on $x_3 = 0$. Now in the group generated by E_1 and E_2, (101) is conjugate with a point on $x_3 = 0$, and an elation with this point for center will move (010) to another point on $x_3 = 0$. Therefore all the

* Cf. Dickson, Math. Annalen, vol. 52 (1899), pp. 561–581; also Linear Groups, Chap. V.

2^k+1 points on $x_3 = 0$ are conjugate, and since all the lines through (100) are conjugate, all the points in the field are conjugate, and every point will be the center of 2^k+1 groups of elations with different axes.

All the elations with axes passing through (100) will generate a group leaving (100) fixed under which $x_1 = 0$ is conjugate with all the lines in the field not passing through (100).

All six permutations are made on the vertices of the triangle (100), (010), (001), hence there will be an homology present of period $(2^k-1)/3$ or (2^k-1) with center at (100) and axis $x_1 = 0$, according as 2^k-1 is or is not divisible by 3, or as k is even or odd.* When k is even, the homology of period 2^k-1 has a determinant that is a not-cube in the field, so in this case the homology with determinant a cube in the field will be of period $(2^k-1)/3$. We denote the period of this homology by $(2^k-1)/\nu$, where $\nu = 1$ or 3.

The group of points on an axis is of order $(2^k+1)2^k(2^k-1)$, so the order of the group leaving (100) and $x_1 = 0$ fixed is $(2^k+1)2^k(2^k-1)^2/\nu$. Under the group leaving (100) fixed, $x_1 = 0$ is conjugate with 2^{2k} lines not passing through (100), hence the group leaving (100) fixed is of order $(2^k+1)2^{3k}(2^k-1)^2/\nu$ and the entire group is of order $(2^k+1)2^{3k}(2^k-1)^2(2^{2k}+2^k+1)/\nu$.

This is the $LF(3, 2^k)$ and consists of all the transformations whose coefficients are in the $GF(2^k)$ and whose determinant is a cube in that field.†

If k is even, a group exists containing the $LF(3, 2^k)$ as a self-conjugate sub-group. It may be generated by $LF(3, 2^k)$ and an homology of period 2^k-1, with center at (100) and axis $x_1 = 0$.

The $LF(3, 2^k)$ cannot be further extended as we have proved that the coördinates of the center of an additional elation must lie in the $GF(2^{2k})$ and if such an elation exists each center must uniquely determine an axis.

Hereafter in this paper we shall denote the condition that there be not present sub-groups of elations with common center and axis as condition B.

Groups containing homologies but no subgroups of elations with common center and axis.

THEOREM 2. *In a group that satisfies condition B, the axis of any two non-commutative homologies must intersect in a point collinear with their centers.*

If the center of each lies on the axis of the other the homologies are commutative.

* H. H. Mitchell, loc. cit., pp. 213–214.

† Dickson, Linear Groups, pp. 75–78.

If two homologies have a common axis and different centers, we may assume that they are of the same order, for, if not, the second would transform the first into a third of the same order as the first with the same axis and a different center. We choose $x_3 = 0$ as the axis of each and (001) and (101) as centers. Then the two homologies will be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \beta+1 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

Their product is

$$\begin{pmatrix} 1 & 0 & \beta+1 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix}.$$

If we choose α the inverse of β this will be an elation with center (100) and axis $x_3 = 0$, and since β may have at least two values we would have two or more elations with common center and axis.

For two homologies to have the same center and different axes is the dual case and can similarly be shown impossible.

If the center A of one homology lies on the axis b of the second and the center B of the second does not lie on the axis a of the first, the homology with center at A will transform the one with center at B into a third with axis b and center not at B or on a, which by a previous case is impossible.

We consider now the case where two homologies with centers and axes distinct are so situated that neither center lies on the axis of the other and the centers are not collinear with the intersection of the axes. We let the center and axis of one be A and a, respectively, and the center and axis of the other B and b. a and b will meet at some point C not on AB, and the two homologies will generate a group leaving fixed C and AB and represented on AB by a binary group of order $(2^k+1)2^k(2^k-1)$ permuting 2^k+1 points, where $k \geqq 2$. There will be $(2^k+1)(2^k-1)$ transformations of order 2, each leaving fixed one and only one of these 2^k+1 points on AB. Hence there are 2^k-1 transformations of order 2 leaving fixed a single point P on AB. Each of these is represented in the plane by one or more transformations leaving points C and P fixed and also line AB and of order 2 on AB. Such a transformation is either an elation with center P and axis PC or a transformation of type II, a power of which is such an elation. Hence we will have 2^k-1 elations with P as center and PC as axis.

Therefore by this process of elimination we conclude that two non-

commutative homologies must have axes that intersect in a point collinear with their centers.

THEOREM 3. *The only group that satisfies conditions A and B and contains homologies is the Hessian* G_{216}.

We let the center and axis of one homology be (010) and $x_2 = 0$ and denote it by E_1, then

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where λ is a mark in the $GF(2^n)$, such that $\lambda^d = 1$.

Since no point or line is to be left invariant, we must have another homology conjugate with E_1. We take as center and axis of this second homology (110) and $x_2 + x_3 = 0$ and denote it by

$$E_2 = \begin{pmatrix} 1 & 1+\lambda' & 1+\lambda' \\ 0 & \lambda' & 1+\lambda' \\ 0 & 0 & 1 \end{pmatrix}$$

where λ', since E_1 and E_2 are conjugate, runs over the same values as λ. If we choose λ' to be the inverse of λ, we find that

$$(E_1 E_2)^2 = \begin{pmatrix} 1 & 0 & \lambda + 1/\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is an elation with center (100) and axis $x_3 = 0$. Unless $\lambda + 1/\lambda$ is invariant when λ is replaced by its powers there will be a group of elations present with common center and axis of order greater than two. Hence we have

$$\lambda + 1/\lambda = \lambda^2 + 1/\lambda^2$$

whence, since $\lambda \neq 1$,

$$\lambda^2 + \lambda + 1 = 0$$

and λ lies in the $GF(2^2)$.

Therefore E_1 and E_2 are each of period 3. E_1 and E_1^2 transform E_2 into two other homologies with centers on $x_3 = 0$ and axes passing through (100). The centers and axes of these two new homologies will be $(1\,\lambda\,0)$, $(1\,\lambda^2\,0)$ and $x_2 + \lambda x_3 = 0$, $x_2 + \lambda^2 x_3 = 0$, respectively.

No further centers of homologies can exist on $x_3 = 0$, for if there be one at $(1\,\mu\,0)$ say, where μ is not in the $GF(2^2)$, by using the conditions

that it is of period 3 and the square of its product with E_1 is the elation determined above, it must take the form

$$S = \begin{pmatrix} 1 & \lambda/\mu & \lambda \\ 0 & \lambda^2 & \lambda\mu \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{where } \lambda^2+\lambda+1 = 0.$$

But in order that the square of the product of S and E_2^2 be the same elation already determined we must have

$$\mu^2+\mu+1 = 0$$

which contradicts our hypothesis that μ is not in the $GF(2^2)$.

Since no point or line is to be left invariant there will be present an homology with center not on $x_3 = 0$ and axis not passing through (100). This can be commutative with not more than one of the four homologies with centers (100), (110), $(1\,\lambda\,0)$ and $(1\,\lambda^2\,0)$. Hence we may assume that it is not commutative with any of the last three. By choosing $x_1 = 0$ to pass through its center we may take $(0\,\mu\,1)$ as its center and $x_1+\varrho x_2+\sigma x_3 = 0$ as its axis. We denote this additional homology by E_3. By using the fact that the axis of E_3 must meet the axes of the three homologies with which E_3 is not commutative in points collinear with $(0\,\mu\,1)$ and the corresponding center, we get that $\mu = 0$, $\varrho = 0$, and $\sigma = 1$. Thus the center and axis of E_3 are (001) and $x_1+x_3 = 0$, and therefore this homology is commutative with E_1. Hence every homology in the group whose center is not on $x_3 = 0$ is commutative with one with center on $x_3 = 0$. Therefore all additional centers of homologies must lie on one of the four axes through (100).

We have from the above and the fact that E_3, being conjugate with E_1, is of period 3, that

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda^2 & 0 & \lambda \end{pmatrix}.$$

The product of E_1 and E_3 is an homology with center (101) and axis $x_1 = 0$. No more centers of homologies can lie on $x_2 = 0$. For such an homology would be commutative with E_1, but not with E_3. Hence its axis would meet the axis of E_3 at (010) which is not on $x_2 = 0$, the line of centers of the additional homology and E_3. Therefore there are two and only two centers of homologies on each of the axes through (100), making 12 in all.

The order of the group leaving the three points (010), (001) and (101) fixed is 9, and since the elation with center (100) and axis $x_3 = 0$ leaves (010)

fixed and interchanges (101) and (001), the order of the group leaving (010) fixed is 18. Since the 12 centers of homologies are all conjugate, the order of the whole group is 12 times 18 or 216, and the group is the Hessian G_{216}.

The system of four triangles (010) (001) (101), (110) $(1\,\lambda\,\lambda)$ $(1\,\lambda^2\,\lambda^2)$, $(11\,\lambda^2)$ $(1\,\lambda\,0)$ $(1\,\lambda^2\,\lambda)$, and $(11\,\lambda)$ $(1\,\lambda^2\,0)$ $(1\,\lambda\,\lambda^2)$ is left invariant, and the whole group is isomorphic with a tetrahedral group permuting these four triangles. It may be generated by E_1, E_2 and E_3.

The sub-group of order 72, which contains no homologies, is the $HO(3,2^2)$ self-conjugate under the G_{216}.

Since we have disposed of all groups containing homologies we shall hereafter designate the condition that there be no homologies present by condition C.

Groups containing commutative elations but no two with the same center and axis and no homologies.

THEOREM 4. *If two elations are commutative, they will have the same center, the same axis, or both.*

The product of two commutative elations is an elation, but the product of two elations with different centers and axes is a transformation of type I with the intersection of the axes of the elations as one vertex of its fixed triangle and the line of centers of the elations as its opposite side. If the axes are distinct and the center of one lies on the axis of the other, their product is a transformation of type III. Hence in either case, two such elations are not commutative.

Conversely, if two elations have the same line as axis (or the same point as center), their product will leave fixed all points on the common axis (or all lines through the common center) and cannot leave fixed a point outside that line (or line not through that point), hence is an elation.

THEOREM 5. *If a group, which satisfies conditions B and C, contains a sub-group of commutative elations having the same axis, and the axis of an additional elation passes through one of their centers, the sub-group of commutative elations is a four group. And, dually, if a group which satisfies conditions B and C, contains a sub-group of commutative elations having a common center and the center of another elation lies on one of their axes, the sub-group of commutative elations is a four group.*

The order of a group of commutative elations, with a common axis, will be of the form 2^k and there will be 2^k-1 centers. We represent these centers by A_1, A_2, A_3, etc. with common axis l. We let the center of an additional elation be A_r and let its axis pass through A_1. If $k>2$ we can choose three elations in the commutative group none of which has its center A_1 which with the identity form a four group. These together with the elation with center at A_r will generate a group leaving A_1

fixed and isomorphic with the binary group on the lines through A_1. Hence the order is $h(2^m+1)2^m(2^m-1)$ where $m>1$ and h is the order of the group leaving fixed all the lines through A_1. This group contains a metacyclic sub-group of order $h\,2^m(2^m-1)$ leaving the line l fixed. This in turn has cyclic sub-groups of order 2^m-1 containing transformations that are not commutative with the elation with center at A_1, and which leave both its center and axis fixed, hence transform it into another elation with the same center and axis. Since we assume that these do not exist we must have $k=2$. The dual theorem may be proved similarly.

THEOREM 6. *The only group, satisfying conditions A, B, and C, and having commutative elations, in which an axis of an elation meets the common axis of a sub-group of commutative elations only at a center (or in which a center of an elation lies on one of the axes of a sub-group of elations with a common center), is the* G_{168}.

We let $x_3=0$ be common axis of a sub-group of elations with centers at (100) and (010). Since the axis of an additional elation E_3 meets $x_3=0$ at a center we let it be $x_2=0$ and let the center be at (001). By Theorem 5 there will be only three centers of elations with axis $x_3=0$. If we choose two of these to be

$$E_1=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix} \quad \text{and} \quad E_2=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

with centers at (100) and (010) respectively, their product will have (110) for center. The square of the product of E_1 and E_3 is an elation with center at (100) and axis $x_2=0$. Similarly every axis passing through (100) will have a center at that point. Hence by Theorem 5 there can be only three axes through (100). Since E_3 transforms $x_3=0$ into the same line into which E_2 transforms $x_2=0$, we must have

$$E_3=\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}$$

Since $(E_2E_3)^3=1$, the three elations generate the octahedral group. There will be two centers other than (100) on each of the axes through (100) and therefore 7 centers in all.

Since (100) is not to be left invariant, there will be still another elation. Its axis must meet $x_3=0$ at either (010) or (110), but since these points are conjugate we may assume that it passes through (010). This new axis

must meet each of the other axes at centers. Hence it will have a center at each of these points of intersection, and cannot have any others. Since the four centers not on $x_3 = 0$ are conjugate under the four group represented on $x_3 = 0$ by the identity, we may assume that the center of the additional elation is at (010) and the axis passes through (001). By using the fact that (100) and (110) must be interchanged, we get that

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Under the whole group (100) is conjugate with the 7 centers hence the order is 7 times 24 or 168. This group consists of all the transformations in the $GF(2)$ and may be identified with the $LF(3,2)$.

If the sub-group of commutative elations originally assumed had had the same center and different axes, similar reasoning would have led to the same group.

This G_{168} contains as a sub-group a dihedral G_6 leaving (100) and $x_1 = 0$ fixed. If the G_{168} were self-conjugate under a larger group, this G_6 would be self-conjugate under a sub-group of the larger group of order greater than 6. Since there are no homologies present and no additional elations, there must then be transformations of type I leaving fixed (100) and $x_1 = 0$ and permuting cyclicly the three centers of elations on $x_1 = 0$. The product of such a transformation, which is assumed not to be in the G_6, and one of the transformations of order three in the G_6, must be an homology, with center (100) and axis $x_1 = 0$. Hence there is no larger group, within the limitations of the theorem, under which the G_{168} is self-conjugate.

THEOREM 7. *In a group containing commutative elations and satisfying conditions A, B and C, a common axis of a commutative sub-group of elations is cut by the axes of all other elations either at a center or at one of two points that are not centers, and the elations in such a sub-group form with the identity a single four group. And, dually, in a group containing commutative elations and satisfying conditions A, B and C, every center of an elation lies either on one of the axes of a sub-group of elations having a common center or on one of two other lines through that center, and such a commutative sub-group of elations is a four group.*

We let l be the common axis of the elations forming a largest sub-group of commutative elations, say of order 2^k. Then there will be $2^k - 1$ centers of elations on l with l for axis. We let these centers be A_1, A_2, A_3, etc. If the axis l' of another elation meets l at one of these

centers, we have, by Theorem 5, that the elations with this axis form a four group. We assume then that l' meets l at some point B not a center of elation with l for axis. B then cannot be a center of elation, for the square of its product with an elation with l for axis would be an elation with center B and axis l. This additional elation must generate with the 2^k-1 commutative elations a group of order $(2^k+1)2^k(2^k-1)$ leaving B fixed and permuting 2^k+1 lines through B, each the axis of (2^k-1) elations. The sub-group leaving l fixed must be the metacyclic group of order $2^k(2^k-1)$, which must also leave fixed another point on l, C say. This group contains cyclic sub-groups of order 2^k-1, the transformations of which permute the 2^k-1 centers on l. Under the group of order $(2^k+1)2^k(2^k-1)$, C would be conjugate with one point on each of the 2^k+1 axes through B, and no three of these 2^k+1 conjugate points can be collinear. For 2^k-1 of them will lie on 2^k-1 distinct lines joining C with the 2^k-1 centers on l' and the other will be still another point on l', hence no line through C can pass through more than one other.

The group of order $(2^k+1)2^k(2^k-1)$ is represented on l by a binary group of order 2^k-1, permuting cyclicly the 2^k-1 centers, and leaving B fixed, and hence leaving fixed another point C on l. Any larger group containing this group as a sub-group must be represented on l by a binary group that is either cyclic or dihedral. For in order for a binary group of order $(2^m+1)2^m(2^m-1)$ to permute 2^k-1 points, $2^m+1 \leqq 2^k-1$ and, since there must be present a cyclic sub-group of order 2^k-1, we must have $m=1$ and the group is the dihedral G_6. A metacyclic group permutes 2^m points, and hence cannot be present.

Any additional elation whose axis does not meet l at B, C or one of the centers of elations will generate with the previously described elations a group of order $(2^m+1)2^m(2^m-1)$ on l which is impossible. Therefore all axes of elations must pass through B, C or one of the 2^k-1 centers on l.

Since an additional axis cannot meet more than two of the 2^k+1 axes through B in points conjugate to C, it must meet the other 2^k-1 in centers of elations. There are then, by Theorem 5, only three centers on each axis, that is $k=2$, and the sub-groups of commutative elations are four groups. The dual theorem follows from a similar line of reasoning.

THEOREM 8. *The only group, satisfying condition A, B, and C, and containing commutative elations, and in which an axis of elation meets a common axis of commutative elations at a non-center (or one in which a center of an elation lies on a line that is not an axis of one of a group of elations having a common center) is the* G^6_{360}.

In the notation of the preceding theorem, the group leaving B fixed is the icosahedral group G_{60}. In the G_{60} there are 15 elations with centers

arranged 3 each on 5 axes through B. If we choose for l the line $x_3 = 0$ and let the centers of the two elations E_1 and E_2 be (100) and (010) we may assume that

$$E_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We choose for center of an elation E_3 whose axis meets $x_3 = 0$ at a non-center as (001), and let its axis meet $x_3 = 0$ at the point $(1\,\lambda\,0)$. Since the icosohedral group is generated by E_1, E_2, and E_3, we choose E_3 so that $(E_1 E_3)^5 = 1$ and $(E_2 E_3)^3 = 1$.

From this we get that

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 1 & 1 \end{pmatrix}$$

and that $\lambda^2 + \lambda + 1 = 0$.

The axis of an additional elation E_4 must meet three of the axes through $(1\,\lambda\,0)$ in centers of elations, hence it will have centers at the points. And since only three centers can lie on an axis all the centers of elations must be at the 15 centers of the G_{60}. We take the center of E_4 to be (010) and assume that its axis meets the axis of E_3 at a center. Now the line from (010) to (001) passes through only three centers while the lines from (010) to the other two centers on the axis of E_3 each pass through five centers, so the axis of E_4 must be $x_1 = 0$. E_4 must interchange (100) and (110). From these conditions we get that

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This interchanges $(1\,\lambda\,0)$ and $(1\,\lambda^2\,0)$, which correspond to B and C respectively in Theorem 7. The point $(1\,\lambda\,0)$, which is left fixed by the G_{60}, is, therefore, conjugate with 5 other points under the whole group. Hence the order of the whole group is 6 times 60 or 360. It is simply isomorphic with the alternating group on 6 letters.

E_1 and E_3 generate a dihedral G_{10}, and by reasoning with this as we did with G_6 in Theorem 6 we conclude that the G_{360} is not self-conjugate under a larger group.

Groups containing no commutative elations and no homologies and those containing only transformations of type I.

The case where there are no commutative elations may be handled with the aid of Diophantine equations, but as the work would be the same

as that of H. H. Mitchell in his treatment of the odd prime case,* except that reflections would be replaced by elations, the work need not be given here. The only groups in this case that satisfy conditions A and C are the Hessians G_{36} and G_{72}.

The only remaining case, namely where all the transformations are of type I, is identical with the odd prime case,† hence there is no group present that does not leave invariant a point, line or triangle.

Summary of subgroups.‡

When k is even, 2^k-1 is divisible by 3, and 2^k+1 is not. When k is odd, the opposite is true. Therefore when k is odd the $LF(3, 2^k)$ consists of all the transformations with coördinates in the $GF(2^k)$, and when k is even the $LF(3, 2^k)$ is a self-conjugate sub-group, of index 3, of the group containing all the transformations in the $GF(2^k)$. When k is odd the $HO(3, 2^{2k})$ is a self-conjugate sub-group, of index 3, of a larger group. The sub-groups listed here are those determined in this paper and those that leave invariant a point, line or triangle. Those that are sub-groups of sub-groups are not listed.

The maximal sub-groups of the $LF(3, 2^k)$ are the following, where $\nu = 1$ or 3 according as k is odd or even.

1. Groups of order $(2^k+1)2^{3k}(2^k-1)^2/\nu$. Each of these groups leaves invariant a point and is isomorphic with a group of order $(2^k+1)2^k(2^k-1)$ permuting the lines through that point.

2. The dual of the above or groups of the same order, each of which leaves invariant a line and is isomorphic with the binary group permuting the points on that line.

3. Groups of order $6(2^k-1)^2/\nu$. Each of these groups leaves invariant a triangle with coördinates in the $GF(2^k)$ and makes all six permutations on its vertices.

4. Groups of order $3(2^{2k}+2^k+1)/\nu$. Each of these groups leaves invariant a triangle with coördinates in the $GF(2^{3k})$ but not in the $GF(2^k)$, and permutes its vertices cyclicly.

5. The linear fractional groups $LF(3, 2^m)$, where k/m is a prime.

6. Groups containing the $LF(3, 2^m)$ as self-conjugate sub-groups of index 3, if m is even and $k = 3m$.

7. The hyperorthogonal groups $HO(3, 2^{2m})$, where $2m = k$.

8. Groups containing the $HO(3, 2^{2m})$ as self-conjugate sub-groups of index 3, if m is odd and $k = 6m$.

* H. H. Mitchell, loc. cit., p. 219.

† loc. cit., p. 225.

‡ Cf. Mitchell, loc. cit., p. 239; Dickson, Linear Groups, chap. 11.

9. Groups of order 360 if $k = 2$.

The following are maximal sub-groups of the $HO(3, 2^{2k})$, where $\nu = 1$ or 3 according as k is even or odd.

1. Groups of order $(2^k+1)2^{3k}(2^k-1)/\nu$. Any such group leaves invariant the center and axis of a group of elations and is isomorphic with a metacyclic group on the axis of order $2^{2k}(2^{2k}-1)/\nu$.

2. Groups of order $(2^k+1)^2 2^k(2^k-1)/\nu$. Any such group leaves invariant the center and axis of an homology and is isomorphic with the binary group on that axis.

3. Groups of order $6(2^k+1)^2/\nu$. Any such group leaves invariant a triangle whose coördinates are in the $GF(2^{2k})$ and makes all six permutations on its vertices.

4. Groups of order $3(2^{2k}-2^k+1)/\nu$. Any such group leaves invariant a triangle whose coördinates lie in the $GF(2^{6k})$, but not in the $GF(2^{2k})$, and permutes its vertices cyclicly.

5. The hyperorthogonal groups $HO(3, 2^{2m})$ where m is a factor of k and k/m is an odd prime. When $m = 1$ this is the Hessian G_{72}.

6. Groups containing the $HO(3, 2^{2m})$ as self-conjugate sub-groups of index 3, if m is odd and $k = 3m$. When $m = 1$ this is the Hessian G_{216}.

7. A group of order 36, which exists as a maximal sub group when $k = 1$.

ON THE OSCILLATION OF HARMONIC FUNCTIONS.*

By Fred W. Perkins.

In the proof of a fundamental existence theorem in the theory of Abelian Integrals, Neumann† uses a property of harmonic functions the formulation of which is facilitated by the

Definition: *Given a function u finite and harmonic within a circle C, let $Du|_C$, the oscillation of u on C, be defined as the difference between the upper and lower bounds of u within C.*

This clearly agrees with the usual definition in cases where the latter is applicable.

Theorem of oscillation: *Given any two concentric circles Γ, C of radii a, A respectively $(a<A)$; and any function u finite and harmonic within C, then*

$$Du|_\Gamma \leqq \frac{4}{\pi}\tan^{-1}\frac{a}{A}\,Du|_C.$$

If u is further continuous on C, then the lower sign can hold only in the trivial case that u is a constant.

Neumann states and proves this merely for the case that u is continuous on C, and does not exclude the lower sign in any case.

Professor Osgood‡ has given, by modern methods, a proof of that part of the result which alone is actually needed in the application to the existence theorem. He has shown that, under the hypotheses of Neumann, there exists a positive quantity q, less than unity, depending on Γ and C, but not on u, such that

$$Du|_\Gamma \leqq q\,Du|_C.$$

In § 1 I give a prove of the theorem (as formulated above in italics) which is based directly on Poisson's Integral. I also show that although

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* Presented to the American Mathematical Society, December 30, 1924.

† Abelsche Integrale, pp. 412–417. Ritter, Math. Ann., vol. 45 (1894), p. 485 ff., gives a result similar to Neumann's, and treats a problem remotely resembling that of § 3 of this paper. Harnack, Logarithmische Potentiale, p. 63 ff., gives a theorem which, interpreted geometrically, yields a special case of the theorem of § 2. See also E. R. Neumann, Math. Ann., vol. 56 (1902), p. 107, § 11, and Carathéodory, Mathematische Abhandlungen, H. A. Schwarz gewidmet, p. 22, § 6.

‡ Annals of Math., (2), vol. 25 (1924), p. 238.

the equality sign in this relation can never hold when u is continuous on C but not a constant, this appraisal of $Du|_\Gamma$ is the best which is applicable to all such functions.

In § 2 and § 3 I give certain further theorems regarding the oscillation of harmonic functions, which are believed to be new.

I am grateful to Professor Osgood for many helpful suggestions.

1. In proving Neumann's theorem, let us first assume that the function u is continuous on C. We see at once that the theorem is true when u is a constant, so we may exclude this case from further consideration.

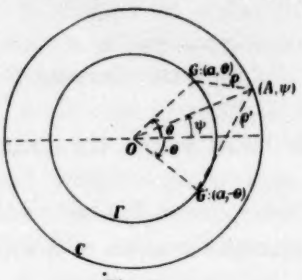

Fig. 1.

Denote the maximum and minimum values of u on C by M, m, respectively. Noting that $M - m \neq 0$, let

$$u' = \frac{u - m}{M - m} \quad \text{or,} \quad u = m + u' Du|_C.$$

Let G, G' be any two distinct points on the circle Γ; set up a polar coördinate system with pole at O, the center of C, and a prime vector so chosen that the polar coördinates of G, G' are respectively (a, θ) and $(a, -\theta)$ where $0 < \theta < \pi$.

Let

$$\varrho = \sqrt{A^2 - 2Aa\cos(\theta - \psi) + a^2}; \quad \varrho' = \sqrt{A^2 - 2Aa\cos(-\theta - \psi) + a^2}.$$

Then by Poisson's Integral,

$$u'(G) - u'(G') = \frac{A^2 - a^2}{2\pi} \int_0^{2\pi} u'|_C \left(\frac{1}{\varrho^2} - \frac{1}{\varrho'^2}\right) d\psi = I_1 + I_2$$

where

$$I_1 = \frac{A^2 - a^2}{2\pi} \int_0^{\pi} u'|_C \left(\frac{1}{\varrho^2} - \frac{1}{\varrho'^2}\right) d\psi, \quad I_2 = \frac{A^2 - a^2}{2\pi} \int_\pi^{2\pi} u'|_C \left(\frac{1}{\varrho^2} - \frac{1}{\varrho'^2}\right) d\psi.$$

Now

$$\frac{1}{\varrho^2} - \frac{1}{\varrho'^2} = \frac{2Aa}{\varrho^2 \varrho'^2} [\cos(\theta - \psi) - \cos(\theta + \psi)] = \frac{4Aa}{\varrho^2 \varrho'^2} \sin\theta \sin\psi,$$

and so

$$\begin{cases} \dfrac{1}{\varrho^2} - \dfrac{1}{\varrho'^2} > 0 & \text{when } 0 < \psi < \pi, \\ \dfrac{1}{\varrho^2} - \dfrac{1}{\varrho'^2} < 0 & \text{when } \pi < \psi < 2\pi. \end{cases}$$

Denoting the polar coördinates of an arbitrary point P within C by (r, φ), set*

$$U(P) = \frac{1}{2\pi}\int_0^{\pi} \frac{A^2-r^2}{A^2-2Ar\cos(\varphi-\psi)+r^2}\,d\psi = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{2Ar\sin\varphi}{A^2-r^2}.$$

We have

$$I_1 \leqq U(G)-U(G'); \qquad I_2 \leqq 0.$$

Moreover, since u is continuous on C, the lower sign cannot hold in both these relations. Consequently,†

$$u(G)-u(G') = [u'(G)-u'(G')]\,Du|_C < [U(G)-U(G')]\,Du|_C \leqq DU|_\Gamma \cdot Du|_C.$$

In particular, if we choose as the points G, G', points at which u takes on respectively its maximum and minimum values on Γ, we have

$$u(G)-u(G') = Du|_\Gamma,$$

and so

$$Du|_\Gamma < DU|_\Gamma \cdot Du|_C.$$

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*The evaluation of the integral here given can be obtained in the following manner: Let

$$V = \frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{A^2-r^2}{A^2-2Ar\cos(\varphi-\psi)+r^2}\,d\psi = \frac{1}{2\pi}\int_0^{\pi} \frac{A^2-r^2}{A^2+2Ar\cos(\varphi-\psi)+r^2}\,d\psi,$$

and note that

$$\begin{cases} U-V = \dfrac{1}{2\pi}\displaystyle\int_0^{\pi} \frac{4Ar(A^2-r^2)\cos(\varphi-\psi)}{(A^2-r^2)^2+4A^2r^2\sin^2(\varphi-\psi)}\,d\psi = \dfrac{2}{\pi}\tan^{-1}\dfrac{2Ar\sin\varphi}{A^2-r^2}, \\ U+V = 1. \end{cases}$$

On adding, and dividing by 2, we obtain the desired result.

It will be noted that U is bounded $(0<U<1)$ and harmonic, within C.

†We have

$$U(G)-U(G') = \frac{2}{\pi}\tan^{-1}\frac{2Aa\sin\theta}{A^2-a^2} \gtrless 0.$$

The polar coördinate system, and hence also the function U, depend on the choice of the points G, G'. If we interchange the rôles of G and G' in the entire discussion, the effect is clearly to replace θ by its supplement. Hence the value of $U(G)-U(G')$ is not changed. However, the sign of the quantity $u(G)-u(G')$ is reversed. This clearly means that the relation

$$u(G)-u(G') < [U(G)-U(G')]\,Du|_C$$

implies

$$|u(G)-u(G')| < [U(G)-U(G')]\,Du|_C.$$

Since, however, we are interested at present in a case in which $u(G)-u(G')>0$, we do not need to use the absolute value signs here.

Since U assumes its greatest and its least values on Γ when $\varphi = \frac{\pi}{2}$, $\varphi = \frac{3\pi}{2}$, respectively, we have

$$DU|_\Gamma = \frac{2}{\pi} \tan^{-1} \frac{2Aa}{A^2 - a^2} = \frac{4}{\pi} \tan^{-1} \frac{a}{A},$$

and consequently

$$Du|_\Gamma < \frac{4}{\pi} \tan^{-1} \frac{a}{A} Du|_C.$$

This completes the proof for the case that u is continuous on C.

In the case that the function u is finite and harmonic within C, but discontinuous on C, draw a circle C_1, concentric with the given circle, and of radius A_1 so chosen that $a < A_1 < A$.

The point $P:(r, \varphi)$ being any point within C_1, let

$$U_1(P) = \frac{1}{2\pi} \int_0^{\pi} \frac{A_1^2 - r^2}{A_1^2 - 2A_1 r \cos(\varphi - \psi) + r^2} d\psi.$$

Then

$$|u(G) - u(G')| < [U_1(G) - U_1(G')] Du|_{C_1} \leqq \frac{4}{\pi} \tan^{-1} \frac{a}{A_1} Du|_{C_1}.$$

Now allow A_1 to approach A. Then $U_1(G)$, $U_1(G')$ approach $U(G)$, $U(G')$ respectively. $Du|_{C_1}$ clearly approaches the difference between the upper and the lower bounds of u within C, that is, $Du|_C$. Hence

$$|u(G) - u(G')| \leqq [U(G) - U(G')] Du|_C \leqq \frac{4}{\pi} \tan^{-1} \frac{a}{A} Du|_C.$$

Specializing the points G, G' as in the previous case, we obtain

$$Du|_\Gamma \leqq \frac{4}{\pi} \tan^{-1} \frac{a}{A} Du|_C.$$

This completes the proof of the theorem.

The points G, G' in C being given, the number $U(G) - U(G')$ is the smallest possible choice of the constant k such that the inequality

$$|u(G) - u(G')| < k Du|_C$$

holds for all functions u, other than constants, which are harmonic within C and continuous on C.

To prove this, we will show that, given any pair of points $G:(a, \theta)$, $G'(a, -\theta)$ and any $\epsilon > 0$, there exists a $\delta > 0$ such that the function

$$u = u_2 = \frac{1}{2\pi}\int_0^{\pi} \frac{(A+\delta)^2 - r^2}{(A+\delta)^2 - 2(A+\delta)\, r\cos(\varphi - \psi) + r^2}\, d\psi$$

satisfies the relation

$$u_2(G) - u_2(G') > \{[U(G) - U(G')] - \epsilon\}\, Du_2|_C.$$

It is sufficient to show that, for a suitable choice of δ,

$$\frac{[U(G) - U(G')]\, Du_2|_C - [u_2(G) - u_2(G')]}{Du_2|_C} < \epsilon,$$

that is,

$$\frac{\frac{2}{\pi}\tan^{-1}\frac{2Aa\sin\theta}{A^2 - a^2}\cdot\frac{4}{\pi}\tan^{-1}\frac{A}{A+\delta} - \frac{2}{\pi}\tan^{-1}\frac{2(A+\delta)a\sin\theta}{(A+\delta)^2 - a^2}}{\frac{4}{\pi}\tan^{-1}\frac{A}{A+\delta}} < \epsilon.$$

Now the function

$$f(x) = \frac{\frac{2}{\pi}\tan^{-1}\frac{2Aa\sin\theta}{A^2 - a^2}\cdot\frac{4}{\pi}\tan^{-1}\frac{A}{x} - \frac{2}{\pi}\tan^{-1}\frac{2xa\sin\theta}{x^2 - a^2}}{\frac{4}{\pi}\tan^{-1}\frac{a}{x}}$$

is continuous at $x = A$ and vanishes there. Consequently, there exists a positive δ so small that $f(A+\delta)$ (which is precisely the left hand member of the last inequality above) is less than ϵ.

In particular, if we choose G, G' so that $\theta = \frac{\pi}{2}$, then

$$u_2(G) - u_2(G') = Du_2|_\Gamma, \qquad U(G) - U(G') = DU|_\Gamma = \frac{4}{\pi}\tan^{-1}\frac{a}{A},$$

and so in this case we know that for any given circle Γ and any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$Du_2|_\Gamma > \left\{\frac{4}{\pi}\tan^{-1}\frac{a}{A} - \epsilon\right\} Du_2|_C.$$

Hence, the circles Γ, C being given, the quantity $\frac{4}{\pi}\tan^{-1}\frac{a}{A}$ is the best possible choice for a constant q such that the relation

$$Du|_\Gamma < q\, Du|_C$$

holds for all functions u harmonic within C and continuous on C, excepting constants.

2. THEOREM. *Let P_1, P_2 be any two points within a circle C, and draw the two circles C_1, C_2 through P_1 and P_2 which are tangent to C. Denote by A_1, A_2, the angles subtended in these circles by the chord P_1P_2, the vertices being at the points of tangency of C with C_1, C_2, respectively. Let u be any function which is finite and harmonic within C. Then*

$$|u(P_1)-u(P_2)| \leqq \frac{A_1+A_2}{\pi} Du|_C.$$

If P_1, P_2 are distinct, and u is continuous on C, but not a constant, then the upper sign always holds.

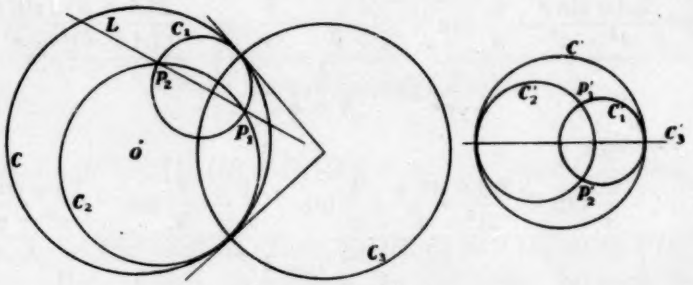

Fig. 2.

Let us exclude for the present the case that P_1 and P_2 are equidistant from O, the center of C. Then the common tangent of the circles C, C_1 is not parallel to that of the circles C, C_2.

With center at the intersection of these tangents draw the circle C_3 passing through the points of tangency. Invert in a circle with center at any point on C_3 which is exterior to C. C goes into a circle C'; C_3 goes into a straight line C_3' orthogonal to C', that is, a diameter of C'; C_1, C_2 go into circles C_1', C_2' tangent internally to C' at the points where C_3' cuts C'; P_1, P_2 are carried respectively into P_1', P_2', the intersections of C_1', C_2'.

Suppose that we are given a function u finite and harmonic within C. This is transformed by the inversion described above into a function v finite and harmonic within C'.

Now P_1', P_2' are clearly equidistant from the center of C', and so

$$|v(P_1')-v(P_2')| \leqq [U(P_1')-U(P_2')]\, Dv|_{C'}.$$

(The function U is now formed for the circle C'; the points P_1', P_2' play the rôles of G, G' respectively.)

If we again invert, so as to return to the original figure, v is carried back into u; U is carried into a function U^* which is bounded and harmonic within C, taking on the boundary value 1 on one of the arcs into which C is divided by the points of tangency of C_1, C_2, (that which lies on the same side of C_3 as does P_1), and the boundary value 0 on the remainder of C. Hence,

$$|u(P_1)-u(P_2)| \leqq [U^*(P_1)-U^*(P_2)]\, Du|_C.$$

If P_1, P_2 are equidistant from O, we define the function U^* to be the function U, formed for the circle C, the points P_1, P_2 playing the rôles of G, G' respectively. The relation above and the remainder of the discussion apply to this special case, as well as to the general case.

In the next part of the proof, we shall use Bôcher's interpretation of Poisson's Integral.†

Given a point P within a circle C, make an inversion with center at the point conjugate to P with respect to C. C is transformed into a circle which has its center at that point into which the point P is carried. This enables us to state the following generalization of Gauss' Mean Value Theorem:

Let P be any point within a circle C, and Q any point on C. Draw c, that circle or straight line through P and Q which cuts C orthogonally. Denote by $\overset{\smile}{PQ}$ that directed arc on c which joins P and Q and lies within C.

Choosing some fixed point Q_0 on C, let ψ be the directed angle which the arc $\overset{\smile}{PQ}$ forms with the arc $\overset{\smile}{PQ_0}$. Let v be any function finite and harmonic within C, assuming the boundary values‡ $v(Q)$ on C. Then the value of v at P is given by

$$v(P) = \frac{1}{2\pi}\int_0^{2\pi} v(Q)\, d\psi.$$

We now turn to the proof of the following lemma:

Let $\overset{\smile}{RS}$ be an arc on a circle C and P_1, P_2 a pair of points within C so situated that the points R, S lie on opposite sides of the line P_1, P_2; let v be the function which is finite and harmonic within C, and takes on the

† Bôcher, Bull. Amer. Math. Soc., (2), vol. 4 (1897–98), p. 424; Annals of Math., (2), vol. 7, (1906), p. 92. Osgood, Lehrbuch der Funktionentheorie, 2d edition, p. 635.

‡ These boundary values need not be continuous. From the method by which the theorem is obtained the result is clearly just as general in this respect as is Gauss' Mean Value Theorem. It is sufficient if we can divide the circle C into a finite number of arcs on each of which the boundary values are continuous.

boundary value 1 *on the arc* $\overset{\frown}{RS}$ *and* 0 *on the remainder of the circle. Denote by* A_1, A_2, *respectively, the undirected angles* $\sphericalangle P_1RP_2$ *and* $\sphericalangle P_1SP_2$. *Then*

$$|v(P_1) - v(P_2)| = \frac{A_1 + A_2}{\pi}.$$

Let P be an arbitrary point within C, and O the center of C. Draw the lines OR, OS, PR, PS. Draw the circular arcs through P meeting C orthogonally. Let α be the angle subtended by arc $\overset{\frown}{RS}$, from P. Denote by χ, ω the angles $\sphericalangle ORP$, $\sphericalangle OSP$.

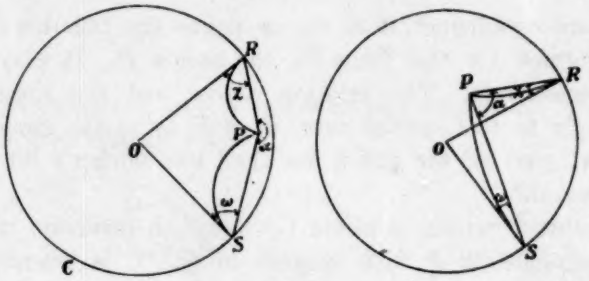

Fig. 3.

We will make the following conventions as to signatures: α is always positive; χ is positive if P lies on the same side of the diameter through R as does the arc $\overset{\frown}{RS}$,* negative if P lies on the other side of this diameter; the signature of ω depends in like manner on the relative positions of P and $\overset{\frown}{RS}$ with respect to the diameter through S.

Then from Bôcher's theorem, it follows that

$$v(P) = \frac{\alpha + \chi + \omega}{2\pi}.$$

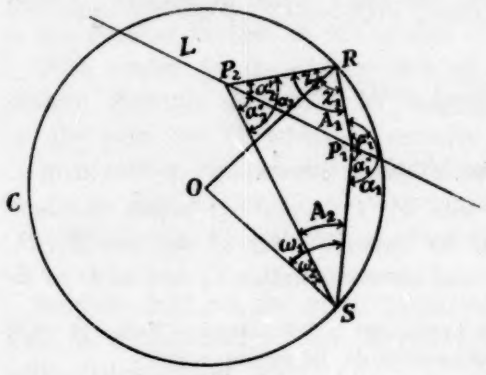

Fig. 4.

Suppose now that we consider two points P_1, P_2 as specified in the hypothesis of the theorem. We will designate by α_1, χ_1, ω_1, the quantities α, χ, ω, for the case that P is choosen as P_1, and by α_2, χ_2, ω_2, the same quantities for the point P_2. Since the points R, S lie on opposite sides of the line L through P_1 and P_2, this line divides

* If $\overset{\frown}{RS}$ is greater than a semicircle, we consider (in determining the sign of χ) merely that part of $\overset{\frown}{RS}$ which lies near R.

the angle α_1 into two parts, α_1' and α_1'', and the angle α_2 into two parts, α_2' and α_2'', as indicated in the figure.

Now

$$v(P_1) - v(P_2) = \frac{1}{2\pi}\{[\alpha_1 + \chi_1 + \omega_1] - [\alpha_2 + \chi_2 + \omega_2]\}$$

$$= \{[(\alpha_1' - \alpha_2') + (\chi_1 - \chi_2)] + [(\alpha_1'' - \alpha_2'') + (\omega_1 - \omega_2)]\}.$$

But (using the conventions regarding signatures)

$$\begin{cases} \alpha_1' - \alpha_2' = \chi_1 - \chi_2 = \pm A_1 \\ \alpha_1'' - \alpha_2'' = \omega_1 - \omega_2 = \pm A_2. \end{cases}$$

Furthermore, since R, S lie on opposite sides of L, it follows that when a point P traces this line, its angles χ, ω either increase together, or decrease together, and so $\chi_1 - \chi_2$ and $\omega_1 - \omega_2$ have the same sign. Consequently,

$$|v(P_1) - v(P_2)| = \frac{A_1 + A_2}{\pi}.$$

To apply this lemma to the difference† $U_1^*(P_1) - U_1^*(P_2)$ we need to know that the points of tangency of C with C_1, C_2 lie on opposite sides of the line L. If we invert in a circle with center at P_1, L is carried into itself; C is carried into a circle cutting L; C_1, C_2 are carried into a pair of tangents to this circle, intersecting on L. In the new figure the points of tangency lie on opposite sides of L, hence the same property holds in the original figure.

We have now, for any function u finite and harmonic within C,

$$|u(P_1) - u(P_2)| \leqq \frac{A_1 + A_2}{\pi} Du|_C.$$

From the connection between this general formula and the relation

$$|u(G) - u(G')| \leqq [U(G) - U(G')]\, Du|_C$$

it is readily seen that although the lower sign cannot hold if u is continuous on C, but not a constant, this relation gives the best possible appraisal for $|u(P_1) - u(P_2)|$ which is applicable to all such functions.

† We know that $U^*(P_1) - U^*(P_2) = U(P_1') - U(P_2') > 0$, or else, in a special case, $U^*(P_1) - U^*(P_2) = U(P_1) - U(P_2) > 0$. Hence we do not need absolute value signs here.

3. THEOREM. *Let O be the center and A the radius of a circle C, and let P be a point within C at a distance ϱ from O. Let u be any function finite and harmonic within C, and denote by $\left[\frac{du}{d\xi}\right]_P$ the directional derivative of u at P in an arbitrary direction ξ. Then*

$$\left|\frac{du}{d\xi}\right|_P \leqq \frac{2}{\pi}\,\frac{A}{A^2-\varrho^2}\,Du|_C.$$

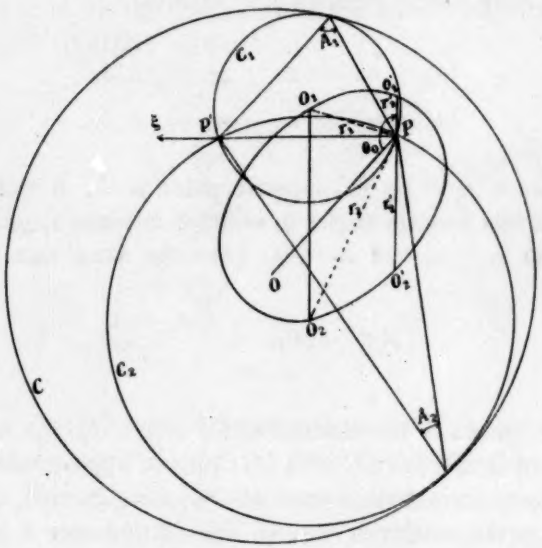

Fig. 5.

We shall obtain this theorem from the result of § 2 by a limiting process. Draw a ray through P in the direction ξ, and let P' be a point near P on this ray. Denoting the length of the line segment PP' by $\Delta\xi$ we have

$$\left|\frac{du}{d\xi}\right|_P = \lim_{P'\to P}\frac{|u(P')-u(P)|}{\Delta\xi}.$$

We will use the same notation as in the preceding section, except that P', P replace P_1, P_2 respectively. Denote by r_1, r_2 the radii of C_1, C_2 and by s_1, s_2 the arcs intercepted by the angles A_1, A_2 in the circles C_1, C_2 respectively.

It is readily shown that although the circles C_1, C_2 do not remain fixed when P' approaches P, we still have*

* $s_1 = 2r_1A_1$, $\Delta\xi = 2r_1\sin A_1$, and $\lim_{P'\to P} A_1 = 0$; hence $\lim_{P'\to P}\frac{s_1}{\Delta\xi} = \lim_{A_1\to 0}\frac{A_1}{\sin A_1} = 1$. Similarly, $\lim_{P'\to P}\frac{s_2}{\Delta\xi} = 1$.

$$\lim_{P' \to P} \frac{s_1}{\Delta \xi} = 1, \quad \text{and} \quad \lim_{P' \to P} \frac{s_2}{\Delta \xi} = 1.$$

By the theorem of § 2

$$\frac{|u(P') - u(P)|}{\Delta \xi} \leqq \frac{1}{\pi} \left[\frac{A_1}{\Delta \xi} + \frac{A_2}{\Delta \xi} \right] Du|_C,$$

and so

$$\left| \frac{du}{d\xi} \right|_P \leqq \frac{1}{\pi} \left[\lim_{P' \to P} \frac{A_1}{s_1} + \lim_{P' \to P} \frac{A_2}{s_2} \right] Du|_C.$$

provided these limits exist.

Since the centers of C_1 and C_2 are each equidistant from the point P and the circle C, we see from a well-known theorem* that they lie on an ellipse of major axis A, with foci at O and P. From the fact that C_1, C_2 pass through P', wee see that their centers, O_1, O_2 are those points in which the ellipse is cut by the perpendicular bisector of PP'. When P' approaches P, these intersections approach O_1', O_2', the points of intersection of the ellipse and the line through P normal to the direction ξ. These points are the centers of the circles tangent to the circle C and to the direction ξ at P. Denote the distances from P to these last points by r_1' and r_2'. The expressions

$$\frac{A_1}{s_1} = \frac{1}{2} \frac{1}{r_1}, \qquad \frac{A_2}{s_2} = \frac{1}{2} \frac{1}{r_2}$$

approach $\frac{1}{2} \frac{1}{r_1'}, \frac{1}{2} \frac{1}{r_2'}$ respectively. Consequently,

$$\left| \frac{du}{d\xi} \right|_P \leqq \frac{1}{2\pi} \left[\frac{1}{r_1'} + \frac{1}{r_2'} \right] Du|_C.$$

Now the value of $\frac{1}{r_1'} + \frac{1}{r_2'}$ is in fact independent of the direction ξ. In proving this, let us consider first the case that P is distinct from O. Set up a polar coördinate system with pole at P and the ray PO as prime vector. In this case, the ellipse described above has distinct foci, and so, in the usual notation, its polar coördinate equation is†

$$r = \frac{em}{1 - e \cos \theta}.$$

O_1', O_2' have polar coördinates of the form (r_1', θ_0), $(r_2', \theta_0 + \pi)$.

* See, for instance, Bôcher, Plane Analytic Geometry, p. 125.

† We make no use of the fact that $e < 1$. In any conic the sum of the reciprocals of the segments into which a focus divides a focal chord is constant.

Hence

$$\left[\frac{1}{r_1'}+\frac{1}{r_2'}\right]=\left[\frac{1-e\cos\theta_0}{em}+\frac{1-e\cos(\theta_0+\pi)}{em}\right]=\frac{2}{em},$$

and so is independent of the direction ξ.

If P and O are coincident, the ellipse becomes a circle. r_1', r_2' are each radii of this circle, and so $\frac{1}{r_1'}+\frac{1}{r_2'}$ is independent of ξ.

We may evaluate $\frac{1}{r_1'}+\frac{1}{r_2'}$ by specializing the direction ξ. If we choose ξ perpendicular to OP, we find

$$\frac{1}{r_1'}+\frac{1}{r_2'}=\frac{1}{\frac{1}{2}(A+\varrho)}+\frac{1}{\frac{1}{2}(A-\varrho)}=\frac{4A}{A^2-\varrho^2}.$$

Consequently,

$$\left|\frac{du}{d\xi}\right|_P\leq\frac{2}{\pi}\,\frac{A}{A^2-\varrho^2}\,Du|_C.$$

HARVARD UNIVERSITY,
January 31, 1925.

TRANSFORMATION OF THE KUMMER CRITERIA IN CONNECTION WITH FERMAT'S LAST THEOREM.*

By H. S. Vandiver.

If

$$u^p + v^p + w^p = 0 \tag{1}$$

is satisfied in integers prime to the odd prime p, it is known that

$$B_n f_{p-2n}(t) \equiv 0, \qquad f_{p-1}(t) \equiv 0 \pmod{p} \tag{2}$$

where $-t \equiv u/v,\ v/u,\ u/w,\ w/u,\ v/w,\ w/v$ modulo p, $f_a(t) = \sum_{i=1}^{p-1} i^{a-1} t^i$ $(a>1,\ n = 1, 2, \cdots, (p-3)/2)$ and $B_1 = 1/6$, $B_2 = 1/30$, etc., are the numbers of Bernoulli. The congruences involving the B's are generally referred to as the Kummer criteria for the solution of (1). In the present paper transformations of (2) will be obtained which are useful in deriving criteria of the type $m^{p-1} \equiv 1 \pmod{p^2}$ although none of this type will be considered here. In particular, if

$$h_n^{(k,l)}(x) = \sum_{s=0}^{p-1} (ks+l)^{n-1} x^s,$$

it is shown that, if (1) is satisfied in integers prime to the odd prime p, then

$$\sum_{l=0}^{k-1} h_n^{(k,l)}(t)\, h_{p-n}^{(k,k-l)}(t) \equiv 0 \pmod{p}, \qquad n = 1, 2, \cdots, p-1, \tag{3}$$

k being any positive integer. For $k = 1$ this reduces to

$$f_n(t)\, f_{p-n}(t) \equiv 0 \pmod{p} \tag{4}$$

which is due to Mirimanoff.†

For $k = 2$ we have

$$h_n^{(2,1)}(t)\, h_{p-n}^{(2,1)}(t) \equiv 0 \pmod{p} \tag{5}$$

after using (4). The relation (5) was previously given by the writer‡ without proof.

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* Presented, in part, to the American Mathematical Society, 1917. The author was enabled to complete this investigation through a grant, relieving him of the duties of teaching, from the Heckscher Foundation for the Advancement of Research established by August Heckscher at Cornell University.

† Journ. für Math., 128, 1905, 45–68.

‡ Bull. Amer. Math. Soc., 28, 1922, 258.

In addition there is derived the fundamental identical congruence (11). This was used in different forms by the writer, Frobenius and Pollazek* in obtaining various criteria of the type $m^{p-1} \equiv 1 \pmod{p^2}$. For $k = 1$ it reduces to a formula employed by Mirimanoff† in obtaining $2^{p-1} \equiv 3^{p-1} \equiv 1 \pmod{p^2}$. The derivation here is much simpler than in any of the previous papers, only the well known property of the Bernoulli numbers

$$1^n + 2^n + \cdots + (p-1)^n \equiv b_n p \pmod{p^2}, \qquad (n < p-1)$$

being employed.

1. Set

$$f_1(x) = \frac{x^p - 1}{x - 1}, \qquad F_1(x) = \frac{x^{pm} - 1}{x - 1},$$

$$F_n(x) = \sum_{r=0}^{pm-1} r^{n-1} x^r \quad (n > 1),$$

where m is any integer. We have the obvious identity

$$\text{(5a)} \qquad \frac{x - y^k}{(x-1)(y-1)} = \frac{y^k}{y-1} - \frac{x(y^k - 1)}{(x-1)(y-1)}.$$

Multiplying by $(x^p - 1)(y^{pm} - 1)$ we have

$$(x - y^k) f_1(x) F_1(y) = y^k (x^p - 1) F_1(y) - x(y^{pm} - 1) f_1(x) \frac{y^k - 1}{y - 1}.$$

Let $x = e^{vk} x$, $y = e^v y$, divide by e^{vk}, differentiate a times with respect to v, put $v = 0$ and the result may be written

$$\text{(6)} \qquad \begin{aligned} &(x - y^k) \sum_{n=0}^{a} \binom{a}{n} k^{a-n} f_{a+1-n}(x) F_{n+1}(y) \\ &\qquad \equiv y^k (x^p - 1) F_{a+1}(y) + a k p y^k x^p F_a(y) \\ &\qquad - x(y^{pm} - 1) \sum_{l=0}^{k-1} y^l h_{a+1}^{(k,l)}(x) - a x p m y^{pm+l} \sum_{l} h_a^{(k,l)}(x) \\ &\qquad + p^2 A(x, y), \end{aligned}$$

where $A(x, y)$ is an integral function of x and y with rational integral coefficients. Assume $a < p$. Let ϱ be any mth root of unity. We have

$$F_n(\varrho) = \sum_{i=0}^{m-1} \sum_{r=0}^{p-1} (pi + r)^{n-1} \varrho^{pi+r},$$

* Journ. für Math., 144, 1914, 314-318 for $k = 2$; Berlin. Sitzungsber., 1914, 653-681; Wien, Sitzungsber., 126. IIa, 1917, 45-59.

† Journ. für Math., 139, 1911, 304-309.

whence

$$F_n(\varrho) = f_n(\varrho)\sum_{s=0}^{m-1}\varrho^{ps} + (n-1)p\sum_{s=1}^{m-1} s\,\varrho^{sp} f_{n-1}(\varrho) + p^2 C(\varrho), \tag{7}$$

where $C(\varrho)$ may be put in the form $c_0 + c_1\varrho + c_2\varrho^2 + \cdots + c_{m-1}\varrho^{m-1}$ in which the c's are rational integers. Hence for $n < p$ and for any of the m distinct values of ϱ we have

$$F_n(\varrho) = pC_1(\varrho), \tag{7a}$$

where C_1 has the same general form as C. In (6) set $y = \varrho$; it becomes in view of (7a)

$$\begin{aligned}(x-\varrho^k)\sum_{n=0}^{a}\binom{a}{n}k^{a-n} f_{a+1-n}(x)\,F_{n+1}(\varrho) \\ = \varrho^k(x^p-1)\,F_{a+1}(\varrho) - axpm\varrho^l h_a^{(k,l)}(x) + p^2 D(x,\varrho),\end{aligned} \tag{8}$$

where D is a polynomial in x whose coefficients have the same form as C. If P is any rational integer and $R(x)$ and $S(x)$ are polynomials in x with rational integral coefficients, such that $R(x) - S(x) = P \cdot M(x)$, $M(x)$ having rational integral coefficients, then we employ the notation $R(x) \equiv S(x) \pmod P$. Multiply both sides of (8) by $(x^m-1)/(x-\varrho^k)$, let ϱ range over all its m distinct values, and sum each member with respect to ϱ. The resulting expressions are polynomials in x with integral coefficients so we may write

$$\begin{aligned}(x^m-1)\sum_{n=0}^{a}\binom{a}{n}k^{a-n} f_{a+1-n}(x)\sum_{\varrho}F_{n+1}(\varrho) \\ \equiv \sum_{\varrho}(x^m-1)\frac{\varrho^k(x^p-1)\,F_{a+1}(\varrho)}{x-\varrho^k} - axpm(x^m-1)\sum_{\varrho}\sum_{l=0}^{k-1}\frac{\varrho^l h_a^{(k,l)}(x)}{x-\varrho^k}\end{aligned} \tag{9}$$

modulo p^2. We also have

$$\sum_{\varrho}F_n(\varrho) \equiv m^n\sum_{r=1}^{p-1} r^{n-1} \pmod{p^2}$$

for $n > 1$, or

$$\sum_{\varrho}F_n(\varrho) \equiv m^n b_{n-1}p \pmod{p^2} \tag{9a}$$

which is also true for $n = 1$ if $b_0 = 1$, $b_1 = -1/2$, $b_{2i} = (-1)^{i-1}B_i$, $b_{2i+1} = 0$, $i > 0$. Making use of this and transposing the term on the left side of (9) which involves $F_{a+1}(\varrho)$, we have, after dividing by p,

$$(x^m-1)\sum_{n=0}^{a-1}\binom{a}{n}k^{a-n}m^{n+1}b_n f_{a+1-n}(x)$$

$$(10)\qquad \equiv \sum_{\varrho}\frac{x(\varrho^k-1)(x^p-1)(x^m-1)F_{a+1}(\varrho)}{p(x-\varrho^k)(x-1)}$$

$$-axm(x^m-1)\sum_{\varrho}\sum_{l=0}^{k-1}\frac{\varrho^l h_a^{(k,l)}(x)}{x-\varrho^k}\pmod p.$$

In this relation put $a=p-1$ and use $\binom{p-1}{n}\equiv(-1)^n \pmod p$; we then obtain for m prime to p

$$(11)\qquad (x^p-1)G_m^{(k)}(x)-(x^m-1)K_m^{(k)}(x)\equiv H_m^{(k)}(x)\pmod p,$$

where

$$K_m^{(k)}(x)=-\sum_{n=0}^{p-2}\left(-\frac{m}{k}\right)^n b_n f_{p-n}(x),$$

$$G_m^{(k)}(x)=\frac{x(x^m-1)}{x-1}\sum_{\varrho}\frac{F_p(\varrho)(1-\varrho^k)}{mp(x-\varrho^k)},$$

$$H_m^{(k)}(x)=x(x^m-1)\sum_{l=0}^{k-1}\sum_{\varrho}\frac{\varrho^l h_a^{(k,l)}(x)}{x-\varrho^k}.$$

We now apply this identity to find criteria for (1). Setting $x=t$ and using (2) we have

$$(t^m-1)K_m^{(k)}(t)\equiv 0\pmod p$$

if we note also that

$$f_p(t)\equiv\frac{t^p-1}{t-1}-1\equiv 0\pmod p,$$

since $t\not\equiv 1 \pmod p$. Now, if $t\equiv -u/v \pmod p$, then $-u/v$, $-v/u$, $-u/w$, $-w/u$, $-v/w$, $-w/v$ may be written in the form

$$(11a)\qquad t,\ \frac{1}{t},\ 1-t,\ \frac{1}{1-t},\ \frac{t}{1-t},\ \frac{1-t}{t}$$

modulo p. Hence from (11) it follows that

$$(12)\qquad (z^p-1)G_m^{(k)}(z)\equiv H_m^{(k)}(z)\pmod p$$

is satisfied by each of the expressions in (11a). The relations (11) and (12) were employed* by Frobenius, Pollaczek and the writer in obtaining criteria of the type $m^{p-1}\equiv 1 \pmod{p^2}$ for small values of m.

* See the references at the beginning of this paper. The writer sent his paper containing the proof $5^{p-1}\equiv 1 \pmod{p^2}$ to Prof. Hensel in March 1914 for publication in the

Since, if $\varrho \neq 1$ and $m \not\equiv 0 \pmod{p}$,

$$\frac{m}{\varrho^p - 1} = \varrho^p + 2\varrho^{2p} + \cdots + (m-1)\varrho^{(m-1)p},$$

we have from (7)

$$F_n(\varrho) = (n-1)p \frac{m f_{n-1}(\varrho)}{\varrho^p - 1} + p^2 C(\varrho). \tag{13}$$

Substituting this in the value of G we have, letting $\sum'$ indicate summation over all values of $\varrho \neq 1$,

$$G_m^{(k)}(x) \equiv \frac{x(x^m-1)}{x-1} \sum' \frac{(1-\varrho^k) f_{p-1}(\varrho)}{(1-\varrho^p)(x-\varrho^k)} \pmod{p}.$$

Inserting this value of G in (11) we have this identity in the form employed by Frobenius.*

2. Multiplying (5a) by $(x^p-1)(y^p-1)$ we have

$$(x-y^k) f_1(x) f_1(y) = y^k (x^p-1) f_1(y) - x(y^p-1) f_1(x) \frac{y^k-1}{y-1}.$$

Set $x = xe^{vk}$, $y = ye^v$, divide by e^{vk}, differentiate $p-2$ times with respect to v, and set $v = 0$; we then obtain modulo p

$$(x-y^k) \sum_{n=1}^{p-1} \binom{p-2}{n-1} k^{p-n-1} f_{p-n}(x) f_n(y) y^k (x^p-1) f_{p-1}(y)$$
$$- x(y^p-1) \sum_{l=0}^{k-1} y^l h_{p-1}^{(k,l)}(x).$$

Setting $y = \varrho$, multiplying by $(x^m-1)/(x-\varrho)$, summing with respect to ϱ, $\varrho \neq 1$, and using (9a) with (13), we obtain

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Journal für die reine und angewandte Mathematik. Within a month from this time he obtained a formula equivalent to (11) (known previously only for $k=2$) but in a more complicated form, together with a proof that

$$G_m^{(1)} + G_m^{(2)} + \cdots + G_m^{(q)} = -\frac{m}{2} q(m) t \frac{t^{m-1}-1}{t-1}$$

$q = (m-1)/2$, $G_m^{(r)} = G_m^{(r)}(t)$, and these results yielded the criteria $11^{p-1} \equiv 1 \pmod{p^2}$ and, if $p \equiv 1 \pmod 6$, $7^{p-1} \equiv 13^{p-1} \equiv 1 \pmod{p^2}$. These proofs were shown to Prof. H. H. Mitchell, who checked them in detail, and the results were also communicated verbally to Prof. G. D. Birkhoff. It was unnecessary to publish this work however as Frobenius, to whom Prof. Hensel sent the writer's paper for report, wrote his paper containing the results just mentioned and this paper appeared on June 11, 1914 several days before the appearance of the writer's paper in the *Journal*.

* Loc. cit., p. 657, (8); p. 658, (14), (15); p. 660, (21).

$$(x^m-1)\sum_{n=1}^{p-2} k^{p-n-1}(m^n-1)(-1)^{n+1} b_n f_{p-n}(x)$$

$$\equiv \frac{x(x^m-1)}{x-1}\sum{}' \frac{(1-\varrho^k) f_{p-1}(\varrho)}{(1-\varrho^p)(x-\varrho^k)} - \frac{x(x^m-1)\sum{}' \sum_{l=0}^{k-1} \varrho^l h_{p-1}^{(k,l)}(x)}{x-\varrho^k},$$

modulo p. This relation may be employed in lieu of (11) to obtain criteria of the type $m^{p-1} \equiv 1 \pmod{p^2}$ *but it does not hold for* $m = 1$, which particular case seems necessary in order to obtain certain transformations of (2).

3. In (10) set $m = 1$; it then reduces to

$$\sum_{n=0}^{a-1} \binom{a}{n} k^{a-n} b_n f_{a+1-n}(x) \equiv -a x \sum_{l=0}^{k-1} \frac{h_a^{(k,l)}(x)}{x-1} \pmod{p}$$

whence

$$(x^p-1)\sum_{n=0}^{a-1} \binom{a}{n} k^{a-n} b_n f_{a+1-n}(x) \equiv -a x \frac{x^p-1}{x-1} \sum_{l=0}^{k-1} h_a^{(k,l)}(x) \pmod{p}.$$

Set $x = x e^{vk}$; then we may write

$$(x^p e^{vkp}-1)\sum_{n=0}^{a-1} \binom{a}{n} k^{a-n} b_n f_{a+1-n}(x e^{vk})$$

$$= -a \sum_{l=0}^{k-1} \left[\left(\sum_{r=0}^{p-1} (kr+l)^{a-1} e^{(kr+l)v} x^r \right) \left(\sum_{r=0}^{p-1} x^r e^{v(kr+(k-l))} \right) \right] + pV(e^{vk} x),$$

where V is a polynomial in $e^{vk} x$ with integral coefficients. Differentiating $p-1-a$ times with respect to v, we have for $v = 0$

$$(x^p-1)\sum_{n=0}^{a-1} \binom{a}{n} k^{a-n} b_n f_{p-n}(x)$$

$$\equiv -a \sum_{l=0}^{k-1} \sum_{i=0}^{p-1-a} \binom{p-1-a}{i} h_{a+i}^{(k,l)}(x) h_{p-(a+i)}^{(k,k-l)}(x) \pmod{p}.$$

Letting $a = 1, 2, \cdots, (p-1)$ in turn we obtain, using (2), the relations (3).

ALGEBRAIC POTENTIAL CURVES.

By Julian L. Coolidge.

1. General form. Let us agree to mean by an *algebraic potential curve* a plane algebraic curve given by the cartesian equation

$$\varphi(x, y) = 0 \tag{1}$$

where φ is a polynomial which identically satisfies Laplace's equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \equiv 0 \tag{2}$$

Curves of this sort have already been considerably studied, and under a bewildering variety of titles. Lucas* calls them stelloids, a harmless but not suggestive title, Holzmüller† uses the objectionable name of hyperbolas, Brooks‡ employs the curious adjective "orthic" which irresistably suggests some sort of a bird, while Kasner§ chooses the title which we have here adopted. It is not the purpose of the present article to find a new name for the curves but to give some properties which previous writers do not seem to have noted. Some of these spring directly from the definition; others come by applying to the curves in question certain known metrical properties of the general algebraic plane curve.

The distinguishing characteristic of a potential curve is that it is apolar to the circular points at infinity, which amounts to saying that the conic polar of an arbitrary point is, in general, a rectangular hyperbola. A potential curve will not, automatically have any singular points

Theorem 1. *If a potential curve have a finite node, the branches cut at right angles.*

Theorem 2. *A real potential curve can not have a real finite cusp.*

This theorem is not as trivial as it seems. Its significance is here. If we have an analytic one-parameter family of curves, and equate the discriminant with regard to this parameter to zero, the resulting curve is the envelop of the system, in general. But if all of the curves of the family have cusps, and these are variable, their locus constitutes part of the

* Journ. de l'Éc. Pol., Cah. 46, 1879.

† Einführung in die Theorie der Isogonalverwandtschaften, Leipzig 1882.

‡ Proc. Amer. Phil. Soc., vol. 43, 1904.

§ Bull. Amer. Math. Soc., vol. 7, 1901.

locus found. In the case of real potential curves there will be no such cusp locus.

For further study of potential curves it is better to replace our cartesian coördinates by isotropic ones given by the equations:

$$z \equiv x + iy, \qquad \bar{z} \equiv x - iy. \tag{3}$$

If the new equation of the curve be

$$\Phi(z, \bar{z}) \equiv 0,$$

then

$$\frac{\partial^2 \Phi}{\partial z \, \partial \bar{z}} \equiv 0,$$

$$\Phi \equiv \psi(z) + \theta(\bar{z}).$$

Assuming the curve to have a real cartesian equation

$$\psi(z) + \theta(\bar{z}) \equiv k\,[\bar{\psi}(\bar{z}) + \bar{\theta}(z)]; \qquad \bar{\psi}(\bar{z}) + \bar{\theta}(z) \equiv \bar{k}\,[\psi(z) + \theta(\bar{z})]$$

$$k\bar{k} = 1,$$

$$k = e^{2i\varphi}.$$

Let

$$e^{i\varphi}\,\psi(z) = f(z)$$

Then

$$e^{i\varphi}\,\theta(\bar{z}) = \bar{f}(\bar{z}),$$

$$f(z) + \bar{f}(\bar{z}) = 0. \tag{4}$$

Here f is the general polynomial of degree n in the complex variable s and $\bar{f}$ is the conjugate polynomial, i. e., that obtained by replacing each coefficient by its conjugate complex value.

2. **Central properties.** The first great advantage of reducing our equation to the type (4) is that by a translation of the axes we can remove simultaneously the terms z^{n-1} and $\bar{z}^{n-1}$. We then write

$$\begin{aligned} f(z) &= a_0 z^n + a_2 z^{n-2} + \cdots + a_n, \\ \bar{f}(\bar{z}) &= \bar{a}_0 \bar{z}^n + \bar{a}_2 \bar{z}^{n-2} + \cdots + \bar{a}_n, \end{aligned} \tag{5}$$

$$a_0 = Re^{i\Phi}, \qquad \bar{a}_0 = Re^{-i\Phi}. \tag{6}$$

If the straight line

$$\alpha z + \bar{\alpha}\,\bar{z} + \beta = 0$$

be an asymptote, it must meet the curve at least twice at infinity. Hence

$$a_0 + \bar{a}_0 \left(-\frac{\alpha}{\bar{\alpha}}\right)^n = 0,$$

$$\frac{n}{\bar{\alpha}^n}(-1)^n [\alpha^{n-1}\beta] = 0,$$

$$\beta = 0.$$

Putting $z = re^{i\theta}$ we have two infinite roots if

$$Rr\,[e^{i[\Phi+n\theta]} + e^{-i[\Phi+n\theta]}] = 0,$$

$$\theta = \frac{1}{n}\left[\left(k+\frac{1}{2}\right)\pi - \Phi\right], \quad k = 0, 1, \cdots, n-1.$$

Two values of θ can never be equal here, so that our curve can not touch the infinite line, nor have an infinite singularity. The asymptotes are all concurrent (in the origin) and make equal angles one with another.

THEOREM 4. *The n asymptotes of a real potential curve of order n are concurrent, and will meet any circle whose centre is the point of concurrence in the vertices of a regular polygon of n sides.*

This theorem is, of course, classical, and explains Holzmüller's name of hyperbola.

Let us define as a "centre" of an algebraic plane curve a point whose line polar is the infinite line. To find a centre of our potential curve (4) we must have

$$f'(z) = \bar{f}'(\bar{z}) = 0.$$

THEOREM 5. *A real potential curve of order n will have, usually, $(n-1)^2$ centres whereof $(n-1)$ are real. They are given by the roots of the derivative of the polynomial in z that appears in the canonical form* (4).

We must next notice that the polynomials f' and $\bar{f}'$ lack terms of order $n-2$ so that the origin is the centre of gravity of the real centres. Moreover the curves

$$f' + \bar{f}' = 0, \qquad if' - i\bar{f}' = 0$$

are two potential curves whose intersections are the totality of the centres and whose asymptotes all pass through the origin. But Waring's theorem* tells us that the centre of gravity of the intersections of two curves, neither of which touches the infinite line, is the centre of gravity of the intersections of their asymptotes:

* Proprietates curvarum algebricarum, ed. 1772, p. 55. I have not been able to see the original, and take the reference from Terquem, Nouv. Ann. de Math. (1), vol. 4, 1845, p. 183.

THEOREM 6. *The point of concurrence of the asymptotes of a real potential curve is the centre of gravity of all the centres, and also of the real centres alone.*

There is another type of centre to an algebraic curve, the "tangential centre" which is the last or point polar of the infinite line with regard to the curve when the latter is expressed in tangential coördinates. Now Roberts* has shown that when the curve does not touch the infinite line this tangential centre is the centre of gravity of the centres as previously defined:

THEOREM 7. *The asymptotes of a real potential curve are concurrent in the tangential centre.*

The first polar of an infinite point with regard to our curve (4) will have an equation of the type:

$$z_1 f'(z) + \bar{z}_1 \bar{f}'(\bar{z}) = 0.$$

THEOREM 8. *The first polar of an infinite point with regard to a potential curve is also a potential curve with the same tangential centre as the original one.*

The line polar of an infinite point is the locus of the centres of gravity of the groups of intersections of the curve with sets of parallel lines through that point. If a line pass through the tangential centre of a potential curve, that centre is, by Waring, the centre of gravity of the intersections.

THEOREM 9. *The tangential centre of a potential curve is the centre of gravity of the intersections with any number of lines through that centre. It lies on the line polars of all infinite points.*

Consider, along with our curve (4), the curve

$$z f'(z) + \bar{z} \bar{f}'(\bar{z}) = 0. \tag{7}$$

This is the locus of the intersections of lines through the tangential centre with the first polars of their infinite points. It is another equation of type (4).

THEOREM 10. *The locus of the intersections of lines through the tangential centre of a potential curve with the first polars of their various infinite points, is a potential curve of the same order as the original one, and with the same tangential centre. It passes through all the centres of the original curve.*

3. The fundamental net of potential curves. Every function of two variables which satisfies Laplace's equation is associated with a second

* Quart. Journ. of Math. vol. 9, 1867, pp. 25 ff.

function, also a solution, called the "conjugate" (not conjugate imaginary) and connected with it by the Cauchy-Riemann differential equations. Thus, from (1) we get

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

In isotropic coördinates, our curve (4) is conjugate to ∞^1 curves

$$f' + \bar{f}' = i(F' - \bar{F}'), \qquad i(f' - \bar{f}') = -(F' + \bar{F}'),$$
$$F' = -if', \qquad \bar{F}' = i\bar{f}'.$$

The general conjugate curve is

$$-i(f+k) + i[\bar{f}+\bar{k}] = 0.$$

If we mean by a "net" of plane algebraic curves a system linearly dependent on three independant ones, then there will be a "net" linearly dependent on a given curve and the system of its conjugates. We shall call this the "fundamental net" associated with the given curve, and write it

(8) $$e^{-i\varphi} f - e^{i\varphi}\bar{f} + Si = 0$$

To find the asymptotic directions for the general curve of the net we proceed exactly as before, merely replacing Φ by $\Phi - \varphi$ so getting

$$\theta = \frac{1}{n}\left[\left(k + \frac{1}{2}\right)\pi - (\Phi - \varphi)\right].$$

If we give to φ the two values φ_1 and φ_2, we have for the smallest angle between the asymptotes of the two curves

$$\theta_1 - \theta_2 = -\frac{\varphi_1 - \varphi_2}{n}.$$

On the other hand, if we take our two curves

$$F_1(z) + \bar{F}_1(\bar{z}) \equiv e^{-i\varphi_1} f - e^{i\varphi_1}\bar{f} + S_1 i = 0,$$
$$F_2(z) + \bar{F}_2(\bar{z}) \equiv e^{-i\varphi_2} f - e^{i\varphi_2}\bar{f} + S_2 i = 0,$$

and seek their angles of intersection, we have

$$\tan\psi = \frac{i[F_1'\bar{F}_2' - F_2'\bar{F}_1']}{F_1'\bar{F}_2' + F_2'\bar{F}_1'}$$
$$= \frac{i[e^{-i\varphi_1}e^{i\varphi_2} - e^{i\varphi_1}e^{-i\varphi_2}]}{e^{-i\varphi_1}e^{i\varphi_2} + e^{i\varphi_1}e^{-i\varphi_2}}$$
$$= \tan(\varphi_1 - \varphi_2).$$

THEOREM 11. *All curves of the same fundamental net have the same tangential centre. If two share one asymptote, they share all asymptotes and have n-point contact at the end of each.*

THEOREM 12. *If two potential curves of the same fundamental net share no asymptote, they cut everywhere at the same angle, which is n times the numerically smallest angle between an asymptote of one, and one of the other.*

THEOREM 13. *All the potential curves of a fundamental net have the same centres, each of which is a node for a singly infinite set of curves of the net.*

We have an obvious n to 1 transformation connected with a fundamental net which we may write

$$f(z) = \zeta. \tag{9}$$

Let us call this an "associated transformation".

THEOREM 14. *An associated transformation will carry the lines of a plane into the curves of the net conformally, parallel lines going into curves with the same asymptotes, and mutually perpendicular lines into mutually conjugate curves.*

Consider two lines of the $(\zeta, \bar{\zeta})$ plane

$$e^{-i\varphi_1}\zeta - e^{i\varphi_1}\bar{\zeta} + S_1 i = 0,$$
$$e^{-i\varphi_2}\zeta - e^{i\varphi_2}\bar{\zeta} + S_2 i = 0.$$

They meet at the point $(\zeta, \bar{\zeta})$ at the angle $\varphi_2 - \varphi_1$. The transformed curves are

$$e^{-i\varphi_1}f(z) - e^{i\varphi_1}\bar{f}(\bar{z}) + S_1 i = 0,$$
$$e^{-i\varphi_2}f(z) - e^{i\varphi_2}\bar{f}(\bar{z}) + S_2 i = 0,$$

or, if we introduce the point $(\zeta, \bar{\zeta})$ which corresponds to the n real intersections of the transformed curves

$$\begin{aligned} e^{-i\varphi_1}[f(z) - \zeta] &= e^{i\varphi_1}[\bar{f}(\bar{z}) - \bar{\zeta}], \\ e^{-i\varphi_2}[f(z) - \zeta] &= e^{i\varphi_2}[\bar{f}(\bar{z}) - \bar{\zeta}]. \end{aligned} \tag{10}$$

If, further, we write

$$f(z) - \zeta \equiv a_0(z - z_1)(z - z_2)\cdots(z - z_n),$$

our first equation (10) will be

$$\frac{(z - z_1)(z - z_2)\cdots(z - z_n)}{(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)\cdots(\bar{z} - \bar{z}_n)} = e^{2i(\varphi_1 - \Phi)}.$$

The left side is $e^{2i\psi}$, where ψ is the sum of the angles from the horizontal direction to the radii vectores from the n real intersections to the

real point $(z, \bar{z})$ of the first given curve. The angles which the asymptotes of the same curve c make with the horizontal direction are

$$\frac{\varphi_1 - \Phi}{n}, \quad \frac{\varphi_1 - \Phi + 2\pi}{n}, \quad \cdots, \quad \frac{\varphi_1 - \Phi + 2(n-1)\pi}{n},$$

and their sum is $\varphi_1 - \Phi$.

THEOREM* 15. *If a point on a potential curve be connected with the real intersections with another curve of the fundamental net, the sum of the angles which these lines make with an arbitrary direction is congruent, modulo 2π to the sum of the angles which the asymptotes make therewith.*

THEOREM 16. *If two arbitrary points be taken, one on each of two potential curves, which belong to the same fundamental net but which have different asymptotes, and if they be connected with the n real intersections of the curves, the sum of the angles subtended at the original points by the n intersections is equal to each of the angles of intersection of the two curves.*

If a complex polynomial f be given, the conjugate curves

$$f + \bar{f} = 0, \qquad if - i\bar{f} = 0,$$

intersect orthogonally at the roots of the polynomial. We thus get what is, perhaps, the prettiest theorem in our whole discussion.

THEOREM 17. *If a complex polynomial be given, and if the real and imaginary parts be equated separately to 0, the angles subtended at any two points one of which is on each of these two curves, by the roots of the polynomial add up to $\pi/2$ modulo 2π.*

4. **The foci.** The usual definition of a focus of a plane curve is the intersection of two tangents to the curve, one of which passes through each circular point at infinity. It is understood here that both lines are finite, and what is more important, neither point of contact should be a circular point. A circular inversion will carry a focus into a focus.

If we take our general curve

$$e^{-i\varphi} f(z) - e^{i\varphi} \bar{f}(\bar{z}) + Si = 0, \tag{8}$$

two complex values $\bar{z}'$ corresponding to a given value z will coalesce if

$$\bar{f}'(\bar{z}') = 0.$$

* Given in incomplete form by Brooks, loc. cit. p. 316. He calls such a set of real points a "complete n-ad", but, as he deals with a single potential curve and not the whole net, he gives no proper description of the set.

We thus get the z coördinates for the foci by eliminating $\bar{z}'$ between

$$e^{-i\varphi} f(z) - e^{i\varphi} \bar{f}(\bar{z}') + Si = 0,$$
$$\bar{f}'(\bar{z}') = 0.$$

Remembering that the equations

$$f'(z') = \bar{f}'(\bar{z}') = 0$$

give us the centres, we get:

THEOREM 18. *If isotropic lines be drawn through the real centres of a real potential curve, and tangents be drawn to that curve at the points where it meets these isotropics, these tangents intersect in pairs in the foci.*

This method of reaching the foci is unsatisfactory, at least when we wish for the real ones, so we give another very neat one due to Brooks.* We take that line in the $(\zeta, \bar{\zeta})$ plane which corresponds to our given curve

$$e^{-i\varphi} \zeta - e^{i\varphi} \bar{\zeta} + Si = 0,$$
$$f'(z') = \bar{f}'(\bar{z}') = 0.$$

Let

$$f(z') = \zeta', \qquad \bar{f}(\bar{z}') = \bar{\zeta}'.$$

The reflection of $(\zeta', \bar{\zeta}')$ in this line is

$$\zeta'' = e^{2i\varphi} \bar{\zeta}' - Si e^{i\varphi}, \qquad \bar{\zeta}'' = e^{-2i\varphi} \zeta' + Si e^{-i\varphi}$$

If $(\zeta'', \bar{\zeta}'')$ be the transform of $(z'', \bar{z}'')$,

$$f(z'') = e^{2i\varphi} \bar{f}(\bar{z}') - Si e^{i\varphi},$$
$$e^{-i\varphi} f(z'') - e^{i\varphi} \bar{f}(\bar{z}') + Si = 0,$$
$$\bar{f}'(\bar{z}') = 0.$$

THEOREM 19. *The real foci of a potential curve are the transforms, under an associated transformation, of the reflections in the corresponding straight line of the transforms of the centres under the inverse transformation.*

It is worth while mentioning, in this connection, that if a real curve do not touch the line at infinity, nor pass through one of the circular points, the tangential centre is the centre of gravity of the real foci.†

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* Loc. cit. p. 316.

† Cf. Siebeck, Crelle, vol. 64, 1865, p. 175.

Suppose that we have a pencil of potential curves of our net which have the same asymptotes. The general form for the equation of one will be

$$f(z)-\bar{f}(\bar{z})+Si = 0;$$

corresponding hereto we shall have a pencil of parallel lines

$$\zeta-\bar{\zeta}+Si = 0.$$

The point (ζ'',ζ'') as defined above traces the straight line

$$\zeta''+\bar{\zeta}'' = [f(z')+\bar{f}(\bar{z}')],$$

so that for $(z'',\bar{z}'')$ we have

$$f(z'')+\bar{f}(\bar{z}'') = f(z')+\bar{f}(\bar{z}').$$

This, however, gives a curve of the net through the centre $(z', \bar{z}')$.

THEOREM 20. *The foci of the potential curves of a fundamental net which have the same asymptotes, trace* $n-1$ *curves of the net, each through one centre.*

5. **Two theorems about angles.** An interesting theorem due to Humbert* may be stated in the following terms. "If neither of two given non-tangent curves touch the line at infinity, and if no asymptote of one be parallel to an asymptote of the other, and if neither touch a minimal line at a point of intersection, then the sum of the cotangents of their angles of intersection is equal to the sum of the cotangents of the angles of their asymptotes."

When we apply this to two potential curves of the same net the result is trivial, but when applied to two not of a net there is more to be found, as they may make different angles at different intersections. On the other hand, the angles of the asymptotes are calculated with equal ease whether they be concurrent in the same point or not. Thus, if we take two curves of our net

$$e^{-i\varphi_1}f(z)-e^{i\varphi_1}\bar{f}(\bar{z})+S_1 i = 0,$$
$$e^{-i\varphi_2}f(z)-e^{i\varphi_2}\bar{f}(\bar{z})+S_2 i = 0,$$

the sum of the cotangents of their angles, or the sum of the cotangents of the angles of their asymptotes, is

$$n^2 \operatorname{ctn}(\varphi_1-\varphi_2).$$

* Liouville (4), vol. 1, 1885, pp. 352 ff.

THEOREM 21. *If two potential curves of order n have no asymptote of one parallel to an asymptote of the other, the sum of the cotangents of their angles of intersection is n^2 multiplied by the cotangent of n times the numerically smallest angle which an asymptote of one makes with an asymptote of the other.*

Let us take the standard potential curve

$$f(z)+\bar{f}(\bar{z}) = 0.$$

Let λ be the cotangent of the angle which an asymptote makes with the x axis;

$$a_0(\lambda+i)^n+\bar{a}_0(\lambda-i)^n = 0,$$

$$e^{i\Phi}(\lambda+i)^n+e^{-i\Phi}(\lambda-i)^n = 0,$$

$$\sum \lambda i = \frac{-ni[e^{i\Phi}-e^{-i\Phi}]}{e^{i\Phi}+e^{-i\Phi}}$$

$$= n \operatorname{ctn} \Phi.$$

THEOREM 22. *The sum of the cotangents of the angles which a potential curve makes with an arbitrary finite non-tangent line is n times the cotangent of the numerically smallest angle, which any asymptote makes with that line.*

CAMBRIDGE, MASS.,
June 1925.

A METHOD OF DERIVING THE INFINITE DOUBLE PRODUCTS IN THE THEORY OF ELLIPTIC FUNCTIONS FROM THE MULTIPLICATION THEOREMS.*

By G. Mittag-Leffler.

In the memoir "En metod att komma i analytisk besittning af de elliptiska funktionerna"† we have shown that the method given by Abel in his "Recherches sur les fonctions elliptiques",‡ which is historically the earliest method of deriving the elliptic functions, is also mathematically sound and leads to the desired result with perfect rigor. The method of Abel has the advantage over all other methods that it does not presuppose any theorems from the general theory of functions. It is consequently the most elementary of all mathematical processes which lead to a real command of the analytical properties of the elliptic functions. A few serious objections can be raised against Abel's own presentation, however. The most important of these is that the passage from the multiplication theorems to the infinite double series and double products is not sufficiently justified; in fact, it is based upon an argument which it is scarcely possible to make mathematically rigorous. This has been pointed out by Broch in his paper "Om de elliptiske Funktioners Rækkeudvikling".§ In this paper, as well as in his learned and suggestive book "Traité Élémentaire des Fonctions Elliptiques",|| there is developed a new derivation of this limiting passage which is perfectly distinct from that of Abel.

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* Authorized translation from the Swedish by Einar Hille. The present paper appeared in Öfversigt K. Vetenskaps-Akademiens Förhandlingar, vol. 33, No. 6 (Stockholm, 1876) with the title "En metod att i teorien för de elliptiska funktionerna härleda de oändliga dubbelprodukterna utur multiplikationsformlerna".

† Helsingfors, I. C. Frenckell & Son, 1876. An english translation of this paper by Einar Hille appeared in these Annals, vol. 24, 1923, pp. 271–351, with the title "An introduction to the theory of elliptic functions". The present paper serves as a complement to this earlier publication. The translator wishes to use this opportunity to amend an oversight in the footnote on p. 343 of "Introductions". The first proof of the existence theorem for differential equations is due to Cauchy who published in 1842 a sequence of memoirs on this question in Comptes Rendus, Acad. Sci., Paris, vol. 14, pp. 1020–1026, vol. 15, pp. 44–59, 86–102, 131–146. It is evident from a remark in the first of these memoirs that Cauchy was in possession of the essential elements of the proof at least as early as 1835.

‡ Œuvres complètes, vol. I, pp. 141–252 [pp. 251–388 in the 2nd edition, Christiania 1881].

§ Stockholm, 1864. P. A. Norstedt & Söner.

|| Christiania, 1876. P. T. Malling.

In our memoir "En metod etc." quoted above* we have later derived the infinite double series which is the general expression for Weierstrass's function $\wp(u)$, starting from the multiplication theorems.† Our derivation follows immediately from the earlier work of Abel and we do not need to resort to auxiliary devices from any extraneous range of ideas. We have also indicated how the infinite double products for the differences $\wp(u)-e_1$, $\wp(u)-e_2$, $\wp(u)-e_3$ can be derived just as easily and directly. Our object in the present note shall be to show how this is to be accomplished.

We have‡

$$\wp(u)-e_\alpha = \frac{G_\alpha\left[\wp\left(\frac{u}{n}\right)\right]}{G_0\left[\wp\left(\frac{u}{n}\right)\right]}$$

and, consequently, *if n is supposed to be an odd integer*,§

$$G_\alpha(x)-[\wp(u)-e_\alpha]G_0(x) = \prod_{m=-\frac{n-1}{2}}^{m=+\frac{n-1}{2}} \prod_{m_1=-\frac{n-1}{2}}^{m_1=+\frac{n-1}{2}} \left[x-\wp\left(\frac{u+2m\omega+2m_1\omega_1}{n}\right)\right]$$

Thus

$$G_\alpha(x) = \prod_{m=-\frac{n-1}{2}}^{m=+\frac{n-1}{2}} \prod_{m_1=-\frac{n-1}{2}}^{m_1=+\frac{n-1}{2}} \left[x-\wp\left(\frac{\omega_\alpha+2m\omega+2m_1\omega_1}{n}\right)\right],$$

$$G_0(x) = n \prod_{m=-\frac{n-1}{2}}^{m=+\frac{n-1}{2}} \prod_{m_1=-\frac{n-1}{2}}^{m_1=+\frac{n-1}{2}} \left[x-\wp\left(\frac{2m\omega+2m_1\omega_1}{n}\right)\right],$$

and consequently we obtain the following three formulas

$$\wp(u)-e_1 = \frac{G_1(x)}{G_0(x)} = \frac{1}{n^2}(x-e_1) \prod_{m=1}^{m=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{2m-1}{n}\omega\right)}{x-\wp\left(\frac{2m}{n}\omega\right)}\right]^2$$

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* § 3 of An introduction etc.

† loc. cit. p. 339.

‡ The same notation is used as in An Introduction etc. The letter α is used temporarily to designate one of the numbers 1, 2, 3. By ω_1, ω_2 and ω_3 we understand for the moment respectively ω, $\omega+\omega_1$ and ω_1.

§ This assumption will be adhered to throughout the paper.

$$\times \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\omega+\frac{2m}{n}\omega_1\right)}{x-\wp\left(\frac{2m_1}{n}\omega_1\right)} \right]^2 \cdot \prod_{m=1}^{m=\frac{n-1}{2}} \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{(2m-1)\omega+2m_1\omega_1}{n}\right)}{x-\wp\left(\frac{2m\omega+2m_1\omega_1}{n}\right)} \right]^2$$

(1)
$$\times \prod_{m=1}^{m=\frac{n-1}{2}} \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{(2m-1)\omega-2m_1\omega_1}{n}\right)}{x-\wp\left(\frac{2m\omega-2m_1\omega_1}{n}\right)} \right]^2 = \frac{1}{n^2}(x-e_1)\cdot A_1\cdot A_2\cdot A_3\cdot A_4,$$

$$\wp(u)-e_2 = \frac{G_2(x)}{G_0(x)} = \frac{1}{n^2}(x-e_2) \prod_{m=1}^{m=\frac{n-1}{2}} \left[\frac{x-\wp\left(\omega_1+\frac{2m-1}{n}\omega\right)}{x-\wp\left(\frac{2m}{n}\omega\right)} \right]^2$$

$$\times \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\omega+\frac{2m_1-1}{n}\omega_1\right)}{x-\wp\left(\frac{2m_1}{n}\omega_1\right)} \right]^2$$

(2)
$$\times \prod_{m=1}^{m=\frac{n-1}{2}} \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{(2m-1)\omega+(2m_1-1)\omega_1}{n}\right)}{x-\wp\left(\frac{2m\omega+2m_1\omega_1}{n}\right)} \right]$$

$$\times \prod_{m=1}^{m=\frac{n-1}{2}} \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{(2m-1)\omega-(2m_1-1)\omega_1}{n}\right)}{x-\wp\left(\frac{2m\omega-2m_1\omega_1}{n}\right)} \right]^2 = \frac{1}{n^2}(x-e_2)B_1\cdot B_2\cdot B_3\cdot B_4,$$

$$\wp(u)-e_3 = \frac{G_3(x)}{G_0(x)} = \frac{1}{n^2}(x-e_3) \prod_{m=1}^{m=\frac{n-1}{2}} \left[\frac{x-\wp\left(\omega_1+\frac{2m}{n}\omega\right)}{x-\wp\left(\frac{2m}{n}\omega\right)} \right]^2$$

(3)
$$\times \prod_{m=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{2m_1-1}{n}\omega_1\right)}{x-\wp\left(\frac{2m_1}{n}\omega_1\right)} \right]^2 \cdot \prod_{m=1}^{m=\frac{n-1}{2}} \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{2m\omega+(2m_1-1)\omega_1}{n}\right)}{x-\wp\left(\frac{2m\omega+2m_1\omega_1}{n}\right)} \right]^2$$

$$\times \prod_{m=1}^{m=\frac{n-1}{2}} \prod_{m_1=1}^{m_1=\frac{n-1}{2}} \left[\frac{x-\wp\left(\frac{2m\omega-(2m_1-1)\omega_1}{n}\right)}{x-\wp\left(\frac{2m\omega-2m_1\omega_1}{n}\right)} \right]^2 = \frac{1}{n^2}(x-e_3)C_1\cdot C_2\cdot C_3\cdot C_4.$$

The twelve products A, B, C are all composed of factors of the form

$$\frac{x-\wp\left(\frac{2\bar{\omega}-W}{n}\right)}{x-\wp\left(\frac{2\bar{\omega}}{n}\right)}.$$

By $2\bar{\omega}$ we understand

$$2\bar{\omega} = 2m\omega \pm 2m_1\omega_1,$$

where m and m_1 are integers between zero and $\frac{n-1}{2}$ which are not simultaneously zero. The quantity W in the eight products A_1, A_3, A_4, B_3, B_4, C_2, C_3 and C_4 is a constant independent of n, m and m_1. In fact, W equals ω in the three A-products, $\omega+\omega_1$ in B_3 and $\omega-\omega_1$ in B_4, ω_1 in C_2 and C_3 and $-\omega_1$ in C_4. In the four remaining products A_2, B_1, B_2 and C_1 the absolute value of W is less than the n-th multiple of a finite positive number independent of n, m and m_1. Actually W equals $-n\omega$ in A_2, $\omega-n\omega_1$ in B_1, $\omega_1-n\omega$ in B_2 and $-n\omega_1$ in C_1.

On the other hand we have*

$$\wp\left(\frac{u}{n}\right)-\wp\left(\frac{v}{n}\right) = \left(1-\frac{n^2}{v^2}\right)\left(\frac{n^2}{u^2}-g\frac{v^2}{n^2}\right)$$

and consequently

$$\frac{\wp\left(\frac{u}{n}\right)-\wp\left(\frac{2\bar{\omega}-W}{n}\right)}{\wp\left(\frac{u}{n}\right)-\wp\left(\frac{2\bar{\omega}}{n}\right)} = \frac{1-\frac{u^2}{(2\bar{\omega}-W)^2}}{1-\frac{u^2}{(2\bar{\omega})^2}}\,\frac{\frac{1}{u^2}-g\frac{(2\bar{\omega}-W)^2}{n^4}}{\frac{1}{u^2}-g_0\frac{(2\bar{\omega})^2}{n^4}} = a\cdot b.$$

Thus every factor of the twelve products A, B, C is decomposed into two sub-factors a and b. To begin with we shall study b somewhat closer. We can write

$$b = 1-\frac{g-g_0}{n^4}\cdot\frac{(2\bar{\omega})^2}{D}+\frac{g}{n^4}\cdot\frac{2\bar{\omega}\cdot 2W}{D}-\frac{g}{n^4}\cdot\frac{W^2}{D}$$

$$= 1+b_0+b_1+b_2, \text{ where } D = \frac{1}{u^2}-g_0\frac{(2\bar{\omega})^2}{n^4}.$$

But

$$1+(|b_0|+|b_1|+|b_2|)>|b|>1-(|b_0|+|b_1|+|b_2|).$$

Further, the difference $g-g_0$ tends to zero with increasing values of n, and the absolute value of $2\bar{\omega}$ is always less than n times a constant independent of n, m and m_1.

* loc. cit. p. 333, formula (15).

W is a constant independent of n, m and m_1 in the eight products A_1, A_3, A_4, B_3, B_4, C_2, C_3 and C_4. Hence in these products

$$|b_0|<\frac{\beta_0}{n^2}, \qquad |b_1|<\frac{\beta_1}{n^3}, \qquad |b_2|<\frac{\beta_2}{n^4},$$

where β_0, β_1, β_2 are finite positive quantities of which β_0 can be made arbitrarily small by increasing the value of n. In each of the six products A_3, A_4, B_3, B_4, C_3 and C_4 there are $\frac{1}{2}(n-1)^2$ factors and in each of the products A_1 and C_2 there are $(n-1)$ factors. Hence in each of these eight products* the sub-product of the b-factors differs from unity by a quantity the absolute value of which is less than

$$\left[1+\frac{1}{n^2}\left(\beta_0+\frac{\beta_1}{n}+\frac{\beta_2}{n^2}\right)\right]^{n^2}-1.$$

This quantity can be made as small as we please by making n sufficiently large.

In the four remaining products A_2, B_1, B_2 and C_1 the quantity W is numerically less than n times a constant independent of n, m and m_1, hence

$$|b_0|<\frac{\beta_0'}{n^2}, \qquad |b_1|<\frac{\beta_1'}{n^2}, \qquad |b_2|<\frac{\beta_2'}{n^2}.$$

Each of these four products contains $(n-1)$ factors. Consequently each of the corresponding b-products differs from unity by an amount which is numerically less than

$$\left[1+\frac{1}{n^2}(\beta_0'+\beta_1'+\beta_2')\right]^n-1$$

and can be made as small as we please by making n large.

Thus each of the twelve products which are composed of b-factors can be brought arbitrarily close to *unity* by increasing the value of n.

Further, the numerators of the four a-products in A_2, B_1, B_2 and C_1,

$$\prod_1^{\frac{n-1}{2}}\left[1-\frac{u^2}{(2\bar{\omega}-W)^2}\right]^2,$$

can be brought as close to unity as we please by increasing n. In fact, each of these products differs from unity by a quantity which is less than

* From this point on the translator has somewhat modified the author's argument in order to make the proof clearer.

$$\left(1+\frac{B}{n^2}\right)^n-1$$

in absolute value, where B is a finite positive quantity independent of n, m and m_1.

In addition, we know that

$$\wp\left(\frac{u}{n}\right)-e_\alpha=\frac{n^2}{u^2}\left[1+g\frac{u^4}{n^4}-e_\alpha\frac{u^2}{n^2}\right]$$

and, consequently, that $\wp\left(\frac{u}{n}\right)-e_\alpha$ is equal to $\frac{n^2}{u^2}$ times a factor which tends to unity when n increases indefinitely.

As the final result of the preceding investigation we find that *the quotients formed by the following three pairs of functions tend to unity as n increases indefinitely, namely*

$$\wp(u)-e_1 \text{ and } \left[\frac{\prod_{m=1}^{m=\frac{n-1}{2}}\left(1-\frac{1}{[(2m-1)\omega]^2}\right)}{\prod_{m=1}^{m=\frac{n-1}{2}}\left(1-\frac{u^2}{(2m\omega)^2}\right)}\cdot\frac{1}{u\prod_{m_1=1}^{m_1=\frac{n-1}{2}}\left(1-\frac{u^2}{(2m_1\omega_1)^2}\right)}\right.$$

$$\left.\times\prod_{m=1}^{m=\frac{n-1}{2}}\prod_{m_1=1}^{m_1=\frac{n-1}{2}}\frac{\left(1-\frac{u^2}{[(2m-1)\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[(2m-1)\omega-2m_1\omega_1]^2}\right)}{\left(1-\frac{u^2}{[2m\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega-2m_1\omega_1]^2}\right)}\right]^2;$$

$$\wp(u)-e_2 \text{ and } \left[\frac{1}{u\prod_{m=1}^{m=\frac{n-1}{2}}\left(1-\frac{u^2}{(2m\omega)^2}\right)\prod_{m_1=1}^{m_1=\frac{n-1}{2}}\left(1-\frac{u^2}{(2m_1\omega_1)^2}\right)}\right.$$

$$\left.\times\prod_{m=1}^{m=\frac{n-1}{2}}\prod_{m_1=1}^{m_1=\frac{n-1}{2}}\frac{\left(1-\frac{u^2}{[(2m-1)\omega+(2m_1-1)\omega_1]^2}\right)\left(1-\frac{u^2}{[(2m-1)\omega-(2m_1-1)\omega_1]^2}\right)}{\left(1-\frac{u^2}{[2m\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega-2m_1\omega_1]^2}\right)}\right]^2;$$

and

$$\wp(u)-e_3 \text{ and}$$

$$\left[\frac{1}{u\prod_{m=1}^{m=\frac{n-1}{2}}\left(1-\frac{u^2}{(2m\omega)^2}\right)}\cdot\frac{\prod_{m_1=1}^{m_1=\frac{n-1}{2}}\left(1-\frac{u^2}{[(2m_1-1)\omega_1]^2}\right)}{\prod_{m_1=1}^{m_1=\frac{n-1}{2}}\left(1-\frac{u^2}{(2m_1\omega_1)^2}\right)}\right.$$

$$\left.\times\prod_{m=1}^{m=\frac{n-1}{2}}\prod_{m_1=1}^{m_1=\frac{n-1}{2}}\frac{\left(1-\frac{u^2}{[2m\omega+(2m_1-1)\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega-(2m_1-1)\omega_1]^2}\right)}{\left(1-\frac{u^2}{[2m\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega-2m_1\omega_1]^2}\right)}\right]^2.$$

Consequently we find that

$$\wp(u)-e_1=\left[\frac{\prod_{m=1}^{m=\infty}\left(1-\frac{u^2}{[(2m-1)\omega]^2}\right)}{\prod_{m=1}^{m=\infty}\left(1-\frac{u^2}{(2m\omega)^2}\right)}\cdot\frac{1}{u\prod_{m_1=1}^{m_1=\infty}\left(1-\frac{u^2}{(2m_1\omega_1)^2}\right)}\right.$$

$$\left.\times\prod_{m=1}^{m=\infty}\prod_{m_1=1}^{m_1=\infty}\frac{\left(1-\frac{u^2}{[(2m-1)\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[(2m-1)\omega-2m_1\omega_1]^2}\right)}{\left(1-\frac{u^2}{[2m\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega-2m_1\omega_1]^2}\right)}\right]^2;$$

$$\wp(u)-e_2=\left[\frac{1}{u\prod_{m=1}^{m=\infty}\left(1-\frac{u^2}{(2m\omega)^2}\right)\prod_{m_1=1}^{m_1=\infty}\left(1-\frac{u^2}{(2m_1\omega_1)^2}\right)}\right.$$

$$\left.\times\prod_{m=1}^{m=\infty}\prod_{m_1=1}^{m_1=\infty}\frac{\left(1-\frac{u^2}{[(2m-1)\omega+(2m_1-1)\omega_1]^2}\right)\left(1-\frac{u^2}{[(2m-1)\omega-(2m_1-1)\omega_1]^2}\right)}{\left(1-\frac{u^2}{[2m\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega+2m_1\omega_1]^2}\right)}\right]^2;$$

$$\wp(u)-e_3=\left[\frac{1}{u\prod_{m=1}^{m=\infty}\left(1-\frac{u^2}{(2m\omega)^2}\right)}\cdot\frac{\prod_{m_1=1}^{m_1=\infty}\left(1-\frac{u^2}{[(2m_1-1)\omega_1]^2}\right)}{\prod_{m_1=1}^{m_1=\infty}\left(1-\frac{u^2}{(2m_1\omega_1)^2}\right)}\right.$$

$$\left.\times\prod_{m=1}^{m=\infty}\prod_{m_1=1}^{m_1=\infty}\frac{\left(1-\frac{u^2}{[2m\omega+(2m_1-1)\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega-(2m_1-1)\omega_1]^2}\right)}{\left(1-\frac{u^2}{[2m\omega+2m_1\omega_1]^2}\right)\left(1-\frac{u^2}{[2m\omega-2m_1\omega_1]^2}\right)}\right]^2.$$

Putting

$$\varphi^2(u)=\frac{1}{\wp(u)-e_2};\quad f^2(u)=\frac{\wp(u)-e_1}{\wp(u)-e_2};\quad F^2(u)=\frac{\wp(u)-e_3}{\wp(u)-e_2},$$

we obtain immediately the product formulae given by Abel in his "Recherches sur les fonctions elliptiques" page 212, in "Œuvres complètes" [page 343 in the 2nd edition].

MISCELLANEOUS QUESTIONS IN THE THEORY OF DIFFERENTIAL EQUATIONS.

1. ON THE METHOD OF FROBENIUS.

By Einar Hille.

The existence of solutions of a linear differential equation in the neighborhood of a regular singular point has been proved by a variety of methods since the days of Fuchs and Frobenius. The original method of Frobenius* had the advantage of giving at the same time an effective method of obtaining the solutions in terms of power series and a proof of the convergence of these series. Many variations of the convergence proof have been given later. Nevertheless the following simple arrangement of the proof seems to possess some slight element of novelty. For the sake of simplicity we give the proof for a system of first order equations, the regular singular point being at the origin.

Our point of departure is the system

$$z\frac{dw_i}{dz} = \sum_{j=1}^{n} P_{ij}(z)\,w_j, \qquad (i = 1, 2, \cdots, n) \tag{1}$$

$$P_{ij}(z) = \sum_{k=0}^{\infty} a_{ijk}\,z^k, \tag{2}$$

where we assume that

$$|a_{ijk}| \leqq MR^{-k}, \tag{3}$$

M and R being fixed positive constants independent of i, j and k. We have to show that this system is satisfied by n sets of series

$$w_i(z) = w_i(z; r_j) = \sum_{k=0}^{\infty} c_{ik}(r_j)\,z^{k+r_j} \quad (i, j = 1, \cdots, n) \tag{4}$$

where r_j is a root of the indicial equation

$$f(r) = |\delta_{ij}\,r - a_{ij0}| = 0, \tag{5}$$

and the series are convergent at least for $0 < |z| < R$.

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* Journal für Mathematik, vol. 76 (1873), pp. 214–235. For the later literature on the subject, see Encykl. d. Math. Wiss. Bd. II_2, Heft 4, Hilb, Lineare Differentialgleichungen im komplexen Gebiet, especially §§ 1 and 3.

The coefficients $c_{ik}(r)$ are determined by the system

$$(6)\quad \sum_{j=1}^{n}[\delta_{ij}(r+k)-a_{ij0}]\,c_{jk}(r)=\sum_{j=1}^{n}\sum_{l=0}^{k-1}a_{ij,k-l}\,c_{jl}(r),\quad \begin{pmatrix} i=1,\cdots,n\\ k=1,2,\cdots\end{pmatrix}.$$

The determinant of the n equations corresponding to a fixed value of k is simply $f(r+k)$. Denoting the co-factor of $\delta_{ij}\,r-a_{ij0}$ in the determinant (5) by $f_{ij}(r)$ we can replace equation (6) by

$$(7)\quad f(r+k)\,c_{ik}(r)=\sum_{m=1}^{n}f_{mi}(r+k)\sum_{j=1}^{n}\sum_{l=0}^{k-1}a_{mj,k-l}\,c_{jl}(r),\quad \begin{pmatrix} i=1,\cdots,n\\ k=1,2,\cdots\end{pmatrix}.$$

Let us restrict the variability of r by assuming $|r|\leqq\varrho$, ϱ being so large that the roots $r_1,\cdots,r_n$ of (5) lie inside the circle $|r|=\varrho$. Let P be the region remaining after deleting the points $r_\nu-p$ $(\nu=1,\cdots,n;$ $p=1,2,\cdots)$ from this circle by means of small circles of radius ε. For the moment we shall assume that the roots of (5) are distinct and incongruent modulo 1. In this case all the points r_ν belong to the interior of P, and $f(r+k)\neq 0$ for all values of r in P, $k=1,2,\cdots$. In view of these assumptions we can find two positive constants μ and δ such that for r in P

$$(8)\quad |f(r+k)|\geqq\delta(\varrho+k)^n,\qquad |f_{mi}(r+k)|\leqq\mu(\varrho+k)^{n-1},\quad \begin{pmatrix} i,m=1,\cdots,n\\ k=1,2,\cdots\end{pmatrix}.$$

Using the inequalities (3) and (8) we conclude that

$$\delta(\varrho+k)^n\,|c_{ik}(r)|\leqq\sum_{m=1}^{n}\mu(\varrho+k)^{n-1}\sum_{j=1}^{n}\sum_{l=0}^{k-1}M\,R^{l-k}\,|c_{jl}(r)|,$$

or

$$(9)\qquad (\varrho+k)\,|c_{ik}(r)|\leqq\frac{\mu}{\delta}\,n\,M\sum_{j=1}^{n}\sum_{l=0}^{k-1}R^{l-k}\,|c_{jl}(r)|.$$

Let us compare these quantities $c_{ik}(r)$ with the constants C_k defined by the equations

$$(10)\quad (\varrho+k)\,C_k=a\sum_{l=0}^{k-1}R^{l-k}C_l\ \text{ where }\ a=\frac{\mu}{\delta}\,n^2M,\ (k=1,2,\cdots).$$

It is evident that we can choose C_0 so large that

$$|c_{ik}(r)|\leqq C_k$$

for all values of i and k that occur, r being in P. This assumes, of course, that we choose the functions $c_{i0}(r)$ which are at our disposal, in such a way that they remain bounded in P.

Replace k by $k+1$ in (10), multiply the result by R and subtract the old equation from the new one, obtaining

$$R(\varrho+k+1)\,C_{k+1}-(\varrho+k)\,C_k = a\,C_k$$

or

$$C_{k+1}=\frac{\varrho+a+k}{\varrho+1+k}\,\frac{1}{R}\,C_k.$$

Hence

$$C_k = \frac{(\varrho+a)(\varrho+a+1)\cdots(\varrho+a+k-1)}{(\varrho+1)(\varrho+2)\cdots(\varrho+k)}\,\frac{1}{R^k}\,C_0, \tag{11}$$

and consequently

$$\sum_{k=0}^{\infty} c_{ik}(r)\,z^k \ll C_0\,F\left(\varrho+a,\,1,\varrho+1,\,\frac{z}{R}\right). \tag{12}$$

Thus the series on the left are convergent at least for $|z|<R$ and uniformly convergent with respect to r in P. If the quantities $c_{i0}(r)$ are holomorphic in P, the same will be true for $c_{ik}(r)$, $k\geqq 1$, and, in view of the uniform convergence, also for

$$w_i(z;r) = \sum_{k=0}^{\infty} c_{ik}(r)\,z^{k+r}, \qquad (i=1,\cdots,n) \tag{13}$$

z being fixed, $0<|z|<R$.

Thus in the case of the equation (5) having distinct incongruent roots, we obtain the system of solutions by equating r in (13) successively to $r_1, r_2, \cdots, r_n$, the resulting series being convergent at least for $0<|z|<R$, in which region the equation (12) gives a common majorant for the solutions. We notice that, if there is a singular point of the equation (1) on the circle $|z|=R$, we have

$$|w_i(z;r_j)|<\text{const}\left[1-\left|\frac{z}{R}\right|\right]^{-a}$$

for approach from the interior.

Concerning the value of ϱ we know through the investigations of I. Schur* that $\sum|r_\nu|^2\leqq\sum|a_{ij0}|^2\leqq n^2M^2$. In particular, $|r_\nu|\leqq nM$ where the equality sign can hold if and only if

$$P_{ij}(0) = a_{ij0} = M e^{(\varphi+\varphi_i-\varphi_j)\sqrt{-1}} \qquad (i,j=1,\cdots,n)$$

with real φ's. Hence we can take $\varrho = nM$ except in the case mentioned.

* Math. Annalen, vol. 66 (1909), pp. 488–510, especially pp. 492–496 where references to the earlier work of Bendixson and Hirsch also are to be found.

If the roots of (5) do not satisfy the restrictions imposed above, the method has to be modified in order to yield the complete system of solutions. It is not our intention to give a discussion of the different cases which arise.* Two auxiliary devices are employed (1) a suitable choice of the $c_{i0}(r)$ as polynomials in r, (2) differentiation of $w_i(z;r)$ — or similar function — with respect to r.

If $f(r_0) = 0$ but $f(r_0+k) \neq 0$ for $k \geqq 1$, there is very little to change in the preceding discussion. Formula (12) still holds for the corresponding solutions and $\frac{\partial^p}{\partial r^p}[w_i(z;r)]$ can be obtained by term-wise differentiation of (13), the resulting series being uniformly convergent in P including $|r-r_0| \leqq \varepsilon$. If, on the other hand, $f(r_0) = f(r_0+k_1) \cdots = f(r_0+k_s) = 0$ but $f(r_0+k) \neq 0$ for $k > k_s$, then some of the equations (7) become illusory for $r = r_0$. The essential thing is now to choose $c_{i0}(r)$ in such a way that, if the equations (7) are divided by $f(r+k)$, the resulting values for $c_{ik}(r)$ shall tend to finite limits as $r \to r_0$. It is obvious that (9) holds for $k > k_s$ and (12) will hold provided C_0 is properly chosen. Term-wise differentiation with respect to r is permitted as above.

* Cf. e. g. Horn, Gewöhnliche Differentialgleichungen (Sammlung Schubert L.), p. 146 et seq.

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NON-MEASURABLE FUNCTIONS CONNECTED WITH CERTAIN FUNCTIONAL EQUATIONS.*

By Henry Blumberg.

1. In the present section, we deal with the functional equation

$$\varphi(f(x), f(y)) = f(\varphi(x, y)),\dagger \tag{1}$$

supposing that $\varphi(x, y)$ is a real, continuous function of x and y, defined for the entire xy-plane and "properly monotone" in these variables taken separately; in other words, if y is constant, φ is a "properly monotone" function of x, i. e., it either always increases or always decreases with x, and likewise, if x is constant, φ is a properly monotone function of y. The function φ is regarded as given, so that (1) is an equation conditioning the real function $f(x)$.

In Section 2, we consider various generalizations.

We shall first prove

Theorem I. *Every discontinuous solution of equation* (1) *is everywhere discontinuous.*‡

Proof. Let $f(x)$ be discontinuous at the point a, so that a sequence $\{a_n\}$ exists with $\lim a_n = a$ and $\lim f(a_n) = h$ (say) $\neq f(a)$. If (a, b, c) is a point on the surface $z = \varphi(x, y)$, we have $c = \varphi(a, b)$, and therefore, according to (1), $f(c) = \varphi(f(a), f(b))$, so that the point $(f(a), f(b), f(c))$ also lies on the surface $z = \varphi(x, y)$. Since $\varphi(x, y)$ is properly monotone in each of its arguments, the curve $\varphi(x, y) = c$ has at most one point on a line $x =$ const. or $y =$ const.; hence the equation $c = \varphi(a_n, b_n)$ determines b_n uniquely,—at any rate, for sufficiently small $|a_n - a|$—and indeed, so that $\lim b_n = b$. From $c = \varphi(a_n, b_n)$, we conclude, by means of (1),

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* Read, in part, at the meeting of the Southwestern Section, American Mathematical Society, Dec. 1, 1917.

† The special case $\varphi(x, y) = x + y$ appears in the author's paper "On Convex Functions", Trans. Amer. Math. Soc., vol. 20 (1919), p. 40. Cf. also Schimmack, *Axiomatische Untersuchungen über die Vectoraddition*, Dissertation (Halle), 1908; Hamel, Math. Annalen vol. 60 (1905), p. 459; Banach, Fundamenta Mathematicae, Tom. I (1920), p. 123; and Sierpiński, l. c., p. 125.

‡ If we assume merely that φ is monotone in each of the variables, the theorem is not valid, as the example $\varphi(x, y) = xy$ shows. Here $\varphi(x, y)$ is properly monotone except for $x = 0$ and $y = 0$, but we have the following solution: $f(x) = 0$, if $x = 0$, and $f(x) = 1$, if $x \neq 0$.

that $f(c) = \varphi(f(a_n), f(b_n))$—in other words, that the point $(f(a_n), f(b_n), f(c))$ lies on the surface $z = \varphi(x, y)$. Now if $f(x)$ is continuous at b, we have $\lim f(b_n) = f(b)$, so that $(h, f(b), f(c))$ is on the surface $z = \varphi(x, y)$. Since, however, $(f(a), f(b), f(c))$ is also on this surface, and $h \neq f(a)$, we have here a contradiction of the proper monotoneity of φ for $y = \text{const}$. Therefore, $f(x)$ is discontinuous at b, and since b is any real number, $f(x)$ is discontinuous everywhere.

We note, moreover, that if $h > f(a)$, then $\lim f(b_n) = k$ (say) is either less than $f(b)$ for every b, or else greater than $f(b)$ for every b. For since $(f(a_n), f(b_n), f(c))$ is on the surface $z = \varphi$, the limit point $(h, k, f(c))$ is also on the surface. Now the surface $z = \varphi$ is, in consequence of the condition of monotoneity, such that $\varphi(x, y)$ is either an increasing function for every constant y, or else a decreasing function for every constant y; for on account of the continuity of $\varphi(x, y)$, the function $\varphi(k + \delta)$ will, if δ is sufficiently small in absolute value, be an increasing (decreasing) function simultaneously with $\varphi(x, k)$, so that an increasing (decreasing) function $\varphi(x, k)$ cannot be the limit of a sequence of decreasing (increasing) functions $\varphi(x, k_n)$. We may therefore make the following classification: (a) both the functions $\varphi(x, k)$ and $\varphi(k, y)$ are increasing (decreasing) for every k; (b) the functions $\varphi(x, k)$ are increasing (decreasing), and the functions $\varphi(k, y)$ are decreasing (increasing) for every k. In case (a), $\varphi(h, f(b))$ is greater (less) than $\varphi(f(a), f(b)) = f(c)$ if $h > f(a)$; and since $\varphi(h, k) = f(c)$, we have $k < f(b)$. In case (b), we conclude similarly that $k > f(b)$.

Likewise, we may show, in case (a), that, if $h < f(a)$, then $k > f(b)$ for every b. Therefore, if $h > f(a)$, we conclude that $k < f(b)$ for every b, and from the latter inequality for any chosen b, that there exists, for every x, a sequence $\{x_n\}$ with $\lim x_n = x$ and $\lim f(x_n) > f(x)$. We thus have, in case (a), for every x—whether $h < f(a)$ or $h > f(a)$—two sequences $\{x'_n\}$ and $\{x''_n\}$ with $\lim x'_n = \lim x''_n = x$, $\lim f(x'_n) > f(x)$, and $\lim f(x''_n) < f(x)$; in other words, the greatest lower bound of f at x is less than $f(x)$, and the least upper bound of f at x is greater than $f(x)$.

In case (b), and under the supposition that $h > f(a)$, we have shown that there exists, for every x, a sequence $\{x_n\}$ with $\lim x_n = x$ and $\lim f(x_n) > f(x)$. Now, if for every x, there were no sequence $\{x_n\}$ with $\lim x_n = x$ such that $\lim f(x_n) < f(x)$, then $f(x)$ would be a lower semi-continuous function, and would consequently possess points of continuity, contrary to Theorem I. Therefore, there is a number a and a sequence $\{a_n\}$ with $\lim a_n = a$ and $\lim f(a_n) < f(a)$; in other words, an a and a sequence $\{a_n\}$ such that $h < f(a)$. But from the latter inequality we may prove, by reasoning parallel to that above, that $k < f(b)$ for every b.

Summarizing, we may state

THEOREM II. *If $f(x)$ is not everywhere continuous, then for every x, the greatest lower bound of f at x is less than $f(x)$, and the least upper bound of f at x is greater than $f(x)$.*

We next prove the following proposition, which is more assertive than Theorem II.

THEOREM III. *If $f(x)$ is not everywhere continuous, it is both positively and negatively unbounded at every point.*

Proof. Let (a, b, c) be a point of the surface $z = \varphi(x, y)$. Since $f(x)$ is not everywhere continuous, there exists, according to Theorem II, a positive number d such that, as near b as we please, there is a number b_1 satisfying the inequality $f(b_1) - f(b) > d$. Let $c_1 = \varphi(a, b_1)$, and let a_1 be determined by the equation $c_1 = \varphi(a_1, b) = \varphi(a, b_1)$. As we have seen above, the number a_1 is uniquely determined, and it will differ from a by as little as we please, provided b_1 is chosen sufficiently near b. Similarly b_2 is chosen to satisfy the inequality $f(b_2) - f(b) > d$; and a_2 is defined by means of the equation $\varphi(a_2, b) = \varphi(a_1, b_2)$; and if b_2 is sufficiently near b, the number a_2 will be uniquely determined and as near a_1 as we please. Continuing in this manner, we define a_n by means of the equation $\varphi(a_n, b) = \varphi(a_{n-1}, b_n)$. Here b_n is such that $f(b_n) - f(b) > d$, and may be chosen so that a_n is as near a_{n-1} as we please. Setting $\varphi(a_n, b) = c_n$, we have, by means of our functional equation, the relation

$$f(c_n) - f(c_{n-1}) = \varphi(f(a_{n-1}), f(b_n)) - \varphi(f(a_{n-1}), f(b)).$$

Let us assume, for the present, that the functions $\varphi(k, y)$ are monotone increasing, so that, in virtue of the inequality $f(b_n) - f(b) > d$, the difference $f(c_n) - f(c_{n-1})$ is positive. Since this difference may be expressed as $\varphi(f(a_n), f(b)) - \varphi(f(a_{n-1}), f(b))$, it follows that $f(a_n)$ is greater or less than $f(a_{n-1})$ according as the functions $\varphi(x, k)$ are monotone increasing or decreasing. In any case, the sequence $\{f(a_n)\}$ is monotone and therefore has a limit, say g, which may be $+\infty$ or $-\infty$. Since $f(b_n) - f(b) > d$, and the functions $\varphi(k, y)$ are increasing, we have

$$f(c_n) - f(c_{n-1}) > \varphi(f(a_{n-1}), f(b) + d) - \varphi(f(a_{n-1}), f(b)).$$

Hence, if g is finite, $f(c_n) - f(c_{n-1}) > k - \varepsilon_n$ where

$$k = \varphi(g, f(b) + d) - \varphi(g, f(b)) > 0,$$

and $\lim_{n \to \infty} \varepsilon_n = 0$. On the other hand,

$$\lim [f(c_n) - f(c_{n-1})] = \lim [\varphi(f(a_n), f(b)) - \varphi(f(a_{n-1}), f(b))] = 0.$$

This contradiction proves that $g = \pm\infty$, the upper sign applying to the case when the functions $\varphi(x, k)$ are monotone increasing, and the lower sign, to the case when they are monotone decreasing.

Now if we modify the argument above by selecting b_n, as we may according to Theorem II, so that $f(b_n)-f(b) < -d$ (instead of $f(b_n)-f(b) > d$), then $f(c_n)-f(c_{n-1})$ will—under the retained assumption that the $\varphi(k, y)$ are increasing—come out negative; and as a consequence, $f(a_n)$ will be less or greater than $f(a_{n-1})$ according as $\varphi(x, k)$ is monotone decreasing or increasing.

Since a_n can, by the proper choice of the b_n, be made to be as near a as we please, we have thus shown that $f(x)$ is both positively and negatively unbounded at $x = a$. In the very same way, we may show that this is also true if the functions $\varphi(k, y)$ are decreasing, so that, since a is arbitrary, our theorem is proved.

THEOREM IV. *If $f(x)$ is not everywhere continuous, then $\varphi(x, y)$ is positively and negatively unbounded for every $x = k$, and every $y = k$. In other words, if $\varphi(x, y)$ has, for some fixed x or y, a finite upper or lower bound, then every solution of equation* (1) *is everywhere continuous.*

Proof. We note first that $\varphi(x, y)$ is positively unbounded if $x = f(a)$. For if (a, b, c) is a point of $z = \varphi(x, y)$, we have, according to equation (1), $f(c) = \varphi(f(a), f(b))$. Since, according to Theorem III, f is positively unbounded at c, there is a sequence $\{c_n\}$ having c as limit such that $\lim f(c_n) = +\infty$. We define b_n by means of the equation $c_n = \varphi(a, b_n)$, whence $f(c_n) = \varphi(f(a), f(b_n))$. The second member of this equation therefore approaches $+\infty$; that is to say, $\varphi(x, y)$ is positively unbounded on the line $x = f(a)$. Furthermore, if φ is positively unbounded on a line $x = h$, it is positively unbounded on every line $x = h + d$, where d is any positive or any negative number according as the curves $\varphi(x, k)$ are increasing or decreasing. The hypothesis of the last statement is true if $h = f(a)$, and since f is positively and negatively unbounded, h may be taken as large or as small as we please; therefore $h + d$ assumes all values from $-\infty$ to $+\infty$, whatever the character of the functions $\varphi(x, k)$ may be. φ is, therefore, positively unbounded if $x = k$. The proof that φ is negatively unbounded for $x = k$ is similar, as is also the proof for $y = \text{const.}$

Theorem IV, then, shows that the existence of discontinuous solutions of equation (1) implies a condition upon the family of curves in which the surface $z = \varphi$ is cut by the planes $x = k$ and $y = k$. The following theorem shows that the existence of discontinuous solutions conditions also the family of curves $\varphi(x, y) = k$.

THEOREM V. *If $f(x)$ is discontinuous, no curve $\varphi(x, y) = k$ has an*

asymptote parallel to the x- or the y-axis. In other words, if there exists a curve $\varphi(x, y) = k$ *that has an asymptote parallel to one of the coördinate axes, every solution of equation* (1) *is everywhere continuous.*

Towards the proof, we may note first that, on account of the continuity and monotoneity of φ, the ordinate of a curve $\varphi(x, y) = k$ is a monotone, continuous function. It is therefore sufficient to show that the x-coördinates and the y-coördinates of the points of the curve are positively and negatively unbounded. This is obviously the case if k is of the form $f(c)$. For if (a, b, c) is a point of $z = \varphi(x, y)$, we have $\varphi(f(a), f(b)) = f(c)$; and as the coördinates a and b of (a, b, c) vary while c is held fast, the quantities $f(a)$ and $f(b)$—since $f(x)$ has no finite lower or upper bound at a and b—are positively and negatively unbounded. Now the numbers $f(c)$ are positively and negatively unbounded, so that for every number k, there are numbers c_1 and c_2 such that $f(c_1) < k < f(c_2)$. Since each of the curves $\varphi(x, y) = f(c_1)$ and $\varphi(x, y) = f(c_2)$ runs from $(-\infty, +\infty)$ to $(+\infty, -\infty)$ or from $(-\infty, -\infty)$ to $(+\infty, +\infty)$—according to whether the functions $\varphi(x, k)$ and $\varphi(k, y)$ are increasing or decreasing simultaneously, or whether the one is increasing and the other decreasing—it follows, on account of the monotoneity condition upon φ, that the curve $\varphi(x, y) = k$ must also run from $(-\infty, +\infty)$ to $(+\infty, -\infty)$, or from $(-\infty, -\infty)$ to $(+\infty, +\infty)$ respectively.

We shall now proceed to prove, that if $\varphi(x, y)$ satisfies an additional condition, every discontinuous solution of our functional equation is non-measurable.

THEOREM VI. *If* $\varphi(x, y)$ *is subjected to the further condition that the curves* $\varphi(x, y) = k$ *are absolutely continuous, then every discontinuous solution of equation* (1) *is non-measurable* (*Lebesgue*).

Proof. Suppose $f(x)$ is discontinuous and nevertheless measurable. Let E_n be the set of points where $|f(x)| > n$, and e_n the measure of E_n; since $f(x)$ is assumed to be measurable, $\lim_{n \to \infty} e_n \neq 0$. Since $f(x)$ is discontinuous, it is, according to Theorem III, unbounded; there exists, therefore, a number c such that $|f(c)| > \max |\varphi(\pm n, \pm n)|$. According to Theorem IV, the curve $\varphi(x, y) = c$ exists. If (a, b) is a point on this curve, then either a or b belongs to E_n. For if a and b both belong to the complement of E_n, we have $|f(a)| \leqq n$ and $|f(b)| \leqq n$; therefore $|f(c)| = |\varphi(f(a), f(b))|$ would be $\leqq \max |\varphi(\pm n, \pm n)|$. The curve $\varphi(x, y) = c$ is, accordingly, such that either the x- or the y-coördinate of each of its points belongs to E_n.

Now since the ordinate of the curve $\varphi(x, y) = c$ is monotone, dy/dx exists everywhere except in a set of measure 0. Moreover, since this ordinate is absolutely continuous, the points where dy/dx exists and is

different from 0 constitute a set of positive measure.* For suppose that the function $g(x)$, defined in the interval (a, b), has 0 as derivative at the points of the set S, which is of measure $b-a$. Every point x of S may be enclosed in an interval (a_x, b_x) such that $|g(b_x)-g(a_x)| \leqq \varepsilon\ (b_x-a_x)$, where ε is independent of x and arbitrarily small. According to Vitali's Theorem, a finite number of non-overlapping intervals (a', b'), $a'', b'')$, $\cdots$, $(a^{(h)}, b^{(h)})$ may be selected from the (a_x, b_x) such that $\sum_{\nu=1}^{h}(b^{(\nu)}-a^{(\nu)}) > b-a-\eta$, where η is as small as we please. If (c', d'), (c'', d''), $\cdots$, $(c^{(k)}, d^{(k)})$ are the intervals of (a, b) complementary to the intervals $(a^{(\nu)}, b^{(\nu)})$, we have $\sum_{\nu=1}^{k}(d^{(\nu)}-c^{(\nu)}) < \eta$. On account of the absolute continuity of $g(x)$, $\sum_{\nu=1}^{k}|g(d^{(\nu)})-g(c^{(\nu)})|$ approaches 0 with η. Since $\sum_{\nu=1}^{h}|g(b^{(\nu)})-g(a^{(\nu)})| \leqq \varepsilon\sum(b^{(\nu)}-a^{(\nu)})$, we may conclude that $|g(b)-g(a)|$ may be made as small as we please, and it is therefore zero. Since this result holds for every interval (a, b) in which $g(x)$ is absolutely continuous, it follows that for an absolutely continuous function to have zero as derivative except in a set of measure zero means that it is identically zero.

Let S be the set of points where $dy/dx > p > 0$ for the curve $\varphi(x, y) = c$; S is, then, of positive measure if, as we suppose, p is sufficiently small. For convenience, we restrict ourselves to the subset T of S that lies in an interval I, and we suppose, as we may, that the measure t of T is positive. Let T_n be the set of points common to T and the complement of E_n. The measure t_n of T_n is at least $t-e_n$, which is positive for (fixed p and) sufficiently large n.

We show that if all the points of T_n are covered by intervals (a_ν, b_ν), then $\sum|b'_\nu-a'_\nu| > p\,t_n$, where a'_ν and b'_ν represent respectively the ordinates of the points of the curve $\varphi(x, y) = c$ whose abscissas are a_ν and b_ν. For each point ξ of T_n may be enclosed in an interval $I_\xi \equiv (a_\xi, b_\xi)$ that lies in one of the intervals (a_ν, b_ν), is of arbitrarily small length, and is such that $(b'_\xi-a'_\xi)/(b_\xi-a_\xi) > p$, where a'_ξ, b'_ξ are the ordinates of $\varphi(x, y) = c$ corresponding to the abscissas a_ξ, b_ξ. According to Vitali's Theorem, a set of non-overlapping intervals (c_ν, d_ν) may be selected from the (a_ξ, b_ξ) covering T_n except for a set of measure zero. Since every interval (c_ν, d_ν) lies in an interval (a_ν, b_ν), and the ordinate of $\varphi(x, y) = c$ is monotone, we have $\sum|b'_\nu-a'_\nu| \geqq \sum|d'_\nu-c'_\nu|$, where c'_ν, d'_ν are the ordinates of $\varphi(x, y) = c$ corresponding to the abscissas c_ν, d_ν. But $\sum|d'_\nu-c'_\nu| > p\sum(d_\nu-c_\nu) \geqq p\,t_n$, since the (c_ν, d_ν) cover T_n except for a set of measure zero. Therefore $\sum|b'_\nu-a'_\nu| > p\,t_n$.

Since the numbers ξ belong to T_n, which is a subset of the com-

* This conclusion cannot be drawn if the condition of absolute continuity is omitted; for as we show in the appended Note, functions may be properly monotone and yet have zero as derivative except in a set of measure zero.

plement of E_n, the corresponding ordinates η of the curve $\varphi(x, y) = c$ must, as we have shown above, belong to E_n; we may therefore cover these η's by means of intervals (α_i', β_i'), $i = 1, 2, \cdots, \infty$ such that $\sum_{i=1}^{\infty}(\beta_i' - \alpha_i') < e_n + \delta$, where δ is arbitrarily small. Let α_i, β_i be the abscissas of the points of the curve $\varphi(x, y) = c$ whose ordinates are α_i', β_i'. The intervals (α_i, β_i) — or (β_i, α_i) in case $\varphi(x, y) = c$ is decreasing — cover all the points of T_n; therefore, according to what we have proved above concerning such a covering of T_n, we have $\sum(\beta_i' - \alpha_i') > p\, t_n \geqq p(t - \varepsilon_n)$. Since e_n, ε_n and δ are infinitesimals independent of p, we have a contradiction. Therefore, $f(x)$ is measurable.

The condition that the curves $\varphi(x, y) = k$ are absolutely continuous may be replaced by the milder condition that the points of the curve $\varphi(x, y) = k$ where the derivative is different from zero constitute a set of positive measure. It is furthermore sufficient to make either of these demands, not for the totality C of curves $\varphi(x, y) = k$, but for some subset C' of C such that the k's of the curves that belong to the complement C'' of C' are not everywhere dense. For then an interval (a, b) exists such that if $a \leqq k \leqq b$, then $\varphi(x, y) = k$ belongs to C'. Since $f(x)$ is, if discontinuous, unbounded at every point, we may, for every n, select c such that $a \leqq c \leqq b$, and $|f(c)| > n$, thus obtaining a curve $\varphi(x, y) = c$ that possesses the properties utilized in the proof of Theorem VI.

2. **Generalizations.** We may note that the proofs of theorems I, II, and III initiate from the fact that the functional equation $\varphi(f(x), f(y)) = f(\varphi(x, y))$ implies that if (a, b, c) is on the surface $z = \varphi(x, y)$, then $(f(a), f(b), f(c))$ is also on the surface. The sequel of the proof does not essentially interrelate the range of (a, b, c) and the range of $(f(a), f(b), f(c))$, so that we may allow the point $(f(a), f(b), f(c))$ to range, not on the same surface $z = \varphi(x, y)$, but on another surface $z = \psi(x, y)$, from which we make the same demands as from $\varphi(x, y)$; namely, that $\psi(x, y)$ shall be a real, continuous function defined for the entire xy-plane, and properly monotone in the arguments taken separately. We are thus led to the functional equation

$$\psi(f(x), f(y)) = f(\varphi(x, y)), \tag{2}$$

which is, indeed, such that if (a, b, c) is on the surface $z = \varphi(x, y)$, then $(f(a), f(b), f(c))$ is on the surface $z = \psi(x, y)$. The reasoning leading to theorems I, II, and III may be repeated for equations (2), with modifications that are apparent. We may thus state the following theorem:

THEOREM III'. *If $f(x)$ is a discontinuous solution of equation* (2), *it is positively and negatively unbounded at every point.*

In the case of theorems IV and V, the generalization for equation (2) will involve only the function ψ:

THEOREM IV'. *If $\psi(x, y)$ has a finite upper or lower bound, for some fixed x or y, then every solution of equation* (2) *is everywhere continuous.*

THEOREM V'. *If there exists a curve $\psi(x, y) = k$ that has an asymptote parallel to one of the coördinate axes, every solution of equation* (2) *is everywhere continuous.*

In the case of Theorem VI, however, the reference in the generalized theorem will be to φ alone:

THEOREM VI'. *If $\varphi(x, y)$ is subjected to the further condition that the curves $\varphi(x, y) = k$ are absolutely continuous, then every discontinuous solution of equation* (2) *is non-measurable.*

And again, as in the case of equation (1), the hypothesis of Theorem VI' may be replaced by the milder condition that the k's of the curves $\varphi(x, y) = k$ for which $dy/dx = 0$ everywhere except in a set of x's of zero measure are not everywhere dense.

Another extension is to three or more variables. For three variables—the case of n (>3) variables is analogous—the functional equation (2) takes the form

$$\psi(f(x), f(y), f(z)) = f(\varphi(x, y, z)).$$

If $z = k$, then $\psi(f(x), f(y), f(k)) = f(\varphi(x, y, k))$, so that f satisfies an equation of type (2).

In the case of one variable, however, the properties deduced for f need not hold. For example, let $\varphi(x) = x^2$. The functional equation $\varphi(f(x)) = f(\varphi(x))$, to which (1) reduces in the case of a single variable, then takes the form $(f(x))^2 = f(x^2)$; and one solution of this equation is $f(x) = x^n$ ($x \neq 0$), $f(0) = 1$.

Note.

The purpose of this note is to show the *existence of properly monotone functions having zero as derivative except in a set of measure zero.*

By means of a one-to-one continuous correspondence between the set of points in the interval (0, 1) and the Cantor ternary set constructed from this interval—a correspondence in which two end points of the same complementary interval of the Cantor set count as a single point—we may, as is well known, define in the interval (0, 1) a continuous, monotone function $f_0(x)$ that varies from $f_0(0) = 0$ to $f_0(1) = 1$ and is constant in the complementary intervals of the Cantor set. If (a, b) is any interval, we denote $(b-a) f_0\left(\frac{x-a}{b-a}\right)$ by $f_{ab}(x)$, $a \leqq x \leqq b$. Let $f_1(x) = \frac{1}{2} a_n + \frac{1}{2} f_{a_n b_n}(x)$, if x is an inner or boundary point of the complementary interval (a_n, b_n); and $f_1(x) = x/2$, if x belongs to no complementary interval. We denote the

intervals of constancy of $f_1(x)$ by (a_n', b_n'). Let $f_2(x) = \frac{1}{4}a_n' + \frac{1}{4}f_{a_n'b_n'}(x)$, if x belongs to an interval (a_n', b_n'); and $f_2(x) = x/4$ if x belongs to no such interval. Generally, we denote the intervals of constancy of $f_\nu(x)$ by $(a_n^{(\nu)}, b_n^{(\nu)})$, and let

$$f_\nu(x) = \frac{1}{2^\nu}(a_n^{(\nu-1)} + f_{a_n^{(\nu-1)}b_n^{(\nu-1)}}(x)),$$

if x belongs to $(a_n^{(\nu-1)}, b_n^{(\nu-1)})$, and

$$f_\nu(x) = \frac{x}{2^\nu},$$

if x belongs to no $(a_n^{(\nu-1)}, b_n^{(\nu-1)})$.

We set

$$g(x) = \sum_{\nu=0}^{\infty} f_\nu(x),$$

and shall prove that $g(x)$ is a continuous, properly monotone, increasing function having a zero derivative everywhere except in a set of measure zero.

Since

$$|f_{a_n^{(\nu-1)}b_n^{(\nu-1)}}(x)| = (b_n^{(\nu-1)} - a_n^{(\nu-1)})\, f_0\left(\frac{x - a_n^{(\nu-1)}}{b_n^{(\nu-1)} - a_n^{(\nu-1)}}\right) \leqq b_n^{(\nu-1)} - a_n^{(\nu-1)},$$

we have $f_\nu(x) \leqq 1/2^\nu$, so that the series $g(x)$ is absolutely and uniformly convergent; and since $f_\nu(x)$ is continuous, $g(x)$ is a continuous function. Moreover, since $f_\nu(x)$ is monotone increasing, $g(x)$ is monotone increasing. Since the intervals of constancy of $f_\nu(x)$ uniformly approach zero in length as ν approaches infinity, there exists for every pair of numbers x_1, x_2, $0 \leqq x_1 < x_2 \leqq 1$, a value ν such that x_1 and x_2 are not in the same interval of constancy of $f_\nu(x)$, so that $f_\nu(x_1) < f_\nu(x_2)$. Consequently $g(x_1) < g(x_2)$. $g(x)$ is therefore properly monotone.

Let $\sum_0^\nu f_\nu(x) = s_\nu(x)$. Since the sum of the lengths of the intervals where $s_\nu(x)$ is constant is 1, we may enclose the points of $(0, 1)$ not lying in the interior of these intervals in a finite number k of non-overlapping intervals (c_i, d_i) such that $\sum_{i=1}^k (d_i - c_i) < \varepsilon$, where ε is arbitrarily small. By taking ν sufficiently large, we may make $g(1) - s_\nu(1)$ less than η, where η is as small as we will. Since $s_\nu(x)$ is constant in an interval complementary to the (c_i, d_i), we have $\sum_{i=1}^k [s_\nu(d_i) - s_\nu(c_i)] = s_\nu(1)$. Therefore

$$g(1) - \sum_{i=1}^k [g(d_i) - g(c_i)]$$
$$= g(1) - s_\nu(1) - \sum [g(d_i) - g(c_i) - s_\nu(d_i) - s_\nu(c_i)] < \eta,$$

since, according to the definition of $g(x)$, we have $g(d_i) - g(c_i) > s_\nu(d_i) - s_\nu(c_i)$.

Now suppose the points where $g'(x) = 0$ constitute a set of measure < 1. Then there exists a positive number p and a set S of positive measure s such that, at every point x of S, one of the derivatives of $g(x)$ at x is greater than p. Let T be the portion of S lying outside the intervals (c_i, d_i). Since $\sum(d_i - c_i) < \varepsilon$, and the measure of S is independent of ε, the measure t of T will be as near s as we please, and therefore positive, if ε is sufficiently small. Every point x of T may be enclosed in an interval $I_x = (\alpha'_x, \beta'_x)$ of arbitrarily small length, such that

$$\frac{g(\beta_x) - g(\alpha_x)}{\beta_x - \alpha_x} > p.$$

We suppose, as we may, that every interval I_x lies entirely in the interior of a complementary interval of the (c_i, d_i). According to Vitali's Theorem, we may select from the I_x a set of non-overlapping intervals $(\alpha^{(\nu)}, \beta^{(\nu)})$, $\nu = 1, 2, \cdots$, that cover T except for a subset of it of zero measure. Therefore $\sum(\beta^{(\nu)} - \alpha^{(\nu)}) \geqq t$. Since the $(\alpha^{(\nu)}, \beta^{(\nu)})$ overlap neither each other nor the (c_i, d_i), we have

$$g(1) - \sum_{\nu=1}^{k}[g(d_i) - g(c_i)] \geqq \sum[g(\beta^{(\nu)}) - g(\alpha^{(\nu)})] > p\sum(\beta^{(\nu)} - \alpha^{(\nu)}) \geqq pt,$$

which contradicts the former inequality $g(1) - \sum[g(d_i) - g(c_i)] < \eta$ for sufficiently small η. This contradiction proves that $g'(x) = 0$ except in a set of measure 0.

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ENTIRE FUNCTIONS DEFINED BY CERTAIN POWER SERIES.

By Julia T. Colpitts.

The present paper deals with entire functions of the type

$$F(z) = \sum_{n=0}^{\infty} f(n) \frac{z^n}{n!}.$$

The coefficients are the values of any given function $f(x)$ which does not become infinite for any positive integral values of x. If $f(x)$ is a many valued function of x, the values $f(n)$ are understood to be uniquely defined by selecting those belonging to a particular branch of $f(x)$ or by some other defining process.

Throughout the paper, k represents a positive integer; $P_k(z)$ and $\varphi_k(n)$ are polynomials of degree k in z and n respectively. The paper will consider chiefly the following case: $f(n)$ a rational function of n, the most general form of which is $R(n)/P(n)$, where $R(n)$ and $P(n)$ are polynomials in n. In this case, $F(z)$ may be written,

$$\sum_{n=0}^{\infty} \varphi_k(n) \frac{z^n}{n!} + \text{terms of type } \sum_{n=0}^{\infty} \frac{c}{(n+a)^s} \frac{z^n}{n!},$$

where s is a positive integer. Series of the first type will be treated in Case I and of the second in Case II. In the latter case we shall go outside the realm of rational functions when the proofs can include rational fractional values of s.

That the genus in both cases is unity, is determined from a theorem of Lindelöf:* *Given* $F(z) = \sum_{n=0}^{\infty} c_n z^n$. *If* $\sqrt[n]{c_n}$ *is of the same order of magnitude as* $n^{-1/\varrho}$ *and if* $n^{1/\varrho} \sqrt[n]{c_n}$ *does not approach* 0 *or* ∞ *when* $n \to \infty$, *the genus of* $F(z)$ *is certainly equal to* ϱ, *if* ϱ *is an integer. If* ϱ *is not an integer, the genus* $= [\varrho]$ = *largest integer in* ϱ.

Case I. The function $F(z) = \sum_{n=0}^{\infty} \varphi_k(n)\, \alpha^n \frac{z^n}{n!}$.

Theorem I. *The necessary and sufficient condition that a function* $f(z)$ *of genus one have a finite number of zeros,* i. e. *that it be of form* $P_k(z)\, e^{\alpha z}$ *is that* $f(z) = \sum_{n=0}^{\infty} \varphi_k(n)\, \alpha^n \frac{z^n}{n!}$.

* Acta Societatis Scientiarum Fennicae, vol. 31, p. 46.

The case in which $\alpha = 1$ will first be considered. Pellet,* in the solution of a problem proposed by Darboux in 1868, proves that

$$(1) \qquad \sum_{n=0}^{\infty} \frac{\varphi(n)}{n!} = e\left[\varphi(0) + \frac{\Delta\varphi(0)}{1!} + \frac{\Delta^2\varphi(0)}{2!} + \cdots + \frac{\Delta^k\varphi(0)}{k!}\right],$$

where $\varphi(n)$ is a polynomial of degree k in n and

$$(2) \qquad \Delta^r\varphi(0) = \varphi(r) - \binom{r}{1}\varphi(r-1) + \binom{r}{2}\varphi(r-2) + \cdots + (-1)^r\varphi(0).$$

Using the same method, it follows that

$$\sum_{n=0}^{\infty} \varphi(n)\frac{z^n}{n!} = \varphi(0) + \frac{\varphi(0)+\Delta\varphi(0)}{1!}z + \frac{\varphi(0)+2\Delta\varphi(0)+\Delta^2\varphi(0)}{2!}z^2 + \cdots$$

$$= \left[\varphi(0) + \Delta\varphi(0)\frac{z}{1!} + \Delta^2\varphi(0)\frac{z^2}{2!} + \cdots + \Delta^k\varphi(0)\frac{z^k}{k!}\right]\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Hence

$$(3) \qquad \sum_{n=0}^{\infty} \varphi_k(n)\frac{z^n}{n!} = P_k(z)e^z.$$

Now consider the converse of the above.

$$z^m e^z = \sum_{n=0}^{\infty} z^m \frac{z^n}{n!} = \sum_{n=m}^{\infty} \frac{z^n}{(n-m)!},$$

that is,

$$(4) \qquad z^m e^z = \sum_{n=0}^{\infty} n(n-1)\cdots(n-m+1)\frac{z^n}{n!}.$$

From this result, it is seen that the function $P_k(z)e^z$ may be put in the form $\sum\varphi_k(n)\frac{z^n}{n!}$.

By substituting $\alpha z = z'$ in the above results, Theorem I is obtained.

From algebra we have: The necessary and sufficient condition that the nth term of a sequence be of form $\varphi_k(n)$ is that the sequence have its kth differences equal. From this follows:

THEOREM II. *The necessary and sufficient condition that a function* $f(z) = \sum_{n=0}^{\infty} a_n\frac{z^n}{n!}$ *be of form* $P_k(z)e^{\alpha z}$ *is that* $a_0, a_1, a_2, \cdots$ *may be written* $b_0\alpha^0, b_1\alpha^1, b_2\alpha^2, \cdots$ *in which* $b_0, b_1, b_2, \cdots$ *have their* k*th differences equal.*

* Nouv. Ann., sér. 2, vol. 9, 1870, pp. 417–419.

The zeros of $f_k(z) = \sum_{n=0}^{\infty} n^k \frac{z^n}{n!}$ will now be studied. In theorems concerning real zeros of a function, only real values of z are considered.

By differentiating $f_{k-1}(z) = \sum_{n=0}^{\infty} n^{k-1} \frac{z^n}{n!}$, it is seen that

$$z f'_{k-1}(z) = f_k(z). \tag{5}$$

THEOREM III. *The function $f_k(z)$ has k zeros, one equal to zero and the others negative. When $k>2$, at least one zero lies between 0 and -1. There is a negative zero greater than and also one less than any of $f_{k-1}(z)$.*

This follows from a theorem of Borel*: *Let $F(z)$ be an entire function of genus zero or one. If all the roots of $F(z)$ are real, the derived function $F'(z)$ has all its roots real and between two consecutive roots of $F(z)$, there is a single root of $F'(z)$. The genus of $F'(z)$ is the same as that of $F(z)$.*

The genus of $f_k(z)$ is unity and the zeros of $f_1(z)$ and $f_2(z)$ are real. Then from Borel's theorem, those of $f_3(z) = z f'_2(z)$ are real; likewise those of $f_4(z) = z f'_3(z)$. Continuing, the zeros of $f_k(z)$ are real for every k. From (5), one zero is 0. The remaining zeros are negative, since the coefficients are all positive.

Since $f_k(z)$ has one more zero than $f_{k-1}(z)$ and, excepting 0, a single zero of $f_k(z)$ lies between each consecutive pair of $f_{k-1}(z)$, then one zero of $f_k(z)$ must be less than any of $f_{k-1}(z)$.

Case II. Functions of the type $\sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \frac{z^n}{n!}$, where s is a positive rational number and a is any real number excepting a negative integer or zero.

Such functions will be denoted by $f_{a,s}(z)$. By differentiating $f_{a,s}(z)$, we obtain

$$f'_{a,s}(z) = f_{a+1,s}(z). \tag{6}$$

In a paper by G. H. Hardy† it is proved that

$$f_{a,s}(z) = \frac{1}{\Gamma(s)} \int_0^1 e^{zt} t^{a-1} \left[\log \frac{1}{t}\right]^{s-1} dt, \quad a>0, \quad s>0. \tag{7}$$

In this paper, Hardy determines asymptotically the zeros of $f_{a,s}(z)$; in the present paper the real zeros of $f_{a,s}(z)$ will be studied.

* É. Borel, Fonctions Entières, 2e éd., p. 32.

† Proc. Lond. Math. Soc., ser. 2, vol. 2 (1904–5), pp. 401–431.

When z is real and $a>0$, $f_{a,s}(z)>0$ since each factor under the integral sign of (7) is positive between the limits. We thus obtain:

THEOREM IV. *When* $a>0$, $f_{a,s}(z)$ *has no real zero.*

The real zeros of $f_{a,s}(z)$ will be discussed for two general cases:

$$(1)\quad s=\frac{2m+1}{2j+1}, \qquad (2)\quad s=\frac{2m}{2j+1},$$

where m and j are positive integers. In each case, only real values of $(n+a)^s$ are considered. The exponent s is called "odd" in the first place and "even" in the second. Only real values of z are considered in the following theorems. When $a<0$, a will often be replaced by $-k-1/q$.

THEOREM V. *When* s *is odd,* $f_{-k-1/q,s}(z)>0$ *when* $z=(q-1)^s(k+1)$, $q>2$, *and when* $z=k+1$, $q\leqq 2$.

$$(8)\quad f_{-k-1/q,s}(z)=-q^s\sum_{n=0}^{k}\frac{1}{(1+qk-qn)^s}\frac{z^n}{n!}+q^s\sum_{k+1}^{\infty}\frac{1}{(qn-qk-1)^s}\frac{z^n}{n!}.$$

Case 1. Given $q>2$. When $z=(q-1)^s(k+1)$, the first positive term is equal to the modulus of the last negative. It will now be proved that for this z, u_{r+1} the $(r+1)$st positive term is, for every value of $r>0$, greater than v_{r+1} the modulus of the $(r+1)$st negative counting from the last negative.

Simplifying $\frac{u_{r+1}}{v_{r+1}}$ and substituting the above value of z, we obtain

$$(9)\quad \frac{u_{r+1}}{v_{r+1}}=\left[\frac{qr+1}{qr+q-1}(q-1)^{2r+1}\right]^s\frac{(k+1)^{2r+1}}{(k-r+1)(k-r+2)\cdots(k+r+1)}.$$

The expression in brackets may be written

$$\frac{qr(q-1)+q-1}{qr+q-1}(q-1)^{2r}>1, \quad \text{since } q>2.$$

In the denominator of the last fraction of (9) are $2r+1$ factors, the middle one of which is $k+1$. Let A and B represent factors equidistant from the ends. Since the geometric mean between two numbers is less than the arithmetic mean,

$$AB<(k+1)^2.$$

The last fraction of (9) is therefore greater than unity. Hence, corresponding to each negative term of (9), there is a positive term numerically greater. In addition, there is an infinite number of positive terms.

Case 2. A similar proof holds.

THEOREM VI. *If s is odd, and a negative, not an integer, then $f_{a,s}(z)$ has one and only one positive zero and this zero is less than that of $f_{a-1,s}(z)$.*

Since $a<0$, $f_{a,s}(z)$ has at least one positive zero. It will now be proved that there is only one.

Take $-1<a<0$, then $a+1>0$ and, from Theorem IV, $f_{a+1,s}(z)>0$ for every z. From (6), $f'_{a,s}(z)=f_{a+1,s}(z)>0$. Then $f_{a,s}(z)$ is always increasing and consequently has but one positive zero, say at $z=c$.

Since $f'_{a-1,s}(z)=f_{a,s}(z)$, then $f_{a-1,s}(z)$, which is negative at $z=0$, decreases until $z=c$, then increases having a zero greater than c. Continuing, we may say that, if $f_{a,s}(z)$ has a positive zero, $z=z_0$, then $f_{a-1,s}(z)$ has a minimum at z_0 and a zero $>z_0$.

In the following work, the positive zero of $f_{a,s}(z)$ will be represented by z_0.

THEOREM VII. *If s is odd and a negative, not an integer, then in each region in which $\beta(a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\frac{z^n}{n!}$ is uniformly convergent, z_0 increases as a increases.*

The function $\beta(a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\frac{z^n}{n!}$ is continuous excepting when a is a negative integer or zero; is convergent in intervals, $-1<a<0$, $-2<a<-1$, and so on; and is uniformly convergent in intervals $-1+\varepsilon<a<0$, $-2+\varepsilon<a<-1$, and so on.

From a well known theorem,* it follows that, in the intervals in which $\beta(a)$ is uniformly convergent,

$$\frac{d}{da}[\beta(a)]=\sum_{n=0}^{\infty}-\frac{s}{(n+a)^{s+1}}\frac{z^n}{n!}$$
$$=-sf_{a,s+1}(z).$$

Now $s+1$ is even and $f_{a,s+1}(z)>0, z>0$. Then as a increases $f_{a,s}(z)$ decreases if $z>0$ and constant. Let $f_{a,s}(z_1)=0$. Take a' greater than a and in the same interval of uniform convergence. Then $f_{a',s}(z_1)<0$. Hence the positive zero of $f_{a',s}(z)$ is greater than z_1.

If $a=-k-1/q$, the positive zero of $f_{a,s}(z)$ increases with q, if k is constant.

The variation of z_0 in any interval between $-k-1$ and $-k$ is given by the theorem: *If $q=1+\varepsilon, \varepsilon$ an infinitesimal, then $z_0\to 0$, its approximate value being $\varepsilon^{s/(k+1)}\left[\frac{(k+1)!}{(k+1)^s}\right]^{\frac{1}{k+1}}$. If $q=1/\varepsilon$, $z_0\to\infty$.*

If $q=1+\varepsilon$, then

$$f_{-k-1/q,s}(z)=\sum_{n=0}^{\infty}\frac{(1+\varepsilon)^s}{[(n-1-k)+(n-k)\varepsilon]^s}\frac{z^n}{n!}.$$

* Cf. Goursat's Mathematical Analysis, vol. II, part I, p. 88.

If $z = e^{s/(k+1)}\left[\frac{(k+1)!}{(k+1)^s}\right]^{\frac{1}{k+1}}$, the first positive term is finite and approximately equal to the modulus of the first negative, while each of the other terms is infinitesimal. The sum of the remaining negative terms is infinitesimal since there is a finite number of them. The remaining positive terms form a series convergent for every z with z^{k+2} as a factor. Their sum is infinitesimal for the above value of z.

If $q = 1/\epsilon$, $f_{-k-1/q,s}(z) = P+R$, where $P = -\sum_{n=0}^{k} \frac{1}{(k+\epsilon-n)^s}\frac{z^n}{n!}$ and $R = \sum_{k+1}^{\infty} \frac{1}{(n-k-\epsilon)^s}\frac{z^n}{n!}$. Since $|P| > \frac{1}{\epsilon^s}\frac{z^k}{k!}$ and $R < e^z$, then z must become infinitely large in order that $R = |P|$.

Approximate value of z_0. In certain cases, an upper limit to z_0 may be found by means of the theorems given below as (A), (B), (C), (D).

(A) *If s is an odd integer, $f_{-1/q,s}(z)$ is positive for any integral value of k satisfying the inequality*

$$k-(s+1/2)\log k > s\log q + 1/2\log\frac{2\pi}{(2s+1)^2} + \frac{\theta}{12k},\quad 0<\theta<1.$$

This may be proved as follows. It is seen that

$$f_{-1/q,s}(z) > -q^s + \sum_{n=1}^{\infty}(1/n)^s\frac{z^n}{n!} \tag{10}$$

We wish to determine a positive integer $k > 2s$, so that the right member of (10) is positive when $z = k$. Consider the particular term $\frac{1}{k^s}\frac{k^k}{k!} = t_k$. This is greater than any following term but less than some preceding it. Let us determine the number of terms which exceed t_k, i. e., determine for what values of r

$$\frac{1}{(k-r)^s}\frac{k^{k-r}}{(k-r)!} > \frac{1}{k^s}\frac{k^k}{k!}$$

or

$$\frac{k(k-1)\cdots(k-r+2)\,[k-(r-1)]}{(k-r)^s\,k^{r-s}} > 1. \tag{11}$$

First, take $r = s-m$, $m = 0, 1, \cdots, (s-1)$. The first member of (11) may be written

$$\frac{k^{m+1}(k-1)\cdots[k-(r-1)]}{(k-r)^{m+r}},$$

which is seen at once to be greater than 1.

Second, take $r = s+m$, $m = 1, 2, \cdots, s$. The denominator of (11) equals $(k^2-kr)^m\,k-r)^{s-m}$. The product of the last $s-m$ factors of the numerator is greater than $(k-r)^{s-m}$. It remains to show that the product of the remaining $2m$ factors is greater than $(k^2-kr)^m$.

The product of the jth factors from beginning and end of the $2m$ factors is

$$(12)\quad (k-j+1)(k-2m+j) = k^2-(2m-1)k+(j-1)(2m-j).$$

Since $2m>j$ and $j\geqq 1$, then $(j-1)(2m-j)\geqq 0$. Also $2m-1<r$. Hence (12) is greater than k^2-kr and (11) is satisfied when $r\leqq 2s$. There are then at least $2s+1$ terms of (10), each greater than or equal to t_k.

The right member of (10), and hence $f_{-1/q,s}(z)$, will be positive when

$$(2s+1)\frac{1}{k^s}\,\frac{k^k}{k!} > q^s;$$

or, using Stirling's formula, when

$$\frac{(2s+1)\,e^k}{k^{s+1/2}\,(2\pi)^{1/2}} > q^s\,e^{\frac{\theta}{12k}},\quad 0<\theta<1,$$

i. e., when $k-(s+1/2)\,\log k > s\log q + 1/2\log\dfrac{2\pi}{(2s+1)^2}+\dfrac{\theta}{12k}$.

(B) *For the function* $f_{-1/q,s}(z)$, $z_0<(q-1)^s$.

If $z=(q-1)^s$, the first positive term of $f_{-1/q,s}(z)$ is equal numerically to the negative term. Then $z_0<(q-1)^s$. If $q=1+\varepsilon$, $z_0<\varepsilon^s$.

(C) *When* $z=\sqrt{q}+1$, $f_{-1/q,1}(z)>0$.

It is seen that

$$f_{-1/q,1}(\sqrt{q}+1) > -q+\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{(\sqrt{q}+1)^n}{n!}.$$

It will be shown that the sum of the first five positive terms is greater than q.

$$\sum_{n=1}^{5}\frac{1}{n}\,\frac{(\sqrt{q}+1)^n}{n!} > \sqrt{q}\left(1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)+\frac{119}{240}\,q+\frac{41}{360}\,q^{3/2}$$

$$+\frac{3}{160}\,q^2+\frac{q^{5/2}}{600} = q\left[\frac{e-1}{\sqrt{q}}+\frac{119}{240}+\sqrt{q}\left(\frac{41}{360}+\frac{3}{160}\sqrt{q}+\frac{q}{600}\right)\right].$$

When $q\leqq 9$, the sum of the first two terms in brackets is greater than one. When $q>9$, the quantity in brackets is greater than

$$\frac{e-1}{\sqrt{q}}+\sqrt{q}\,(.18)+\frac{119}{240}>1.$$

It follows that $f_{-1/q,1}(\sqrt{q}+1)>0$.

(D) *For the function* $f_{-1/q,1}(z)$, z_0 *is less than* $\sqrt{q}$ *excepting in the interval* $6.3 < q < 10.4$ *where it lies between* $\sqrt{q}$ *and* $\sqrt{q}+1$.

Let

$$u(q) = \sum_{n=0}^{\infty} \frac{q}{nq-1} \frac{q^{n/2}}{n!}.$$

Then

$$u'(q) = \sum_{n=0}^{\infty} \frac{(n^2q-n-2)q^{n/2}}{2(nq-1)^2 n!}$$

$$= -1 + \frac{\sqrt{q}(q-3)}{2(q-1)^2} + \text{terms which are positive if } q>1;$$

$$u''(q) = \sum_{n=0}^{\infty} \frac{n^3q^2-2n^2q^2-2n^2q+8q+n+2}{4(nq-1)^3} \frac{q^{n/2-1}}{(n-1)!}.$$

It is seen that $u''(q)>0$ if $q>1$; hence $u'(q)$ is increasing when $q>1$. By substitution, it is found that $u'(q)$ vanishes once at q' where $7<q'<9$. From above and by substitution we find that, as q increases from 1 to q', $u(q)$ decreases from ∞ becoming zero when $q=6.3$ and, as q increases from q', $u(q)$ increases becoming zero when $q=10.4$.

When $s=1$, an approximate value of z_0 is best obtained by using:

(B) when $q-1<\sqrt{q}$, i. e. when $1<q\leqq 2.6$;

(D) when $2.7\leqq q\leqq 49$;

(A) when $q>49$.

The value from (A) for large values of q is illustrated by the following example: When $s=1$ and $q=3000$, the smallest integral value of k satisfying the inequality in (A) is 12 while the actual zero is between 10 and 11.

The *negative zeros of* $f_{-k-1/q,s}(z)$ will now be considered.

THEOREM VIII: *If* s *is odd, then when* $z=-k$, $f_{-k-1/q,s}(z)$ *is* (a) *positive when* k *is odd,* (b) *negative when* k *is even.*

$$f_{-k-1/q,s}(-k) = q^s(P+R), \text{ where } P = \sum_{n=0}^{k} (-1)^{n+1} \frac{1}{[(k-n)q+1]^s} \frac{k^n}{n!}$$

and $R = \sum_{n=k+1}^{\infty} (-1)^n \frac{1}{[(n-k)q-1]^s} \frac{k^n}{n!}$.

Let the modulus of each term of P and R be represented by $u_n = A_n \frac{k^n}{n!}$

In P, u_n increases with n, since A_n increases with n and $\frac{k^n}{n!}$ has its greatest value when $n=k-1$ and k. Similarly, in R, u_n decreases with n. The terms of P and R are alternately positive and negative. (a) The last term of P and the first term of R are each positive. Hence $P>0$ and

$R>0$. (b) The last term of P and first term of R are each negative. Hence $P<0$ and $R<0$.

A closer approximation to the negative zero may sometimes be obtained from

THEOREM IX. *When s is odd, and $s\geqq 1$ and $k>1$, $f_{-k-1/q,s}(1-k)$ is* (a) *positive when k is odd,* (b) *negative when k is even.*

The proof is the same as above excepting that, in this case, the factor $\frac{(1-k)^n}{n!}$ in the last term of P is less than in the preceding term. It can be proved however that $u_k>u_{k-1}$. For

$$\frac{u_k}{u_{k-1}}=(q+1)^s\,\frac{k-1}{k},$$

and since $s\geqq 1$ and $k>1$, then $(q+1)^s\,\frac{k-1}{k}>2\cdot 1/2=1$.

THEOREM X. *When s is odd, the number of negative zeros of $f_{-k-1/q,s}(z)$ is* (a) *k if $k=0$ or 1,* (b) *not more than k if $k>1$; and is odd or even according as k is odd or even. In any case there is no zero less than $-k$.*

For every k, $f_{-k-1/q,s}(0)<0$.

(a) Take $k=0$. From Theorem IV, $f_{1-1/q,s}(z)>0$ for every z. Then from (6), $f_{-1/q,s}(z)$ is always increasing and has no negative zero.

Take $k=1$. From (6) and the above, $f_{-1-1/q,s}(z)$ is decreasing, $z<0$. From Theorem VIII, $f_{-1-1/q,s}(-1)>0$. Hence when $k=1$, there is a zero $z=c$ where $-1<c<0$.

(b) Let $k=2$. We know that $f_{-2-1/q,s}(z)$ is negative when $z=-2$ and that it increases until $z=c$, then decreases. Hence the number of negative zeros is either 0 or 2 and there is no zero less than -2.

Continuing in this way with $k=3, 4$, and so on, it is seen that when k is even, the function always increases at least until $z=-k$ where it is negative; when k is odd, the function decreases at least until $z=-k$ where it is positive. Hence there is no zero less than $-k$.

From Theorem IX, we obtain the theorem: *When s is odd, and $s\geqq 1$ and $k>1$, $f_{-k-1/q,s}(z)$ has no zero less than $1-k$.*

The number of negative zeros when $k\leqq 7$ is determined from the following theorem: *The function $f_{-k-1/q,s}(z)$ has no negative zeros when $k=2, 4$ and 6.*

In proving this, we shall take $z=-y$, and, on account of Theorem X, it will be asssumed that $y<k$. Now $f_{-k-1/q,s}(-y)=q^s\,(P+R)$ where

$$P=\sum_{n=0}^{k}\frac{(-1)^{n+1}}{(1+qk-qn)^s}\,\frac{y^n}{n!}\quad\text{and}\quad R=\sum_{n=k+1}^{\infty}\frac{(-1)^n}{(qn-qk-1)^s}\,\frac{y^n}{n!}.$$

The terms of R are decreasing in absolute value and the first term is negative. Hence $R<0$. Take a series of three consecutive terms of P beginning with a negative and multiply one of the negative terms by r:

$$(13)\quad \frac{-y^{m-1}}{(m-1)!}\left[\frac{1}{(q(k-m+1)+1)^s}-\frac{1}{(q(k-m)+1)^s}\,\frac{y}{m}+\frac{r}{(q(k-m-1)+1)^s}\,\frac{y^2}{m(m+1)}\right].$$

The discriminant of the quantity in brackets is less than

$$\frac{1}{[q(k-m)+1]^{2s}}\left[\frac{1}{m^2}-\frac{4r}{m(m+1)}\right].$$

This is negative or 0 when $r \geqq \frac{m+1}{4m}$. Hence for these values of r, the series in (13) is negative for every y. Accordingly:

If $k=2$, $P<0$.

If $k=4$, first term + second + 1/2 third <0, and
1/2 third + fourth + fifth <0.

If $k=6$, first term + second + 1/2 third <0,
1/2 third + fourth + 2/3 fifth <0, and
1/3 fifth + sixth + seventh <0.

Therefore $P<0$ when $k=2$, 4 and 6.

When $k \leqq 7$ and odd, there is one and only one negative zero.

The following theorem determines the number of negative zeros when $q \to 1$ and when $q \to \infty$. *When k is even and s is odd, $f_{-k-1/q,s}(z)$ has no negative zero when $q=1+\varepsilon$ or $q=1/\varepsilon$, ε an infinitesimal. When k is odd, there is one negative zero which is given approximately by*

$$-\left[\frac{(k+1)!}{(k+1)^s}\right]^{1/(k+1)}\cdot\varepsilon^{s/(k+1)} \text{ when } q=1+\varepsilon; \text{ and by } -\varepsilon^{s/k}\cdot\left[\frac{k!}{k^s}\right]^{1/k} \text{ when } q=1/\varepsilon.$$

From Theorem X, it is unnecessary to consider $z<-k$; we therefore take $-k<z<0$. When $q=1+\varepsilon$ and k is even,

$$\sum_{n=k+3}^{\infty}\frac{(1+\varepsilon)^s}{[(n-k-1)+(n-k)\varepsilon]^s}\,\frac{z^n}{n!}<0,$$

since the terms decrease in absolute value and the first is negative. The sum of the first $(k+2)/2$ positive terms is less than the modulus of $\frac{(1+\varepsilon)^s}{\varepsilon^s}\,\frac{z^{k+1}}{(k+1)!}$ if z is finite; and is less than the modulus of the first term if z is infinitesimal. Hence $f_{-k-1/q,s}(z)<0$ when $z<0$ if k is even and $q=1+\varepsilon$.

When $q=1/\varepsilon$, the proof is similar to the above. If k is odd, the approximate value of the negative zero is obtained by a method similar to that used for the approximate positive zero when $q=1+\varepsilon$.

Some properties of $f_{a,1}(z)$ may be studied by means of the incomplete gamma-function

$$P_z(a) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{a+n}}{(n+a)n!}.$$

It is seen at once that

$$P_z(a) = z^a f_{a,1}(-z). \tag{14}$$

Hence

$$f_{a,1}(z) = \frac{P_{-z}(a)}{(-z)^a}. \tag{15}$$

It has been proved* that

$$P_z(a) = \Gamma(a)\left[1 - e^{-z}\sum_{n=0}^{a-1}\frac{z^n}{n!}\right] \tag{16}$$

where a is a positive integer. Hence

$$f_{a,1}(z) = \frac{\Gamma(a)}{(-z)^a}\left[1 - e^z \sum_{n=0}^{a-1}\frac{(-z)^n}{n!}\right] \tag{17}$$

where a is a positive integer.

Some facts concerning the negative zeros of $f_{a,1}(z)$ are given in the following theorem:

(a) *When k is even and $a = -k - \frac{1}{q}$, $f_{a,1}(z)$ has no negative zero.*

(b) *When k is odd, $f_{a,1}(z)$ has one and only negative zero.*

(c) *There is a zero at $z = -1$ for one and only value of a in each interval*

$$-2m-1-\frac{6}{(2m)!} \leqq a \leqq -2m-1-\frac{1}{(2m)!}$$

and

$$-2m-2+\frac{1}{(2m+1)!} \leqq a \leqq -2m-2+\frac{6}{(2m+1)!}, \quad m = 2, 3\cdots$$

and for no other value of a.

(a) This is determined from the theorem:† If z is any arbitrary, finite, positive number and if p is an integer $\geqq 0$, then $P_z(a)$ is always negative when a is in the interval $-2p-1 < a < -2p$.

If $z \leqq 0$ and k is even, then $P_{-z}(a)$ is negative. From (15), $f_{a,1}(z)$ is also negative.

(b) From (a), if k is odd, $f_{a+1}(z) < 0$ when $a < 0$; then from (6), $f_{a,1}(z)$ is decreasing. From Theorem VIII, it follows that there is one negative zero.

* Nielsen, Theorie der Gammafunktion, p. 28.

† Nielsen, Theorie der Gammafunktion, p. 34.

(c) This follows from the theorem proved by Gronwall:* The real zeros of $P_1(a)$ lie one and only one in each interval

$$-2m-1-\frac{6}{(2m)!} \leqq a \leqq -2m-1-\frac{1}{(2m)!}$$

and

$$-2m-2+\frac{1}{(2m+1)!} \leqq a \leqq -2m-2+\frac{6}{(2m+1)!}, \quad m=2, 3, \cdots.$$

The zeros of $f_{a,s}(z)$ when s is even will now be discussed. It is seen at once that the function has no positive zeros.

THEOREM XI. *If s is even, $f_{-k-1/q,s}(z)$ has at least one negative zero between 0 and $-(k+1)$ if k is even and $1 \geqq (q-1)^s+\left(\frac{q-1}{2q-1}\right)^s$; or if k is odd, and $1 \geqq \frac{1}{(q+1)^s}+\frac{1}{(q-1)^s}$.*

Let the given function be represented by $F(z)$. When s is even, $F(0)>0$. *Take k even.* $F(-k-1)=P+R$ where

$$(18)\quad P=\sum_{n=0}^{k-1}\frac{(-1)^n q^s}{(qn-qk-1)^s}\frac{(k+1)^n}{n!}, \quad R=\sum_{n=k}^{\infty}\frac{(-1)^n q^s}{(qn-qk-1)^s}\frac{(k+1)^n}{n!}.$$

The terms of both P and R are alternately positive and negative. There is an even number of terms in P with the last one negative and, as in Theorem VIII, it may be proved that the terms increase numerically; hence $P<0$. Beginning with the fourth, the sum of the terms of R is negative since they are decreasing numerically and the fourth is negative. The sum of the first three terms of R is negative or zero if

$$(19)\quad \frac{1}{(q-1)^s}\frac{(k+1)^{k+1}}{(k+1)!} \geqq \frac{(k+1)^k}{k!}+\frac{1}{(2q-1)^s}\frac{(k+1)^{k+2}}{(k+2)!}.$$

Since $\frac{(k+1)^{k+1}}{(k+1)!}=\frac{(k+1)^k}{k!}>\frac{(k+1)^{k+2}}{(k+2)!}$, then $R<0$ if

$$(20)\quad 1 \geqq (q-1)^s+\left(\frac{q-1}{2q-1}\right)^s.$$

The last term is positive and $<(1/2)^s$. Then the term $(q-1)^s$ must be less than 1. Hence q has only a small range of values, $1<q<q_0<2$.

* Bull. Amer. Math. Soc., vol. 22 (1916), p. 375.

Take k odd. $F(-k-1) = P+R$ where

$$(21)\quad P=\sum_{n=0}^{k-2}\frac{(-1)^n q^s}{(nq-qk-1)^s}\frac{(k+1)^n}{n!},\quad R=\sum_{n=k-1}^{\infty}\frac{(-1)^n q^s}{(nq-qk-1)^s}\frac{(k+1)^n}{n!}.$$

The proof is similar to that given when k is even. In this case, it may be shown that the sum of the first three terms of R is negative if

$$(22)\qquad 1 \geqq \frac{1}{(q+1)^s}+\frac{1}{(q-1)^s}.$$

Here the only restriction on q is that it must exceed a certain number which is greater than 2.

THEOREM XII. *When s is even, $f_{-k-1/q,s}(z)$ has not more than $k+1$ negative zeros.*

When $k=0$, from (6) and Theorem IV, we find that $f_{-k-1/q,s}(z)$ is always increasing and cannot have more than one negative zero. Then, when $k=1$, the function cannot have more than one maximum or minimum value. Now in a continuous function the number of zeros cannot exceed by more than one the sum of the number of maximum and minimum values of the function. Hence when $k=1$, the function cannot have more than two zeros. Continuing with $k=2, 3$, and so on, it is seen that for any k, there are not more than $k+1$ zeros.

THEOREM XIII. *Let* $F(z)=\sum_{n=0}^{\infty}\left(\frac{b_1}{n+a_1}-\frac{b_2}{n+a_2}\right)\frac{z^n}{n!}$, $a_k>0$, $b_k>0$, $k=1, 2$; *and* $a_1>a_2$. *Then $F(z)$ has no positive zero and not more than one negative zero. If* $\frac{b_1}{a_1}-\frac{b_2}{a_2}\leqq 0$, *$F(z)$ has no real zero.*

To prove this, the following theorem of Laguerre* will be used: *The number of real roots of* $\int_a^b e^{-zt}\varphi(t)\,dt$ *is at most equal to the number of roots of* $\int_a^t \varphi(u)\,du$ *which are included between a and b.*

From (7), $F(-z)=\int_0^1 e^{-zt}(b_1 t^{a_1-1}-b_2 t^{a_2-1})\,dt$. Let $\varphi(t)=b_1 t^{a_1-1}-b_2 t^{a_2-1}$. Take $f(t)=\int_0^t \varphi(u)\,du = t^{a_2}\left[\frac{b_1 t^{a_1-a_2}}{a_1}-\frac{b_2}{a_2}\right]$. Now $f(t)$ can have not more than one positive zero and can have no zero between 0 and 1 if $\frac{b_1}{a_1}-\frac{b_2}{a_2}\leqq 0$.

A necessary condition then that $F(z)$ have a real zero is

$$(23)\qquad \frac{b_1}{a_1}-\frac{b_2}{a_2}>0.$$

* Œuvres, vol. I, p. 29.

Assuming that (23) is satisfied, we shall prove that $F(z)$ has no positive zero.

$$F(z) = \sum_{n=0}^{\infty} \frac{(b_1 - b_2)\, n + a_2\, b_1 - a_1\, b_2}{(n + a_1)\,(n + a_2)} \frac{z^n}{n!}.$$

From (23), $b_1 > (a_1/a_2)\, b_2 > b_2$. Also $a_2\, b_1 - a_1\, b_2 > 0$. Hence $F(z) > 0$, when $z \geqq 0$.

THEOREM XIV. *Let* $F(z) = \sum_{n=0}^{\infty} \left(\frac{b_1}{n + a_1} - \frac{b_2}{n + a_2} - \frac{b_3}{n + a_3} \right) \frac{z^n}{n!}$, $a_k > 0$, $b_k > 0$, $k = 1, 2, 3$; *and* $a_1 > a_2 > a_3$. *Then* $F(z)$ *has no positive zero and not more than one negative zero. It has no real zero if* $\frac{b_1}{a_1} - \frac{b_2}{a_2} - \frac{b_3}{a_3} \leqq 0$.

Here $f(t) = t^{a_3} \left(\frac{b_1\, t^{a_1 - a_3}}{a_1} - \frac{b_2\, t^{a_2 - a_3}}{a_2} - \frac{b_3}{a_3} \right)$. As before, by use of Laguerre's theorem, it may be proved that $F(z)$ has not more than one real zero, and that a necessary condition for a real zero is

$$\frac{b_1}{a_1} - \frac{b_2}{a_2} - \frac{b_3}{a_3} > 0. \tag{24}$$

Assuming (24) to be true, it will be proved that $F(z) > 0$ when $z \geqq 0$.

$$F(z) = \sum_{n=0}^{\infty} \frac{An^2 + Bn + C}{(n + a_1)\,(n + a_2)\,(n + a_3)} \frac{z^n}{n!},$$

where

$$A = b_1 - b_2 - b_3,$$
$$B = b_1 (a_2 + a_3) - b_2 (a_1 + a_3) - b_3 (a_1 + a_2),$$
$$C = b_1\, a_2\, a_3 - b_2\, a_1\, a_3 - b_3\, a_1\, a_2.$$

From (24), $C > 0$. Now $a_2\, a_3 (b_1 - b_2 - b_3) > C > 0$; hence $A > 0$. B may be written:

$$\frac{1}{a_3} [(b_1\, a_2\, a_3 - b_2\, a_1\, a_3 - b_3\, a_2\, a_1) + (b_1 - b_2 - b_3)\, a_3^2 + b_3\, (a_2 - a_3)\, (a_1 - a_3)].$$

Then $B > 0$ since each quantity in parentheses is positive.

The function $F(z) = \sum_{n=0}^{\infty} \frac{dn + e}{(n + b)^2 + c^2} \frac{z^n}{n!}$ where b, c, d, and e are all real, will now be considered. It may be written $(e - bd)\, P_{b,c}(z) + d\, Q_{b,c}(z)$, where

$$P_{b,c}(z) = \sum_{n=0}^{\infty} \frac{1}{(n + b)^2 + c^2} \frac{z^n}{n!} \text{ and } Q_{b,c}(z) = \sum_{n=0}^{\infty} \frac{n + b}{(n + b)^2 + c^2} \frac{z^n}{n!}.$$

It will now be shown that $P_{b,c}(z)$ and $Q_{b,c}(z)$ may each be represented by integrals when $b>0$. From the formula $\int_0^\infty \sin ct(e^{-t})^n dt = c/(n^2+c^2)$, $n>0$, we have

$$\int_0^\infty \sin ct\,(e^{-t})^{n+b}\,dt = \frac{c}{(n+b)^2+c^2},\quad n+b>0.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{(n+b)^2+c^2}\,\frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{c}\int_0^\infty \sin ct\, e^{-bt}\,\frac{(ze^{-t})^n}{n!}\,dt. \tag{25}$$

Now apply the theorem that

$$\sum_{n=0}^{\infty}\int_a^\infty \varphi(x) f_n(x)\,dx = \int_a^\infty \varphi(x)\left[\sum_{n=0}^{\infty} f_n(x)\,dx\right]$$

if $\sum_{n=0}^{\infty} f_n(x)$ *is uniformly convergent for all values of* $x>a$ *and* $\int_a^\infty |\varphi(x)\,dx|$ *is convergent.* The conditions of this theorem are satisfied in (25) since $\sum_{n=0}^{\infty}\frac{(ze^{-t})^n}{n!}$ is uniformly convergent and $\int_0^\infty |\sin ct\, e^{-bt}dt|$ is convergent. Hence

$$\sum_{n=0}^{\infty} \frac{1}{(n+b)^2+c^2}\,\frac{z^n}{n!} = \frac{1}{c}\int_0^\infty \sin ct\, e^{-bt}\, e^{ze^{-t}}\,dt,\ b>0. \tag{26}$$

Similarly, by using the formula $\int_0^\infty e^{-nt}\cos ct\,dt = n/(n^2+c^2)$, $n>0$, we obtain

$$\sum_{n=0}^{\infty} \frac{n+b}{(n+b)^2+c^2}\,\frac{z^n}{n!} = \int_0^\infty \cos ct\, e^{-bt}\, e^{ze^{-t}}\,dt,\ b>0. \tag{27}$$

APPROXIMATION OF CURVES AND SURFACES BY ALGEBRAIC CURVES AND SURFACES.

By P. A. Smith.

1. Let J be a simple closed curve lying in the (x, y)-plane. By the Theorem of Jordan, J separates the plane into two regions, — the interior and exterior regions. The object of this paper is to obtain a polynomial series

$$\sum_{n=1}^{\infty} A_n(x, y) \tag{1}$$

which will converge uniformly in a region R containing J to a continuous function which is 0 on J, and which, in R, is > 0 at points exterior to J and < 0 at interior points. The series (1) moreover is to give rise (by equating successive sums to zero) to a sequence of *non-singular algebraic ovals* converging to J in a manner explicitely described in Theorem 1.

Analogous results will be obtained for $(n-1)$ dimensional manifolds in n-space (for example, simple closed surfaces in 3-space) but only for a restricted class of such manifolds (§ 7). The approximation of the most general manifolds is discussed in § 8.

A special type of approximation of curves, namely by ovals of *unicursal* algebraic curves, is treated in § 5. It will not follow however that the series (1) can be chosen so as to give rise to a sequence of unicursal curves (see § 6).

Let $J(x, y)$ be a "distance function" with respect to J defined as follows: if $\delta(x, y)$ is the shortest distance from the point (x, y) to J, then $J(x, y) = \delta(x, y)$ for (x, y) exterior to J, $J(x, y) = -\delta(x, y)$ interior to J, and $J(x, y) = 0$ for (x, y) on J. It is evident that $J(x, y)$ is bounded and continuous in any closed region of the plane.

Let R be the rectangular region $a < x < b$, $c < y < d$, where a, b, c, d are so chosen that J will lie completely in R, and let R' designate R plus its boundary.* The existence of the series (1) will follow from

Theorem 1. *Let $\varepsilon_1, \varepsilon_2, \cdots$ be a sequence of positive numbers converging to zero. There exists for each n a polynomial $P_n(x, y)$ such that*

$$|P_n(x, y) - J(x, y)| < \varepsilon_n$$

* The accent will have this significance throughout this paper when the accented letter denotes a region.

in R', while the real part of the curve $P_n(x, y) = 0$ in R consists of a non-singular oval in (1,1) *continuous correspondence with J, the distance between corresponding points being $< \varepsilon_n$.*

This last situation will be described more briefly by the statement: "The oval is mapped (ε_n) on J".

For the series (1) we may obviously take

$$P_1(x, y) + \sum_{n=1}^{\infty} (P_{n+1}(x, y) - P_n(x, y))$$

since the sum of the first n terms of this series is $P_n(x, y)$.*

2. **Polynomial approximation.** We shall make use of the following approximation theorem:

Let S be the region defined by $c_1 < x < c_2$, $d_1 < y < d_2$ and let $f(x, y)$ be a function defined and continuous at every point of S' and such that $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous at every point of T' where T is a region which with its boundary lies in S. Then there exists a sequence of polynomials $\varphi_n(x, y)$ which converges uniformly to $f(x, y)$ in S' while $\partial \varphi_n/\partial y$ and $\partial \varphi_n/\partial y$ converge uniformly to $\partial f/\partial x$ and $\partial f/\partial y$ respectively in T'.

A proof of this theorem for one variable using for $\varphi_n(x)$ the polynomial

$$\frac{k_n}{2} \int_0^1 f(u) [1 - (u - x)^2]^n \, du, \qquad k_n = \frac{3 \cdot 5 \cdots (2n + 1)}{2 \cdot 4 \cdots 2n}$$

is given by de la Vallée-Poussin.† The extension of this method to two (or n) variables using for φ_n the polynomial

$$\left(\frac{k_n}{2}\right)^2 \int_0^1 \int_0^1 f(u, v) [1 - (u - x)^2]^n [1 - (v - y)^2]^n \, du \, dv$$

is automatic. It is assumed for this purpose that $0 < c_1 < c_2 < 1$ and $0 < d_1 < d_2 < 1$, which is permissable since S may be transformed by a suitable linear transformation into a region lying inside the square $0 < x, y < 1$. We further assume that the region of definition of $f(x, y)$ has been extended in such a way that $f(x, y)$ is defined and continuous not only in S but at every point of the region $0 \leqq x, y \leqq 1$.‡

* By known theorems on polynomial approximation (see for example Borel, Leçons sur les Fonctions de Variables Réelles, p. 62) there exist polynomials $Q_n(x, y)$ converging to $J(x, y)$. But in general the real branches of $Q_n = 0$ will have complicated singularities and hence Theorem 1 is not a trivial application of polynomial approximation theory.

† Bull. de l'Acad. Belg., Cl. des Sci., 1908, p. 193. Landau (Rend. del Circ. Mat. di Palermo, vol. 25, 1908, p. 337) uses polynomials similarly defined and proves their convergence for n variables but does not treat the convergence of the derivatives.

‡ We may define $f(x, y)$ outside of S' as follows: let Z be a fixed point of S and let an arbitrary ray having Z for an end-point cut the boundary of S at the point Z_1, (x_1, y_1)

3. Proof of Theorem 1. We shall prove this theorem by establishing a sequence of propositions which we shall mark with Roman numerals for reference.

I. *Let η_1 be an arbitrary positive number. There can be inscribed in J a simple polygon Π which can be mapped (η_1) on J.*

Proof. Let a parametric representation of J be given by

$$x = f(t), \qquad y = g(t)$$

where f and g are continuous for all finite values of t and each possesses the period ω. The distance between two points of J whose parameter values are t_1 and t_2 is

$$+\sqrt{[f(t_1)-f(t_2)]^2+[g(t_1)-g(t_2)]^2}$$

and, since f and g are continuous, we can choose a number $\delta > 0$ sufficiently small such that, if any two t's differ by less than δ, the distance between the corresponding points of J is $< \eta_1/2$.

Let us divide the interval $(t_0, t_0+\omega)$ into a finite number of sub-intervals each of length $< \delta/2$. J is correspondingly divided into the arcs $\alpha_1, \alpha_2, \cdots, \alpha_p$ named in circular order and two points on any α or on two successive α's are within a distance of $\eta_1/2$ from each other. We can now inscribe in J a simple polygon Π in such a way that at least one vertex of Π shall fall on each α and such that the vertices of Π shall follow each other in the same order with respect to Π as they do with respect to J.*

Let $V_1, V_2, \cdots, V_k, V_{k+1} = V_1$ be the vertices of Π named in the order of t increasing and call e_i the edge $V_i V_{i+1}$ $(i = 1, 2, \cdots, k)$. As t increases continuously from t_{ν_1}† to t_{ν_2} an arc β_1 is described on J. The points V_1

---

and the boundary of the region $0 < x, y < 1$ at the point Z_2; then define $f(x, y) = f(x_1, y_1)$ on the segment $Z_1 Z_2$.

The method of de la Vallée-Poussin leads to more general results than those here stated. Granting these results for the convergence of the $\varphi_n(x, y)$ and making certain assumptions concerning the derivatives of $f(x, y)$, the convergence of the derivatives of the $\varphi_n(x, y)$ can be proved very simply. We define $f(x, y)$ to be zero for points not in S'. Then on integrating $\dfrac{\partial \varphi_n}{\partial x}$ by parts, we obtain a polynomial of the form φ_n with $f(u, v)$ replaced by $\dfrac{\partial f(u, v)}{\partial u}$.

* A proof of this statement is given by de la Vallée-Poussin in his Cours d'Analyse Infinitésimale, 3rd ed., vol. 1, p. 379. The first step in the proof is to divide the plane into small rectangular regions q_i by a net of straight lines. There is a slight ambiguity in the proof in case any vertex of a q_i falls on J. This may be avoided however by slightly altering the net in the neighborhood of such vertices.

† t_{ν_1} being the value of t corresponding to V_1 in the interval $(t_0, t_0+\omega)$.

and V_2 are on the same α or on two successive α's. In either case the distance V_1V_2 is smaller than $\eta_1/2$. Furthermore, if C (fig. 1) is any point of β_1, the distance V_1C is $<\eta_1/2$, since $|t_{V_1}-t_c|<\delta/2$. Hence each point of e_1 is within a distance η_1 from every point of β_1. Hence e_1' ($=e_1$ plus its end-points) can be mapped (η_1) on β_1', and this can be done in such a way that V_1 and V_2, considered as end-points of β_1, correspond respectively to themselves considered as end-points of e_1.

The remaining edges $e_2, \cdots, e_k$ are now mapped in the same way on the corresponding arcs $\beta_2, \cdots, \beta_k$. Thus Π has been mapped (η_1) on J and the proof of I completed.

Consider now the closed curve $\sigma_i = e_i + \beta_i + V_i + V_{i+1}$ which may have multiple points. Let O be a point of the plane lying on neither Π nor J

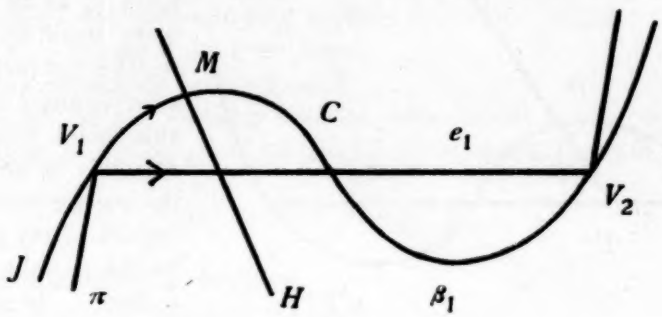

Fig. 1.

and let a ray be pivoted on O and pass through a movable point M of σ_i. As M describes σ_i once, the total angular variation of the ray is either zero or a non-zero multiple of 2π.

Let L be the set of points of the plane which lie on neither Π nor J and let $C_i (i = 1, \cdots, k)$ consist of those points of L for which the total angular variation with regard to σ_i is a non-zero multiple of 2π. Let C be the logical sum $C_1 + \cdots + C_k$ and D the set $L-C$.*

II. *Each point of D is either interior to both J and Π or exterior to both.*

For, let H be a point of D. Call v_J^1 the angular variation of a ray pivoting on H as the movable pointpasses from V_1 to V_2 along β_1 and call v_π^1 the corresponding variation when the movable point describes e_1 instead of β_1 (fig. 1).

The total angular variation with respect to H as the movable point completely describes σ_1 is evidently $v_J^1 - v_\pi^1$. But this variation is zero since H is not in C_1. Hence $v_J^1 = v_\pi^1$. Similarly $v_J^2 = v_\pi^2$, and so on.

* This set is not null; it contains for example the points of the plane not in R'.

Therefore the variation with respect to H as the movable point completely describes Π is exactly equal to the variation as the point describes J. Now, the total variation with respect to H as the movable point describes any simple closed curve not on H is either 0 or 2π according as H is exterior or interior to that curve.* Hence H is either exterior to both Π and J, or interior to both.

III. *Each point N of the set C is within a distance η_1 from Π and from J.*

For N is a point of some C_i, say C_1. Let E be a point of e_1. Draw the linear segment EN and extend it indefinitely beyond N forming the ray $EN\infty$ (fig. 2). The ray $N\infty$ must meet β_1. For if not, $N\infty$ does not touch σ_1 at all and hence there would be a position which a ray pivoting on N never assumes as the movable point describes σ_1; and this is absurd since the angular variation with respect to any point of C_1 as the movable point describes σ_1 is a non-zero multiple of 2π. Call, then, F a point which $N\infty$ has in common with β_1. Then $EF<\eta_1$ by the construction of Π and hence $EN<\eta_1$ and $NF<\eta_1$. This establishes III.

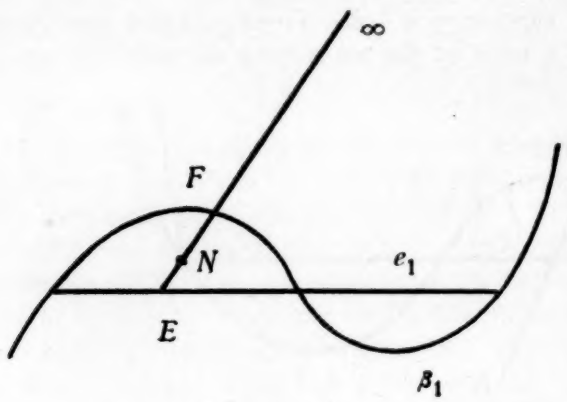

Fig. 2.

Let $\Pi(x, y)$ be the distance function with respect to Π. Then

IV. *The inequality*

$$|\Pi(x, y) - J(x, y)| < 2\eta_1$$

holds for all finite points of the plane.

For if (x, y) is on both Π and J, then $\Pi(x, y) = J(x, y) = 0$. Next, if (x, y) is interior to one of Π or J and exterior to the other, then it belongs to the set C and hence by III

$$|\Pi(x, y)| < \eta_1, \qquad |J(x, y)| < \eta_1.$$

Finally, suppose that (x, y) is interior to both Π and J or exterior to both. Then $\Pi(x, y)$ and $J(x, y)$ are of like sign and their absolute values differ by less than η_1. For since each point of Π is within a distance of η_1

* See Kerékjártó, Vorlesungen über Topologie I, p. 84.

from J and conversely, the shortest distance from (x, y) to J can not differ from the shortest distance to Π by more than η_1.

We now construct a "rounded-off" polygon K by fitting arcs of tangent circles into the corners of Π as shown in fig. 3.

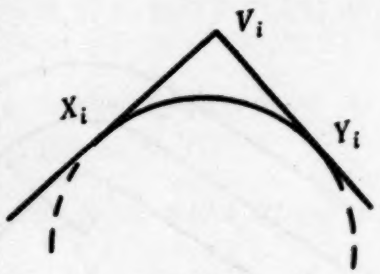

Fig. 3.

Let η_2 be an arbitrary positive number. Manifestly, if the circles used in constructing K are sufficiently small, then (a) K will be free from multiple points, and (b) K can be mapped (η_2) on Π. For (b) it is necessary to take the circles so small that each arc $X_i Y_i$ (fig. 3) can be mapped (η_2) on the corresponding broken line $X_i V_i Y_i$. By virtue of I we have

V. *K can be mapped $(\eta_1+\eta_2)$ on J.*

Let $K(x, y)$ be the distance function with respect to K. By applying the reasoning in the proofs of III and IV to the curves Π and K, we obtain

$$|\Pi(x, y)-K(x, y)|<2\eta_2.$$

Combining this with IV, we have

VI. *For all finite points of the plane*

$$|K(x, y)-J(x, y)|<2(\eta_1+\eta_2). \tag{2}$$

K consists alternately of linear segments and circular arcs, together with all the end-points. Referring alike to linear segments and arcs as edges, each edge to include its end-points, let p be the shortest distance between any two non-consecutive edges of K. Let q be the length of the smallest radius of the circles used in constructing K, and let $\mu>0$ be smaller than each of $p/2$, q.

Now let P be an arbitrary point of K and s a linear segment of length 2μ, normal to K and having P for its mid-point. When P is fixed, we shall call s together with its end-points a *normal segment.*

VII. *Two normal segments having distinct mid-points can not meet.*

This is obvious if the two mid-points P_1 and P_2 are on the same linear segment of K. If they are on the same arc or on a linear segment and its adjacent are, VII holds since $\mu<q$. Suppose then that P_1 and P_2 are on two non-consecutive edges of K. Then if the corresponding normal segments meet, the distance between P_1 and P_2 would be $\leq 2\mu<p$, contrary to the definition of p.

VIII. *A normal segment with mid-point P has no other point in common with K.*

This is a corollary of VII.

We now let P describe a complete circuit on K. It follows* from VIII that one end-point of s remains interior to K and the other exterior. Hence the end-points of s trace two closed curves,—one, Γ_e lying wholly exterior to K and the other, Γ_i, wholly interior. Since $\mu < q$, Γ_e and Γ_i will have no corners. They will in fact, consist of linear segments parallel to those of K and circular arcs concentric with the arcs of K (fig. 4).

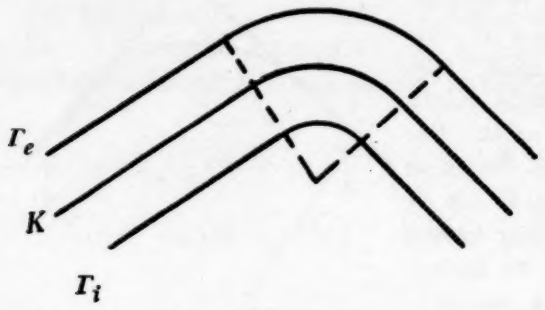

Fig. 4.

IX. *Γ_e and Γ_i possess no multiple points.*

For a multiple point of either curve would be on two or more normal segments through distinct points of K, contrary to VII.

The totality of normal segments constitutes a ring-shaped region U containing K and having the boundary $\Gamma_e + \Gamma_i$.

X. *Each point of U' is on one and only one normal segment.*

Each point of U is on at least one by construction of U. That it can be on only one follows from VII.

We now consider the distance function $K(x, y)$. From the nature of K and of the region U we see that the surface $z = K(x, y)$ will consist, for (x, y) in U of pieces of planes and pieces of right circular cones joined alternately,—the planes and the elements of the cones making uniformly an angle of 45° with the (x, y)-plane. A normal to the surface at the point (x, y, z) would vary continuously as (x, y) moves continuously in U; that is, its direction cosines are continuous functions of x and y in U. Hence $\frac{\partial K}{\partial x}$ and $\frac{\partial K}{\partial y}$ exist and are continuous at each point of U.

Let $\eta_3 > 0$ be arbitrarily small and let $\bar{\mu} > 0$ be smaller than each of μ, η_3. A region $\bar{U}$ is now to be constructed precisely as we constructed

* It is a theorem proved by Lennes (Theorems on Simple Finite Polygon and Polyhedron, Am. Journ., 1911, p. 43) that, if a linear segment AB meets a polygon in only one point C not a vertex, then AC and BC lie one in the interior region of the polygon and one in the exterior region. It is a simple matter to extend this theorem to a polygon such as K, with C any point of the polygon and AB not tangent to one of the arcs.

U except that we shall now take s to be of length $2\bar{\mu}$. Let $\bar{\Gamma}_e$ and $\bar{\Gamma}_i$ be the new curves corresponding to the curves Γ_e and Γ_i. $\bar{U}$ together with its boundary lies wholly in U. Hence $\frac{\partial K}{\partial x}$ and $\frac{\partial K}{\partial y}$ are continuous at each point of $\bar{U}$ *and its boundary*. When (x, y) is on $\bar{\Gamma}_e$, $K(x, y) = \bar{\mu}$, while for (x, y) on $\bar{\Gamma}_i$, $K(x, y) = -\bar{\mu}$. Moreover

$$(3)\qquad \begin{aligned} K(x, y) &> \bar{\mu} \qquad \text{for } (x, y) \text{ exterior to } \bar{\Gamma}_e, \\ K(x, y) &< \bar{\mu} \qquad \text{for } (x, y) \text{ interior to } \bar{\Gamma}_i. \end{aligned}$$

By the theorem of § 2 we can obtain now a polynomial $P(x, y)$ such that in R'*

$$(4)\qquad |P(x, y) - K(x, y)| < \bar{\mu}$$

while in $\bar{U}'$,

$$(4a)\qquad \left|\frac{\partial P}{\partial x} - \frac{\partial K}{\partial x}\right| < \frac{1}{2}, \qquad \left|\frac{\partial P}{\partial y} - \frac{\partial K}{\partial y}\right| < \frac{1}{2}.$$

By (2) and (4) we shall have, since $\bar{\mu} < \eta_3$,

$$(5)\qquad |P(x, y) - J(x, y)| < 2(\eta_1 + \eta_2) + \eta_3$$

uniformly in R'.

By virtue of (3) and (4) for (x, y) in R'

$$\begin{aligned} P(x, y) &> 0 \qquad \text{on and exterior to } \bar{\Gamma}_e, \\ P(x, y) &< 0 \qquad \text{on and interior to } \bar{\Gamma}_i. \end{aligned}$$

Hence the points of R' at which $P(x, y)$ is zero lie in $\bar{U}$. These points constitute a set Δ of one or more real branches of the curve $P(x, y) = 0$.

XI. Δ *meets each normal segment of* $\bar{U}$ *at one and only one point.*

Proof. Each normal segment intersects Δ at least once; for $P(x, y) > 0$ at that end of the segment which lies on $\bar{\Gamma}_e$, while $P(x, y) < 0$ at the other end.

Each normal segment meets Δ at most once. To prove this we note first that

$$\frac{dP}{dn} - \frac{dK}{dn} = \left(\frac{\partial P}{\partial x} - \frac{\partial K}{\partial x}\right)\cos\alpha + \left(\frac{\partial P}{\partial y} - \frac{\partial K}{\partial y}\right)\sin\alpha$$

where $\frac{dP}{dn}$ and $\frac{dK}{dn}$ denote derivatives along a normal to K in an out-

* We can obviously assume that R contains all the geometric elements which we have constructed up to this point, in particular, Γ_e and Γ_i. For if this were not the case, we could here increase the size of R.

ward direction, α being the angle which the normal makes with the x-axis. Hence

$$\left|\frac{dP}{dn}-\frac{dK}{dn}\right| \leqq \left|\frac{\partial P}{\partial x}-\frac{\partial K}{\partial x}\right| |\cos\alpha| + \left|\frac{\partial P}{\partial y}-\frac{\partial K}{\partial y}\right| |\sin\alpha|$$

and hence by (4a)

$$\left|\frac{dP}{dn}-\frac{dK}{dn}\right| < \frac{1}{2}(|\sin\alpha|+|\cos\alpha|) < 1. \tag{6}$$

Now if $P(x, y)$ is zero at two distinct points S_1 and S_2 of a normal segment, then at a third point S_3 lying on the segment $S_1 S_2$ we would have $\frac{dP}{dn} = 0$ by Rolle's Theorem. Then at S_3

$$\left|\frac{dP}{dn}-\frac{dK}{dn}\right| = 1$$

since $\frac{dK}{dn} = 1$ at every point of $\bar{U}$. This contradicts (6) and hence XI is proved.

XII. *Δ can be mapped (η_3) on K.*

Proof. By making each point P of K correspond to the point of Δ which lies on the normal segment through P, we obtain a (1,1) correspondence between the points of Δ and those of K. We must show that this correspondence is continuous. Suppose that $T_1, T_2, \cdots$ is a sequence of points of K converging to the unique limit point T of K, and let $T_1', T_2', \cdots$ be the corresponding points of Δ. This latter sequence will have one or more limit points which must lie on the normal segment through T. But there can not be more than one such limit point, for otherwise Δ, being a closed set, would contain these points and would thus intersect the normal segment through T at more than one point which can not happen, by XI. Therefore $T_1', T_2', \cdots$ converge to a unique limit point T' which is the point of Δ corresponding to T.

This proves that the correspondence is continuous. The distance between any pair of corresponding points is $< \eta_3$ since $\mu < \eta_3$. Hence XII is completely established.

COROLLARY. *Δ is a simple closed curve.*

By V and XII we have

VIII. *Δ can be mapped $(\eta_1 + \eta_2 + \eta_3)$ on J.*

Since η_1, η_2, η_3 were arbitrary numbers, we can assume that

$$2(\eta_1 + \eta_2) + \eta_3 < \varepsilon_n.$$

ε_n being defined in Theorem 1. Hence (5) may be written

$$|P(x, y)-J(x, y)|<\varepsilon_n$$

and by XIII Δ can be mapped (ε_n) on J. Hence $P(x, y)$ may be taken for the polynomial $P_n(x, y)$ of Theorem 1, which completes the proof of that theorem.

4. **Behavior of $P_n(x, y)$ outside of R'.** Let Q be the square region $0<x, y<1$. The lines $x=\frac{1}{3}$, $x=\frac{2}{3}$, $y=\frac{1}{3}$, $y=\frac{2}{3}$ divide Q into nine square regions of equal area. Call the central region S_0 and the others $S_1, S_2, \cdots, S_8$.

Let c be a simple closed curve in S_0 consisting of a finite number of analytic arcs and let $f(x, y)$ be a function defined and continuous at every finite point, equal to zero on c, >0 exterior to c, and <0 interior to c. Call C the region interior to c and let M be the maximum value of $-f(x, y)$ in C'. Assume finally that for every point not in S_0,

$$f(x, y)>M. \tag{8}$$

LEMMA. *The inequality*

$$\left(\frac{k_{2n}}{2}\right)^2 \iint_Q f(u, v)\,[1-(u-x)^2]^{2n}\,[1-(v-y)^2]^{2n}\,du\,dv>0 \tag{9}$$

holds for every finite point not in S_0', and for every value of $n>0$.

We have

$$\iint_Q = \iint_C + \iint_{Q-C}.$$

Obviously, for any finite point not in S_0' and for any positive n,

$$\iint_C<0, \quad \iint_{Q-C}>0,$$

from the definition of $f(x, y)$ and the fact that the other factor of the integrand is never negative. Hence $\iint_Q>0$ if

$$\iint_{Q-C}>-\iint_C$$

or, *a fortiori*, if

$$\iint_{S_i}>-\iint_C, \tag{10}$$

S being any one of the regions $S_1, \cdots, S_8$, since $\iint_{Q-C-S_i} > 0$. Hence it is sufficient to show that for each finite (x, y) not in S_0', the inequality (10) holds for at least one S_i regardless of the value of n.

But this is true because (a) the area of each S_i is greater than the area of C; (b) if (u, v) is any point of C and (u_i, v_i) any point of S_i, then

$$f(u_i, v_i) > M > -f(u, v)$$

by (8) and the definition of M; (c) if (x, y) is any finite point not in S_0', then for each n there is at least one S_i such that

$$[1-(u_i-x)^2]^{2n}\,[1-(v_i-y)^2]^{2n} > [1-(u-x)^2]^{2n}\,[1-(v-y)^2]^{2n}$$

(u, v) and (u_i, v_i) having the same meaning as in (b). To prove (c) we observe first that for a fixed x not in the interval (0, 1) the function

$$k(u) = [1-(u-x)^2]^{2n}$$

has no maximum in the interval (0, 1) but is either monotone or has a single minimum. For

$$\frac{dk}{du} = -4n(u-x)\,[1-(u-x)^2]^{2n-1}.$$

Since $u \neq x$ in the interval (0, 1) $\frac{dk}{du}$ can only be zero if at all, for $u = x+1$ or $u = x-1$. One, but not both, of these values may fall in the interval (0, 1). For that value, $k(u) = 0$ and hence that value is a minimum since $k(u)$ is never negative.

From this it is clear that for any x not in (0, 1) $k(u)$ is greater point for point in one at least of the intervals $(0, \frac{1}{3})$ $(\frac{2}{3}, 1)$ (end-points excepted) than it is in $(\frac{1}{3}, \frac{2}{3})$. (c) follows easily.

THEOREM 2. *The $P_n(x, y)$ of Theorem 1 can be so chosen that for each n, $P_n(x, y) > 0$ at every finite point not in R'.*

Proof. Let K_n be the rounded-off polygon used in obtaining $P_n(x, y)$ (§ 3) and $K_n(x, y)$ the corresponding distance function. Let us now choose the rectangular region R so large that on its boundary

$$K_n(x, y) > N_n \qquad (n = 1, 2, \cdots)$$

where N_n is the maximum value of $-K_n(x, y)$ in the interior region of K_n.*

* It is obvious that R can be so chosen if we recall that the functions $K_n(x, y)$ converge to the function $J(x, y)$.

Let us recall that $P_n(x, y)$ is a polynomial which approximates $K_n(x, y)$ uniformly in R' to a pre-determined degree of accuracy, while the first derivatives of $P_n(x, y)$ approximate those of $K_n(x, y)$ in the neighborhood of K_n. According to § 2, $P_n(x, y)$ can be obtained by first transforming R linearly into a region which we can take to be S_0, lying in Q. $K_n(x, y)$ is thereby transformed into $\bar{K}_n(x, y)$ which can be approximated in S_0' as closely as is necessary by a certain polynomial $\bar{P}_n(x, y)$ of the type (9) whose derivatives approximate those of $\bar{K}_n(x, y)$. On transforming back, the desired $P_n(x, y)$ is obtained.

On the boundary of S_0 and hence also at every point not in S_0', we will have

$$\bar{K}_n(x, y) > N_n.$$

Hence all the conditions of the Lemma are satisfied and it follows that

$$\bar{P}_n(x, y) > 0$$

for every finite point not in S_0', and hence

$$P_n(x, y) > 0$$

for every finite point not in R'.

Since $P(x, y)$ is zero in R' only on the oval approximating J, we have the

COROLLARY. *The $P_n(x, y)$ of Theorem* 1 *can be so chosen that for each n, the corresponding approximating oval is the only real branch of the curve $P_n(x, y) = 0$.*

5. **Approximation by unicursal curves.** THEOREM 3. *Given an arbitrary positive number ε, there exists a unicursal algebraic curve whose only real branch is a non-singular oval which can be mapped (ε) on J.*

Proof. We shall let K, η_1, η_2, retain the meanings already given them in § 3. Let a parametric representation of K be given by

$$x = F(\lambda), \quad y = G(\lambda)$$

where $\lambda = \dfrac{2\pi}{S} s$, S being the total length of arc of K, and s the length of arc measured in a fixed sense from a fixed point of K. We shall assume that F and G are defined for all finite values of λ. Hence

(a) F and G are continuous for all finite values of λ and periodic, each possessing the period 2π.

Let $\theta(\lambda)$ be the angle which a tangent to K at the point $(F(\lambda), G(\lambda))$ makes with the x-axis. Owing to the rounded nature of K, $\theta(\lambda)$ is continuous for all values of λ while $\sin\theta(\lambda)$ and $\cos\theta(\lambda)$ considered as functions of λ each possess the period 2π.

We have

$$(11)\qquad \begin{aligned} F'(\lambda) &= \frac{dx}{d\lambda} = \frac{S}{2\pi}\frac{dx}{ds} = \frac{S}{2\pi}\cos\theta(\lambda), \\ G'(\lambda) &= \frac{dy}{d\lambda} = \frac{S}{2\pi}\frac{dy}{ds} = \frac{S}{2\pi}\sin\theta(\lambda). \end{aligned}$$

Hence

(b) $F'(\lambda)$ and $G'(\lambda)$ are continuous and periodic, each of period 2π.

(c) F, G, F', G', are bounded and each possesses only a finite number of maxima and minima when λ varies over any interval of length 2π.

Now let α be a number such that

$$(12)\qquad 0 < \alpha < \frac{S}{8\pi},$$

and let $\delta > 0$ be such that, if λ_i and λ_j are any two values of λ satisfying the inequality

$$|\lambda_i - \lambda_j| < \delta,$$

then

$$(13)\qquad \begin{aligned} |F'(\lambda_i) - F'(\lambda_j)| &< \alpha, \\ |G'(\lambda_i) - G'(\lambda_j)| &< \alpha. \end{aligned}$$

Having thus defined δ, let η_4 be an arbitrary positive number and let ξ be such that

$$(14)\qquad 0 < \xi < \eta_4$$

and such that, if P_1 and P_2 are any two points of K with $\overline{P_1P_2} < 2\xi$, then

$$|\lambda_1 - \lambda_2| < \delta$$

where λ_1 and λ_2 are suitably* chosen values of λ corresponding to P_1 and P_2. The existence of ξ follows easily from continuity considerations and the fact that K is without multiple points.

By virtue of (a), (b), and (c) we can approximate the functions F and G uniformly by Fourier Sums whose derivatives approximate F' and G' respectively. More precisely, there exist functions $\varphi(\lambda)$ and $\psi(\lambda)$ defined by

* i. e. with regard to addition of suitable multiples of 2π.

$$\varphi(\lambda) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos n\lambda + b_n \sin n\lambda,$$

(15)

$$\psi(\lambda) = \frac{c_0}{2} + \sum_{n=1}^{N} c_n \cos n\lambda + d_n \sin n\lambda,$$

where N is sufficiently large and the a_n, b_n, c_n, d_n are Fourier coefficients, such that

$$|F(\lambda) - \varphi(\lambda)| < \frac{\xi}{\sqrt{2}},$$

(16)

$$|G(\lambda) - \psi(\lambda)| < \frac{\xi}{\sqrt{2}},$$

and

$$|F'(\lambda) - \varphi'(\lambda)| < \alpha,$$

(17)

$$|G'(\lambda) - \psi'(\lambda)| < \alpha,$$

for all values of λ.

I. *The curve Ω represented by the equations*

$$(\Omega) \qquad x = \varphi(\lambda), \qquad y = \psi(\lambda)$$

can be mapped (η_4) on K and is therefore a simple closed curve.

A continuous correspondence between the points of K and those of Ω may be defined by letting the point of Ω with parameter value λ correspond to the point of K having the same parameter value. The distance $\Delta(\lambda)$ between corresponding points is given by

$$\Delta^2(\lambda) = [F(\lambda) - \varphi(\lambda)]^2 + [G(\lambda) - \psi(\lambda)]^2$$

and by (16) and (14)

$$\Delta(\lambda) < \xi < \eta_4 \tag{18}$$

for all values of λ.

It remains to show that the correspondence established between Ω and K is (1, 1). Let us consider as identical, values of λ differing by integer multiples of 2π. Then to each point of K there corresponds only one value of λ and hence one point of Ω. We must then show that to each point of Ω there corresponds only one value of λ and hence only one point of K. To prove this suppose that to the point P_0 of Ω there correspond two values of λ, λ_1 and λ_2. Let Q_1 and Q_2 be the two points of K corresponding to these parameter values. By (18)

$$\overline{Q_1 P_0} < \xi, \qquad \overline{Q_2 P_0} < \xi$$

and hence $\overline{Q_1 Q_2} < 2\xi$. Hence by choice of ξ

$$|\lambda_1 - \lambda_2| < \delta.$$

Now by assumption

$$\varphi(\lambda_1) = \varphi(\lambda_2), \qquad \psi(\lambda_1) = \psi(\lambda_2).$$

Hence there must exist by Rolle's Theorem numbers λ_3 and λ_4 where

$$\lambda_1 < \lambda_3 < \lambda_2$$
$$\lambda_1 < \lambda_4 < \lambda_2$$

such that

$$\varphi'(\lambda_3) = 0, \qquad \psi'(\lambda_4) = 0.$$

Hence by (17)

(19) $$F'(\lambda_3) < \alpha, \qquad G'(\lambda_4) < \alpha.$$

Since $|\lambda_3 - \lambda_4| < \delta$, we have by (13)

$$|F'(\lambda_3) - F'(\lambda_4)| < \alpha$$

and hence by (19, i)

(20) $$F'(\lambda_4) < 2\alpha.$$

Combining (11) with (19, ii) and (20)

$$|\cos\theta(\lambda_4)| < \frac{2\pi}{S} 2\alpha, \qquad |\sin\theta(\lambda_4)| < \frac{2\pi}{S}\alpha.$$

Hence by (12)

$$|\cos\theta(\lambda_4)| < \frac{1}{2}, \qquad |\sin\theta(\lambda_4)| < \frac{1}{4}.$$

But this is absurd, and hence to no point of Ω can there correspond two values of λ. This establishes I.

As a consequence of I together with V of § 3, we have

II. *Ω can be mapped $(\eta_1 + \eta_2 + \eta_4)$ on J.*

Since η_1, η_2, η_4 are arbitrary positive numbers, we can assume that

$$\eta_1 + \eta_2 + \eta_4 < \varepsilon,$$

where ε is given in Theorem 3. In order now to complete the proof of this theorem, it remains to show that Ω is the only real branch of a unicursal curve.

Since $\sin n\lambda$ and $\cos n\lambda$ are expressible as rational functions of $\tan\frac{\lambda}{2}$, we can, by making the substitution

$$u = \tan\frac{\lambda}{2}$$

reduce the equations (Ω) to

$$x = R_1(u), \qquad y = R_2(u), \tag{21}$$

where R_1 and R_2 are rational functions with real coefficients. As λ varies from $-\pi$ to $+\pi$, u varies from $-\infty$ to $+\infty$ and the corresponding point of Ω describes Ω exactly once. Hence, with the exception of the point which corresponds to $\pm\infty$, there is a $(1, 1)$ correspondence between the points of Ω and the real values of u. Hence Ω is a complete real branch of the unicursal curve which is represented by (21) when u is allowed to assume all complex values. By the Theorem of Harnack a curve of genus p can have at most $p+1$ real branches.* Hence Ω is the *only* real branch of U, and Theorem 5 is thus completely established.

6. **Discussion of results for curves.** The work of § 5 does not preclude the possibility of the unicursal curve U having real isolated points. It is conceivable for example, that on demanding that Ω approximate J more and more closely (corresponding, say, to the sequence $\varepsilon_1, \varepsilon_2, \cdots$ of Theorem 1) the set of real isolated points tends to become everywhere dense in some region containing J. Suppose that for each ε_n the corresponding U has a polynomial representation $S_n(x, y) = 0$. The polynomials $S_n(x, y)$ would certainly not converge uniformly in any region containing J to a function of the type $J(x, y)$.

On the other hand, we have obtained no information concerning the genus of the curves $P_n(x, y) = 0$ of Theorem 1. In other words, we can not conclude from our methods that Theorems 1 and 3 can be combined.

7. **Approximation of manifolds in n-Space.** We extend Theorem 1 to $(n-1)$-dimensional manifolds† M^{n-1} in a number space S^n, admitting of a local parametric representation of the form

$$x^i = \varphi^i(u^1, \cdots, u^{n-1}), \qquad (i = 1, 2, \cdots, n).$$

The points $(u^1, \cdots, u^{n-1})$ describe an $(n-1)$-cell C^{n-1} in which the φ^i together with their first derivatives are continuous, the φ^i being single-valued. At no point of C^{n-1} do the Jacobians

* See for example, Severi-Löffler: Vorlesungen über Algebraische Geometrie, p. 239. It is implied in this proof that a real branch is any set of real points of the curve whose image on the Riemann Surface is a 1-circuit. Ω obviously satisfies this definition.

† As defined by O. Veblen, Analysis Situs, p. 88.

$$\frac{\partial(x^i, x^j, \cdots, x^k)}{\partial(u^1, u^2, \cdots, u^{n-1})}$$

all vanish. By these assumptions M^{n-1} possesses at each point a unique normal which varies continuously with the point.

By the Jordan-Brouwer Theorem* S^n is separated by M^{n-1} into two connected regions. Hence a distance function $M^{n-1}(x^1, \cdots, x^n)$ may be defined with respect to M^{n-1}.

Let R^n be a generalized rectangular region containing M^{n-1}.

THEOREM 4. *Given an $\varepsilon > 0$, there exists a polynomial $A(x^1, \cdots, x^n)$ such that*

$$| M^{n-1}(x^1, \cdots, x^n) - A(x^1, \cdots, x^n) | < \varepsilon$$

in R_n, while the real finite points of the algebraic manifold

$$A(x^1, \cdots, x^n) = 0 \tag{22}$$

constitute a manifold which can be mapped (ε) on M^{n-1}.

$A(x^1, \cdots, x^n)$ may be obtained precisely as we obtained the corresponding polynomial for the rounded-off polygon K, § 3.† The only detail which must be considered is the construction of a region U^n by means of normal segments, analogous to the ring-shaped region U of § 3, in which the first derivatives of $M^{n-1}(x^1, \cdots, x^n)$ are continuous. It will be sufficient to prove the following propositions:

I. For each point P of M^{n-1}, there exists a number $\eta > 0$ and an $(n-1)$-cell B_P on M^{n-1} containing P, such that no two normal segments of length 2η and mid-points distinct points of B_P meet.

Proof. In a small neighborhood V_P of P on M^{n-1} corresponding to $(u^1, \cdots, u^{n-1})$ in a cell C_P^{n-1}, each normal is given a unique sense by bringing a variable, sensed normal into coincidence with it. To each of the points on these normals is associated one or more sets

$$(u^1, \cdots, u^{n-1}, n),$$

where n is the distance, with proper sign, from M^{n-1} to the point along one of the normals through it. The corresponding x-coördinates are given, say, by

$$x^i = \psi^i(u^1, \cdots, u^{n-1}, n), \qquad (u^1, \cdots, u^{n-1}) \text{ in } C_P^{n-1}. \tag{23}$$

* See J. W. Alexander, Proof and Extension of the Jordan-Brouwer Theorem, Trans. Amer. Math. Soc., vol. 23 (1922), p. 333.

† The approximation theorem of § 2 generalizes without difficulty, and an obvious extension of § 4 allows us to choose $A(x^1, \cdots, x^n)$ so that its finite real points constitute but a single manifold.

Now at P, the Jacobian

$$J = \frac{\partial(\psi^1, \psi^2, \cdots, \psi^n)}{\partial(u^1, \cdots, u^{n-1}, n)}$$

is equal to

$$\sum \frac{\partial \psi^i}{\partial n} A^i = \sum \frac{\partial x^i}{dn} A^i = \sum \alpha^i A^i,$$

where the α^i are the direction cosines of the normal at P, and A^i is the cofactor of $\frac{\partial \psi^i}{dn}$ in J. The A^i are proportional to the α^i; and hence

$$\sum \alpha^i A^i = K \sum (\alpha^i)^2, \qquad K \neq 0.$$

Hence at P, $J \neq 0$. But the derivatives $\frac{\partial \psi^i}{\partial u^j}$ are continuous since the $\frac{\partial \varphi^i}{\partial u^j}$ are; also the derivatives $\frac{\partial \psi^i}{\partial n}$ are continuous. Hence there exists an an $(n-1)$-cell D_P contained in C_P^{n-1}, and a number $\eta_1 > 0$ such that for, $(u^1, \cdots, u^{n-1})$ in D_P and $|n| < \eta_1$, $J \neq 0$. Hence for these points (23) possesses a single-valued inverse. Hence if B_P is the image of D_P on M^{n-1}, no two normal segments of length 2η, $\eta < \eta_1$, and distinct mid-points in B_P intersect.

II. A value μ for η exists which serves for the whole manifold M^{n-1}. For if not, we could easily show the existence of a point for which I does not hold. As a corollary:

III. A normal segment of length 2μ has only its mid-point in common with M^{n-1}.

IV. One end-point of a normal segment of length 2μ and variable mid-point remains always interior to M^{n-1}, the other exterior. This follows from III.*

We take for U^n the totality of normal segments of length 2μ.

V. The first derivatives of $M^{n-1}(x^1, \cdots, x^n)$ are continuous in U^n. For at a point P in U^n, $\frac{\partial M^{n-1}}{\partial x^i}$ is the rate of change of the normal distance from P to M^{n-1}. This is easily seen to be equal to the cosine of the angle which the normal makes with the x^i-axis, and hence it is continuous.

8. **More general manifolds.** Theorem 4 obviously holds for any closed manifold H^{n-1} in S^n for which there exists a manifold of type M^{n-1}

* For, if AB is a normal segment with variable mid-point P, then by III PA (P not included) is always on the same side of M^{n-1}; similarly for PB. Now the totality $\{PA\}$ of segments PA plus the totality $\{PB\}$ constitutes a complete neighborhood of M^{n-1}, which consists of two disjoint parts since M^{n-1} is two-sided. Hence $\{PA\}$ is on one side of M^{n-1} and $\{PB\}$ on the other.

upon which it may be mapped (ϵ). It is not known that the most general H^{n-1} can be thus approximated. We shall indicate briefly however how this may be done for 2 and 3-dimensional polyhedra in S^3 and S^4 respectively, and point out what prevents our method from extending to n dimensions.

Let Π^{n-1} ($n = 3$ or 4) be a non-singular $(n-1)$-dimensional polyhedron* in S^n.

THEOREM 5. *There exists a manifold K^{n-1} of type M^{n-1} (§ 7) which can be mapped (ϵ) on Π^{n-1}, $n = 3$ or 4.*

Proof. K^{n-1} will be obtained by "rounding off the corners" of Π^{n-1} by fitting in pieces of tangent surfaces. These surfaces will always be generated by an arc of a circle or a portion of a sphere. The resulting manifold will possess a continuously turning tangent plane and will be seen therefore to possess the properties of an M^{n-1}. We shall treat separately the cases $n = 3$ and $n = 4$.

$n = 3$. By introducing new edges when necessary, we may assume that the faces of Π^2 are triangular. Since Π^2 is non-singular, two faces abut on each edge.

Let it be understood once and for all that the upper bound of each radius mentioned in the following is to be so small that the statements which will be made concerning mapping will be true.

Now let P be a variable point of an edge e_1 whose ends are the vertices V_1 and V_2. A plane perpendicular to e_1 at P, intersects the two faces abutting on e_1 in a pair of lines whose angle θ is constant. We may assume $\theta \neq \pi$; for, if $\theta = \pi$, there is no rounding to be done along e_1 and we proceed to another edge. Into θ we fit an arc σ of a tangent circle of radius ϱ (see fig. 3, § 3) and σ can be mapped (ϵ) on the broken line which it replaces. In the neighborhood of V_1, ϱ, which is a function of P, is to approach zero *linearly* as P approaches V_1; similarly at V_2. Elsewhere on e_1, ϱ is to vary so that $\frac{d\varrho}{d\xi}$, $\xi = V_1 P$, is continuous. The resulting surface S_{e_1} generated by σ, consists near V_1 (also V_2) of a portion of a circular cone with apex V_1. S_{e_1} possesses a continuously turning tangent plane and can be mapped (ϵ) on the part of Π^2 which it replaces.

We treat the remaining edges in a similar manner, choosing the upper bounds of the radii so small that no two surfaces S_e intersect. We thus obtain a surface Π_1^2 which can be mapped (ϵ) on Π^2 and which possesses a continuously turning tangent plane except at the vertices.

* For a definition of polyhedron see O. Veblen, On the Decomposition of n-Space by a Polyhedron, Trans. Amer. Math. Soc. 14 (1913), p. 65. By *non-singular* we shall mean that, if P is any point of Π^{n-1}, the points of Π^{n-1} interior to and on an $(n-1)$-sphere of sufficiently small radius and center P constitute an $(n-1)$-cell and its boundary.

Now let a sphere O of radius τ be placed about V_1 for center. For τ sufficiently small the points of Π^2 interior to O constitute a 2-cell. It is easily seen that the same statement holds for Π_1^2, the 2-cell being, say, c^2. We further take τ so small that O cuts the surfaces S_ϵ adjacent to V_1 only in the portions where they are conical. With τ so chosen, we see that O is *orthogonal* to Π_1^2, cutting it in a simple rounded-off polygon L. We replace c^2 by one of the two 2-cells on O, call it d^2, which L bounds. c^2 can be mapped (ϵ) on d^2 and this is to be done in such a way that L considered as the boundary of c^2 corresponds to itself considered as the boundary of d^2.

There is no difficulty now in rounding off the corner along L which d^2 makes with $\Pi_1^2 - c^2$, by fitting a 2-dimensional collar which can be mapped (ϵ) on the part of Π_1^2 which it replaces. The collar is generated precisely like a surface S_ϵ but with a generating arc of constant radius.

Treating the remaining vertices of Π_1^2 in like manner, we have the desired K^2.

$n = 4$. We merely outline the argument here which follows closely that of the previous case.

By introducing new 2-faces when necessary, we may assume that all the 3-faces of Π^3 are tetrahedral, and hence all 2-faces triangular. Two 3-faces abut at each 2-face.

Let P be a variable point of a 2-face F_1^2 at which the 3-faces F_1^3 and F_2^3 abut. An S^2 perpendicular to F_1^3 and F_2^3 at P cuts them in a pair of lines forming a constant angle which we may assume to be $\neq \pi$. We round off this angle by an arc of radius $r = \varphi(P)$ where $\varphi(P)$, a single-valued function of P, has the property that it approaches zero *linearly* as P approaches the boundary B of F_1^2 along a line either perpendicular to an F^1 on B or through an F^0 on B. Elsewhere on F_1^2, $\frac{\partial r}{\partial \xi}$ and $\frac{\partial r}{\partial \eta}$ are to be continuous, where ξ and η are the coördinates of P with regard to a cartesian system set up in the plane of F_1^2. Finally $\varphi(P)$, always positive, is to have an upper bound small as we please.*

Having treated all 2-faces in like manner, we obtain a manifold Π_1^3 having the desired properties except along its edges and vertices. An S^3 perpendicular to an edge F^1 at a variable point Q, intersects Π_1^3 near Q in a set of points whose structure is precisely that of a neighborhood of a vertex

* Such a function may be defined by means of a tetrahedral surface T erected on F_1^2 for its base, the fourth vertex D having its orthogonal projection on F_1^2. The edges meeting at D are rounded off by the methods for obtaining Π_1^3. Then D is rounded off by a suitable sphere O and collar. In order that the function shall be single-valued, it will be necessary that the cell of O which replaces the neighborhood of D shall be the one interior to T.

of a Π_1^2, and is accordingly rounded off by means of a sphere of radius s and a 2-dimensional collar the radius of whose generating arc is t. s and t are to vary with Q precisely as ϱ varied with P in the construction of Π_1^2. The remaining edges are similarly treated.

We now have a manifold Π_2^3 which has the desired properties except at the vertices. Owing to the linear nature of the variation of the various radii in the neighborhood of a vertex, it is clear that the structure of Π_2^3 near a vertex F^0 is that of a sheaf of lines through F^0. Hence a hypersphere O^3 of small radius and center F^0 is *orthogonal* to Π_2^3 intersecting it in a closed surface L^2 without corners. The points of Π_2^3 interior to and on O^3 constitute a 3-cell c^3 and its boundary. Hence L^2 is of genus zero and hence, by a theorem due to J. W. Alexander,* L^2 bounds two 3-cells on O^3, one of which we call d^3. We replace c^3 by d^3 and round off the corner between d^3 and $\Pi_2^3 - c^3$ by fitting in 3-dimensional collar of obvious construction.

When the remaining vertices of Π_2^3 are similarly treated, the desired K^3 will be obtained.

The above method demands for its extension to any n that the theorem of Alexander just referred to hold for spaces of any dimensionality, and this is still an open question.

* On the Subdivision of 3-Space by a Polyhedron, Proc. Nat. Acad. Sci., vol. 1 (1924), p. 6.

NOTE ON MATRICES IN A GIVEN FIELD.

By J. H. M. Wedderburn.

The object of this note is to give two theorems on matrices whose coefficients lie in a given field or domain of rationality. The first deals with a well known result concerning real symmetric matrices and Hermitian matrices. The form of proof is suggested by the discussion given by Tait* but has the formal advantage of not employing vectors. The second gives a normal form for a matrix which is rational in a given field which differs from that already given by Professor Dickson† in that it is analogous to the form usually employed in the field of complex numbers.

1. **Real Hermitian matrices.** Before proceeding to the main theorem we require the following lemmas.

Lemma 1. *If $a, b, c, \cdots$ are matrices such that*‡

$$a\bar{a}' + b\bar{b}' + c\bar{c}' + \cdots = 0, \tag{1}$$

then $a, b, c, \cdots$ are all zero.

For the trace of (1) is the sum of the squares of the absolute values of the coefficients of $a, b, c, \cdots$.

Since $\bar{a}' = a$ when a is Hermitian, we have:

Lemma 2. *If $a, b, c, \cdots$ are Hermitian matrices such that $a^2+b^2+c^2+\cdots=0$, then $a, b, c, \cdots$ are all zero.*

Theorem 1. *The roots of a Hermitian matrix are real and its elementary divisors are simple.*

Let $x = \bar{x}'$ be a Hermitian matrix and $g(x) = 0$ its reduced equation;§ since $0 = \bar{g}(\bar{x}) = \bar{g}(x')$ and x' has the same reduced equation as x, it follows that the coefficients of g are real. Suppose that $\xi_1 = \alpha + i\beta$ $(\beta \neq 0)$ is a root of $g(x)$; then $\xi_2 = \alpha - i\beta \neq \xi_1$ is also a root and, if λ is a scalar variable, we may write

$$g(\lambda) = (\lambda - \xi_1)(g_1(\lambda) + ig_2(\lambda)) = (\lambda - \xi_2)(g_1(\lambda) - ig_2(\lambda)) \tag{2}$$

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* P. G. Tait, Note on the reality of the symbolical cubic etc., Proc. Roy. Soc. Edinb., vol. 6 (1867), pp. 92–93, Sc. Papers, vol. 1, pp. 74–75.

† L. E. Dickson, A new theory of the rational equivalence of linear transformations of pairs of bilinear forms (so far published in abstract only), Bull. Amer. Math. Soc., vol. 31 (1925), p. 198.

‡ Here a' and $\bar{a}$ denote, respectively, the transverse and the conjugate imaginary matrices related to a.

§ Readers who are not familiar with the theory of the reduced equation, which is due to Frobenius, will find a short discussion of it in Professor Dickson's "Algebras and their arithmetrics", p. 109.

where g_1 and g_2 are real polynomials, of lower degree than g, neither of which is identically zero since g is real and we are assuming that ξ_1 is complex.

Now

$$(3) \qquad [g_1(x)]^2 + [g_2(x)]^2 \equiv (g_1(x) + ig_2(x))(g_1(x) - ig_2(x)) = 0$$

since from (2) together with $\xi_1 \neq \xi_2$ it follows that ξ_1 is a root of $g_1(\lambda) - ig_2(\lambda)$ and hence $(g_1(\lambda) + ig_2(\lambda))(g_1(\lambda) - ig_2(\lambda))$ is divisible by $g(\lambda)$. But, since the coefficients of g_1 and g_2 are real, $g_1(x)$ and $g_2(x)$ are Hermitian matrices, so that from (3) and Lemma 2 we conclude that $g_1(x)$ and $g_2(x)$ are zero; and this is impossible since $g_1(\lambda)$ and $g_2(\lambda)$ are not identically zero and are of lower degree than $g(\lambda)$. Hence the hypothesis that g has a complex root leads to a contradiction, so proving the first part of the theorem.

To prove that the elementary divisors are simple it is only necessary to show that $g(\lambda)$ has no multiple roots. Let

$$g(\lambda) = (\lambda - \xi)^r h(\lambda), \qquad h(\xi) \neq 0,$$

and, assuming $r > 1$, set $g_1(\lambda) = (\lambda - \xi)^{r-1} h(\lambda)$; then $[g_1(\lambda)]^2$ is divisible by $g(\lambda)$ and hence the square of the Hermitian matrix $g_1(x)$ is zero, so that, by Lemma 2, $g_1(x)$ itself is zero, which is impossible since the degree of g_1 is lower than that of g. It follows that r cannot be greater than 1, which completes the proof of the theorem.

2. Normal form of a matrix which is rational in a given field. The notion of an elementary divisor, which is usually introduced with reference to the field of all complex numbers, generalizes readily when this field is replaced by any field F. The formal definition is as follows. *If $E_1, E_2, \cdots, E_n$ are the invariant factors of a matric polynomial in a scalar variable λ, and if*

$$E_i = [\alpha_1(\lambda)]^{r_{i1}} [\alpha_2(\lambda)]^{r_{i2}} \cdots [\alpha_\nu(\lambda)]^{r_{i\nu}}, \qquad i = 1, 2, \cdots, n,$$

where each $\alpha_k(\lambda)$ is an irreducible, rational, scalar polynomial in F, such of the factors $[\alpha_k(\lambda)]^{r_{ik}}$ as are not constants are called the elementary divisors of the matric polynomial in F.

If

$$(4) \qquad a(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_m$$

is a scalar polynomial in a scalar variable λ, then the matrix of degree m defined by*

* So far as I am aware, this form was first given by Frobenius. It is perhaps most readily derived by taking the polynomial basis $1, x, x^2, \cdots, x^{m-1}$ and using the relations $x \cdot x^{m-1} = -(a_1 x^{m-1} + \cdots + a_m)$, $x \cdot x^{m-2} = x^{m-1}, \cdots, x \cdot 1 = x$.

$$(5) \qquad x_a = \begin{matrix} -a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{matrix}$$

has $\alpha(\lambda)$ as its characteristic function i. e., $\alpha(\lambda) = |\lambda - x_a|$; and $\alpha(x_a) = 0$ is the reduced equation of x_a.

Let 1_a denote the identity matrix of degree m and consider the matrix of degree rm defined by

$$(6) \qquad x_a = \begin{matrix} x_a & 1_a & & & \\ & x_a & 1_a & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & x_a & 1_a \\ & & & & x_a \end{matrix} \qquad (r \text{ rows and columns})$$

where x_a represents the block of m^2 terms defined by (5), where 1_a is a block of m^2 terms with 1 in each term of the main diagonal and 0 elsewhere, and where all other terms are 0. It is readily seen* that, if if $f(\lambda)$ is a scalar polynomial, then

$$(7) \qquad f(x_a) = \begin{matrix} f(x_a) & f'(x_a) & \cdots & \dfrac{f^{(r-2)}(x_a)}{(r-2)!} & \dfrac{f^{(r-1)}(x_a)}{(r-1)!} \\ & f(x_a) & \cdots & \dfrac{f^{(r-3)}(x_a)}{(r-3)!} & \dfrac{f^{(r-2)}(x_a)}{(r-2)!} \\ & & \cdots & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & f(x_a) & f'(x_a) \\ & & & & f(x_a) \end{matrix}$$

where all terms on a parallel to the main diagonal on the right are the same as the corresponding element in the first row, and all terms to the left of the main diagonal are 0.

* Cf., for instance, Ann. of Math. (2), vol. 23 (1921), pp. 123–124.

Now, if $f(x_a) = 0$, each of the coefficients in (7) must vanish and, in particular, $f^{(r-1)}(x_a) = 0$. If $f(x_a) = 0$ is the reduced equation of x_a, $f^{(r-1)}(\lambda)$ must equal $\alpha(\lambda)$; for x_a can satisfy no equation of lower degree than that of $\alpha(\lambda)$. It follows immediately that $f(\lambda) \equiv [\alpha(\lambda)]^r = 0$ is the reduced equation of x_a, and hence, from the theory of invariant factors, $\lambda - x_a$ can be reduced by means of rational elementary transformations to the form

$$\begin{matrix} [\alpha(\lambda)]^r & & & \\ & 1 & & \\ & & 1 & \\ & & & \cdot \\ & & & \cdot \\ & & & 1 \end{matrix}$$

We can now state our conclusion as follows:

THEOREM 2. *Let $\alpha_1(\lambda), \alpha_2(\lambda), \cdots, \alpha_k(\lambda)$ be polynomials, not necessarily distinct, which are rational and irreducible in a field F and whose degrees are $m_1, m_2, \cdots, m_k$ respectively; let $r_1, r_2, \cdots, r_k$ be any positive integers such that*

$$r_1 m_1 + r_2 m_2 + \cdots + r_k m_k = n;$$

then, if x_{α_i} is the matrix of degree $r_i m_i$ formed from $\alpha_i(\lambda)$ in the same way as x_a in (6) is formed from $\alpha(\lambda)$, the matrix of degree n defined by

$$\begin{matrix} x_{\alpha_1} & & & \\ & x_{\alpha_2} & & \\ & & \cdot & \\ & & & \cdot \\ & & & \cdot \\ & & & x_{\alpha_k} \end{matrix}$$

has $\alpha_1(\lambda), \alpha_2(\lambda), \cdots, \alpha_k(\lambda)$ as its elementary divisors in F.

ON THE ARITHMETICAL APPLICATIONS OF THE POWER SERIES FOR THETA QUOTIENTS.

By E. T. Bell.

1. The arithmetical consequences of the trigonometric series for the elliptic functions and other theta quotients (elliptic functions of the rth kind, $r>1$) have been extensively investigated, those of the power series for the same functions scarcely at all. The systematic use of the latter series amounts to a distinct technique in the application of elliptic functions to the theory of numbers, yielding many interesting theorems of new kinds which it would be difficult to obtain otherwise. For example, it was shown in a former note* that four far reaching theorems on representations as sums of squares, published without proof by Liouville† follow easily from the power series for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, $\operatorname{sn}^2 u$. In a little known paper the late V. Bulyguin‡ by a skilful use of the same series and their derivatives obtained, for the first time, a *complete* and simple solution of the problem of determining for any positive integers n, r the number of representations of n as a sum of $2r$ squares. These two applications appear to exhaust the literature.

In the present paper we shall sketch a general method and indicate by a few examples how specially prepared lists of what were elsewhere§ called the arithmetical expansions for theta quotients can be profitably used in this connection.

2. The singular feature of the arithmetical theorems obtainable by this method is seen in the following illustration, which also fixes the notation for the quadratic forms used later.

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* Bull. Amer. Math. Soc., vol. 26 (1919), pp. 19–25.

† Journ. des Math. (2), vol. 6 (1861), pp. 233–8; vol. 6. pp. 369–376.

‡ Bull. de l'Acad. impér. des Sci. de St. Petersburg (1914), pp. 389–404. This remarkable paper seems to have escaped adequate notice. The abstracts (presented by G. Humbert) in the Paris Comptes Rendus, vol. 158 (1914), pp. 328–330, ibid., vol. 161 (1915), pp. 28–31, on which is based the account in Dickson's *History*, vol. 2, p. 317, are insufficient. Humbert was unaware of the existence of the completed memoir, for six years after it was in print he enquired whether Bulyguin (died 1919) had ever finished his important research on the problem of squares. I am indebted for my knowledge of Bulyguin's work to Professor J. V. Ouspensky of Leningrad.

§ Trans. Amer. Math. Soc., vol. 22 (1921), pp. 198–201; Messenger of Math., vol. 53 (1924), pp. 166–176, hereafter cited as (M). To save space we shall use without further reference the m, n, d, τ, t, notation in all series and quadratic forms in this paper, as explained in (M), p. 167.

Let U_a, V_b, W_c denote respectively sums of a uneven, b even, c arbitrary, (uneven or even) squares of integers $\gtreqless 0$, and let (a_i, α_i) $(i = 1, 2, \cdots)$ be distinct pairs of integers, so that $(a_i, \alpha_i) = (a_j, \alpha_j)$ only when $a_i = a_j$, $\alpha_i = \alpha_j$, and similarly for (b_i, β_i) and (c_i, γ_i). The number of representations of n in the form

$$F \equiv \sum_{i=1}^{A} a_i U_{\alpha_i} + \sum_{j=1}^{B} b_j V_{\beta_j} + \sum_{k=1}^{C} c_k W_{\gamma_k}$$

will be written

$$N(n) \equiv N(n:(a_1, \alpha_1), \cdots, (a_A, \alpha_A);\ (b_1, \beta_1), \cdots, (b_B, \beta_B);\ (c_1 \gamma_1), \cdots, (c_C, \gamma_C)),$$

in which note that the symbols containing a, b, c refer respectively to uneven, even, arbitrary squares, and the order of the a, b, c within the complete symbol is fixed. When all the squares of a given kind, say the W_c, are lacking in F, their place in $N(n)$ is marked by a star, thus

$$N(n:(a_1, \alpha_1), \cdots, (a_A, \alpha_A);\quad (b_1, \beta_1), \cdots, (b_B, \beta_B);\ *).$$

Call $f(n)$ *simple* if it is an algebraic function of the real divisors alone of a single integer, and likewise call the sum of a finite number of simple functions simple. Keep the a, b, c in F fixed; let the α_i, β_j, γ_k run through a range of sets of values $S_t (t = 1, 2, \cdots, T)$,

$$S_t \equiv (\alpha_1^{(t)}, \cdots, \alpha_A^{(t)};\ \beta_1^{(t)}, \cdots, \beta_B^{(t)};\ \gamma_1^{(t)}, \cdots, \gamma_C^{(t)}),$$

and let the particular $N(n)$, for n fixed, pertaining to the S_t just written be $N_t(n)$. Then, although in general no one of $N_t(n)$ $(t = 1, 2, \cdots, T)$ is simple, it may be possible to choose rational numbers (not necessarily integers), ν_t, *not depending upon the n whose representations are concerned*, such that

$$\sum_{t=1}^{T} \nu_t N_t(n)$$

is simple, and moreover T does not depend upon n. The situation is analogous to a familiar one in the integral calculus, or to the algebraic addition theorems for certain transcendental functions.

The analysis gives at once the forms F, the systems S_t, and the numbers T, ν_t for which such theorems exist. It will be evident as we proceed that the number of these theorems is infinite. In what follows we do not state the explicit forms of the ν_t. For a *specific* F the ν_t can always be found; the *general* discussion of these coefficients appears to be a matter of considerable difficulty. Our results therefore are in the nature of existence theorems (like those of Liouville cited in § 1), although the analysis in each case

shows how the precise forms of the coefficients can be found if required (an important detail not indicated by Liouville).

3. To arrive at theorems of the type described in § 2, the power series for the theta quotients must be obtained in two essentially different ways. Using Jacobi's notation we have

$$k = \vartheta_2^2/\vartheta_3^2, \qquad k' = \vartheta_0^2/\vartheta_3^2, \qquad 2K = \pi\vartheta_3^2,$$

and, with $z \equiv u/\vartheta_3^2$,

$$\operatorname{sn} u = \frac{\vartheta_3\,\vartheta_1(z)}{\vartheta_2\,\vartheta_0(z)}, \quad \operatorname{cn} u = \frac{\vartheta_0\,\vartheta_2(z)}{\vartheta_2\,\vartheta_0(z)}, \quad \operatorname{dn} u = \frac{\vartheta_0\,\vartheta_3(z)}{\vartheta_3\,\vartheta_0(z)},$$

whose Maclaurin developments can be written in the convenient symbolic forms

$$\operatorname{sn} u = \sin Su, \qquad \operatorname{cn} u = \cos Cu, \qquad \operatorname{dn} u = \cos Du,$$

where S^j, C^j, D^j $(j = 0, 1, \cdots)$ are to be replaced after expansion by S_j, C_j, D_j, and $C_0 = D_0 = 1$,

$$S_{2r-1} = \sum_{t=0}^{r-1} s_t k^{2t} = \sum_{t=0}^{r-1} s_t\,\vartheta_2^{4t}\,\vartheta_3^{-4t} \qquad (r>0),$$

$$C_{2r-2} = \sum_{t=0}^{r-2} c_t k^{2t} = \sum_{t=0}^{r-2} c_t\,\vartheta_2^{4t}\,\vartheta_3^{-4t} \qquad (r>1),$$

$$D_{2r-2} = \sum_{t=0}^{r-2} c_t k^{2r-2t-2} = \sum_{t=0}^{r-2} c_t\,\vartheta_2^{4r-4t-4}\,\vartheta_3^{-4r+4t+4} \qquad (r>1),$$

in which the s_t, c_t are integers >0. For completeness s_t, c_t should carry a double suffix, $s_{2r-1,t}$, $c_{2r-1,t}$, but we can neglect this detail. The symbolic notation $\sin Su$, etc., leads to a powerful algorithm for expanding

$$F(u) \equiv \operatorname{sn}^a u\,\operatorname{cn}^b u\,\operatorname{dn}^c u,$$

where a, b, c are any integers. As this was fully discussed in a previous paper* we shall assume here the few simple results required for illustrating the general method.

The second power series development of $F(u)$ is obtained by writing $F(u)$ in the form $F(z)$, where

$$F(z) \equiv \frac{\vartheta_3^{a-c}\,\vartheta_0^{b+c}}{\vartheta_2^{a+b}} \cdot \frac{\vartheta_1^a(z)\,\vartheta_c^b(z)\,\vartheta_3^c(z)}{\vartheta_0^{a+b+c}(z)}.$$

* Trans. Amer. Math. Soc., vol. 24 (1922), pp. 89–112.

Now suppose, as generally is the case, that the arithmetical development of $F_1(z) \equiv GF(z)$, where G is a theta constant (independent of z), can be found more readily than that of $F(z)$. Then, expanding the *circular* functions in the arithmetical development of $F_1(z)$, we must always reach (as is evident from the papers cited) series of one or other of the forms

$$A(z) + \sin Pz, \qquad A(z) + \cos Pz,$$

where $P^j = P_j$ is a power series in q, the sin or cos form occurring according as $F(z)$ is an odd or an even function of z (or u), and $A(z)$ is either a constant or a series in $z^{\pm 1}$ containing at most a finite number of powers of z^{-1}, and moreover the coefficients of the positive powers in $A(z)$ are readily determined functions of Bernoulli or Euler numbers.*

Finally, equating coefficients of like powers of u in this development of $F(z)$ and the Maclaurin expansion, we obtain an identity of the form $\Pi_j = P_j$, where Π_j is a finite polynomial in $\vartheta_\alpha^{\pm 1}$ ($\alpha = 0, 2, 3$) with rational coefficients. Let us suppose for simplicity, as usually is the case, that no negative powers of ϑ_α occur in Π_j, and further that only even powers of ϑ_0 occur. Then ϑ_0 may be eliminated by the transformation of the second order and by means of $\vartheta_0^4 + \vartheta_2^4 = \vartheta_3^4$. The coefficient of q^n in Π_j is now seen by inspection to be of the form $\sum \nu_t N_t(n)$ in § 2; equating like powers of q in Π_j we get a theorem of the type there discussed.

When the above assumptions regarding Π_j are violated, the method is essentially the same, after clearing the equation of negative powers of ϑ_α, but the theorems are not simple in the sense of § 2. They relate, among other interesting circumstances, to representations in quadratic forms in an odd number of indeterminates.

Obvious modifications of the foregoing outline lead to precisely similar conclusions regarding the twofold expansion of any rational function of elliptic theta functions. In the particular case where the function expanded is elliptic, the final arithmetical theorems are not simple as defined if the denominator of the function consists of more than a single product of powers of elliptic functions. We proceed to a few illustrations of the general process.

4. To connect the elliptic functions with the arithmetical expansions given in (M), we require the forms of sn u, etc., in § 3 and the following from (M),

* In the illustrations of this paper we pay no attention to the arithmetical theorems obtainable by considering $A(z)$.

$$\vartheta_2\vartheta_3\frac{\vartheta_1(x)}{\vartheta_0(x)} = 4\sum q^{m/2}[\sum \sin dx],$$

$$\vartheta_0\vartheta_2\frac{\vartheta_2(x)}{\vartheta_0(x)} = 4\sum q^{m/2}[\sum (-1|\delta)\cos dx],$$

$$\vartheta_0\vartheta_3\frac{\vartheta_3(x)}{\vartheta_0(x)} = 1+4\sum q^{n}\ [\sum(-1|\tau)\cos 2tx].$$

In these we expand the sines and cosines, getting

$$\operatorname{sn} u = 4\ \vartheta_2^{-2}\sum\frac{(-1)^{r-1}u^{2r-1}}{(2r-1)!}[\vartheta_3^{2-4r}\sum q^{m/2}\zeta_{2r-1}(m)],$$

$$\operatorname{cn} u = 4\vartheta_2^{-2}\sum\frac{(-1)^{r-1}u^{2r-2}}{(2r-2)!}[\vartheta_3^{4-4r}\sum q^{m/2}\xi'_{2r-2}(m)],$$

$$\operatorname{dn} u = \vartheta_3^{-2}+4\sum\frac{(-1)^{r-1}u^{2r-2}}{(2r-2)!}[\vartheta_3^{2-4r}\sum q^{n}2^{2r-2}\xi'_{2r-2}(n)],$$

in which the outer $\sum$ refers to $r=1$ to ∞, the inner is as in (M), viz., it refers to all odd integers $m>0$, or to all integers $n>0$, and $\zeta_s(N)$ = the sum of the sth powers of all divisors of N, $\xi'_s(N)$ = the excess of the sum of the sth powers of all those divisors of N whose conjugates are of the form $4t+1$ over the like sum in which the conjugates are of the form $4t-1$.

Comparing these with the Maclaurin developments in § 3, we have the following expansions of the S, C, D polynomials:

$$S_{2r-2} = 4\vartheta_2^{-2}\ \vartheta_3^{2-4r}\ \sum q^{m/2}\ \zeta_{2r-1}(m),$$

$$C_{2r-2} = 4\vartheta_2^{-2}\ \vartheta_3^{4-4r}\ \sum q^{m/2}\ \xi'_{2r-2}(m),$$

$$D_{2r-2} = 4\vartheta_3^{2-4r}\ 2^{2r-2}\ \sum q^{n}\ \xi'_{2r-2}(n),$$

and therefore finally

$$\sum_{t=0}^{r-1} s_t\vartheta_2^{4t+2}\vartheta_3^{4r-4t-2} = 4\sum q^{m/2}\zeta_{2r-1}(m),$$

$$\sum_{t=0}^{r-1} c_t\vartheta_2^{4t+2}\vartheta_3^{4r-4t} = 4\sum q^{m/2}\xi'_{2r}(m),$$

$$\sum_{t=0}^{r-1} c_t\vartheta_2^{4r-4t}\vartheta_3^{4t+2} = 2^{2r+2}\sum q^{n}\ \xi'_{2r}(n),$$

the summations on the right referring to all odd integers $m>0$, or to all integers $n>0$.

The last three give the simplest theorems of the type in § 2. For, changing q into q^4 and comparing coefficients of like powers of q, we have

$$\text{(I)} \quad \sum_{t=0}^{r-1} s_t N(2m:(1,4t+2);\ (1,4r-4t-2);*) = 4\zeta_{2r-1}(m),$$

$$\text{(II)} \quad \sum_{t=0}^{r-1} c_t N(2m:(1,4t+2);\ (1,4r-4t);*) = 4\xi'_{2r}(m),$$

$$\text{(III)} \quad \sum_{t=0}^{r-1} c_t N(4n:(1,4r-4t);\ (1,4t+2);*) = 2^{2r+2}\xi'_{2r}(n).$$

5. The expansions of § 4 alone combined with the processes of elliptic functions and the series of (M) suffice to give an endless variety of theorems of the type considered. For example, by Jacobi's imaginary transformation and § 3,

$$\operatorname{sn}\left(k'iu, \frac{1}{k'}\right) = ik' \frac{\operatorname{sn} u}{\operatorname{cn} u} = i\sum_{r=1}^{\infty} k'^{2r-1} S_{2r-1}\left(\frac{1}{k'^2}\right) \frac{u^{2r-1}}{(2r-1)!},$$

where it is now necessary to indicate that the argument of S is $1/k'^2$, not k^2 as before. The expansion of the second function is given in (M):

$$\vartheta_0\vartheta_3\vartheta_1(x)/\vartheta_2(x) = \tan x - 4\sum q^{2n}\left[\sum(-1)^{d+\delta}\sin 2dx\right],$$

and we have

$$\frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{1}{\vartheta_0^2}\cdot\vartheta_0\vartheta_3\frac{\vartheta_1(z)}{\vartheta_2(z)}, \quad z \equiv u/\vartheta_3^2.$$

To obtain the necessary expansion of this function as a power series in z (or of the preceding in x) we require

$$\tan x = \frac{1}{2x}\cos 2Gx = \sum_{r=1}^{\infty} \frac{(-1)^r x^{2r-1}}{(2r)!} 2^{2r-1} G_{2r},$$

where the cosine is symbolic and the G's are the numbers of Genocchi in the even suffix notation,* $G_{2n} = 2(1-2^{2n})B_{2n}$, the B's being the Bernoulli numbers in a similar notation. Proceeding with the coefficient of q^{2n} as in § 4 we get finally, after several easy reductions,

$$\vartheta_0\vartheta_3\frac{\vartheta_1(x)}{\vartheta_2(x)} = \sum_{r=1}^{\infty} \frac{(-1)^r x^{2r-1} 2^{2r-1}}{(2r-1)!}\left[\frac{G_{2r}}{2r} + 4\sum q^{2n}\psi_{2r-1}(n)\right],$$

* Cf. Lucas, *Théorie des Nombres*, p. 256.

the inner $\sum$ referring as before, by (M), to all integers $n>0$, and ψ being

$$\psi_r(n) \equiv \sum(-1)^{d+\delta} d^r = -[1+(-1)^n](2^{r-1}-1)\zeta_r\left(\frac{n}{2}\right) - [1+2(-1)^n]\zeta'_r(n),$$

where $\zeta_r(N)$ is as defined in § 4, and $\zeta'_r(N)$ is the sum of the rth powers of all odd divisors of N. The d, δ here refer (by (M)) to all pairs of conjugate divisors of N. This function is of frequent occurrence. Comparing expansions as before we now have

$$k'^{2r-2} S_{2r-1}\left(\frac{1}{k'^2}\right) = \frac{(-1)^r 2^{2r-1}}{\vartheta_0^2 \vartheta_3^{4r-2}}\left[\frac{G_{2r}}{2r} + 4\sum q^{2n} \psi_{2r-1}(n)\right].$$

In precisely the same way* we find from

$$\operatorname{cn}\left(k' iu, \frac{1}{k'}\right) = \frac{\operatorname{dn} u}{\operatorname{cn} u}, \qquad \operatorname{dn}\left(k' iu, \frac{1}{k'}\right) = \frac{1}{\operatorname{cn} u},$$

the following,

$$k'^{2r-2} C_{2r-2}\left(\frac{1}{k'^2}\right) = \frac{(-1)^{r-1}}{\vartheta_3^{4r-2}}[E_{2r-2} + 4\sum q^n \xi_{2r-2}(n)],$$

$$k'^{2r-2} D_{2r-2}\left(\frac{1}{k'^2}\right) = \frac{(-1)^{r-1}}{\vartheta_0^2 \vartheta_3^{4r-2}}[E_{2r-2} + 4\sum q^n (-1)^n \xi_{2r-r}(n)],$$

where the E's are Euler numbers in the even suffix notation of Lucas. Replacing k' by its theta equivalent, and referring to § 3 for the polynomials, we get from these three,

$$\sum_{t=0}^{r-1} s_t \vartheta_0^{4r-4t-2} \vartheta_3^{4t+2} = (-1)^t 2^{2r-1}\left[\frac{G_{2r}}{2r} + 4\sum q^{2n} \psi_{2r-1}(n)\right],$$

$$\sum_{t=0}^{r-1} c_t \vartheta_0^{4r-4t} \vartheta_3^{4t+2} = (-1)^r [E_{2r} + 4\sum q^n \xi_{2r}(n)],$$

$$\sum_{t=0}^{r-1} c_t \vartheta_0^{4t+2} \vartheta_3^{4r-4t} = (-1)^r [E_{2r} + 4\sum q^n (-1)^n \xi_{2r}(n)],$$

the last two of which mutually check by changing the sign of q. In these, cf. (M), $\xi_s(N)$ is the excess of the sum of the sth powers of all those divisors of N which are of the form $4t+1$ over the like sum for the $4t-1$ divisors.

* As the calculations, although straightforward, are rather detailed, I have checked all twice.

The arithmetical interpretations read off from these as they stand relate to differences of numbers of representations. It is more interesting however to eliminate ϑ_0 first. For example, using

$$\vartheta_0^2 = \vartheta_3^2(q^2) - \vartheta_2^2(q^2), \qquad \vartheta_3^2 = \vartheta_3^2(q^2) + \vartheta_2^2(q^2)$$

on the first, we reduce it to the form

$$\sum_{t=0}^{2r} s_t'' \vartheta_3^{4r-2t}(q^2)\,\vartheta_2^{2t}(q^2) = (-1)^r 2^{2r-1}\left[\frac{G_{2r}}{2r} + 4\sum q^{2n}\psi_{2r-1}(n)\right],$$

in which the s_t'' are integers. This gives

$$\text{(IV)}\quad \sum_{t=0}^{2r} s_t'' N(4n:(1,2t);\ (1,4r-2t);\ *) = (-1)^r 2^{2r-1}\psi_{2r-1}(n).$$

In (I)–(IV) all the quadratic forms concerned are sums of squares (with coefficients 1). A process similar to the above gives corresponding theorems in which some of the squares have the coefficient 2, on replacing ϑ_0^2 as above, and leaving ϑ_3^2 as it was. Treating instead the third theta identity as suggested, we find, with the c_{ta} integers,

$$\text{(V)}\quad \sum_{t=0}^{r-1}\sum_{a=0}^{2t+1} c_{ta} N(2n:(1,2a);\ (1,4t+2-2a);\ (2,4r-4t)) = 4(-1)^{n+r}\xi_{2r}(n).$$

6. As another example we shall apply the transformation of the second order to the theta identity in § 4 from which (I) was derived, getting

$$\sum_{t=0}^{r-1} s_t 2^{2t+1}\vartheta_3^{2t+1}(q^2)\,\vartheta_2^{2t+1}(q^2)\,\vartheta_3^{4r-4t-2} = 4\sum q^{m/2}\zeta_{2r-1}(m),$$

which gives

$$\text{(VI)}\quad \sum_{t=0}^{r-1} 2^{2t+1} s_t N(m:(1,2t+1);\ (1,2t+1);\ (2,4r-4t-2)) = 4\zeta_{2r-1}(m).$$

Or, before applying the transformation we may combine terms equidistant from the ends of the theta sum, since $s_{n-t} = s_n$, and apply $2\vartheta_2\vartheta_3 = \vartheta_2^2(\sqrt{q})$. Separation of cases according to odd or even r then gives the following,

$$\sum_{t=0}^{r-1} 2^{4r-4t-4} s_t \vartheta_2^{8t+4}(\sqrt{q})(\vartheta_3^{8r-8t-4} + \vartheta_2^{8r-8t-4}) = 2^{4r}\sum q^{m/2}\zeta_{4r-1}(m),$$

$$s_{r-1}\vartheta_2^{8r-4}(\sqrt{q}) + \sum_{t=0}^{r-2} 2^{4r-4t-4} s_t \vartheta_2^{8t+4}(\sqrt{q})(\vartheta_3^{8r-8t-8} + \vartheta_2^{8r-8t-8}) = 2^{4r}\sum q^{m/2}\zeta_{4r-3}(m).$$

The first gives

$$\text{(VII)}\quad \sum_{t=0}^{r-1} 2^{4r-4t-4} s_t [N(4m:(1,8t+4);\ (2,8r-8t-4);\ *) + N(4m:(1,8t+4),\ (2,8r-8t-4);\ *;\ *)] = 2^{4r}\zeta_{4r-1}(m);$$

and the second,

$$\text{(VIII)}\quad s_{r-1} N(4m:(1,8r-4);\ *;\ *) + \sum_{t=0}^{r-2} 2^{4r-4t-4} s_t [N(4m:(1,8t+4);(2,8r-8t-8);\ *) + N(4m:(1,8t+4),(2,8r-8t-8);\ *;\ *)] = 2^{4r}\zeta_{2r-1}(m).$$

7. In the paper cited on anharmonic polynomial generalizations of the Bernoulli and Euler numbers, it was shown how to obtain the coefficient (a polynomial in k, k') of the general term in the Maclaurin series for any elliptic function. The general form of these polynomial coefficients (degree in k or k', etc.) can be readily determined by the methods there given, but the explicit forms of the rational number coefficients as functions of the rank and degree are not easily found. A knowledge of the arithmetical form of these numerical coefficients, which have engaged the attention of many writers on elliptic functions, including Weierstrass, Hermite and Hurwitz, would throw much light on the necessary and sufficient conditions that an integer may be simply representable in a given quadratic form. The methods outlined above, combined with expansions such as those in (M) seem sufficient for the algebraic discussion. For the further investigation of the underlying arithmetic, the method of Gruder* for calculating the numerical coefficients explicitly is more promising than previous attempts, all of which prove illusory in practice.

* Wiener Berichte, vol. 126 (1917), pp. 125–183.

CONDITIONS FOR SELF DUAL CURVES.*

By Temple Rice Hollcroft.

1. **Introduction.** In former papers† in which the term "self dual curve" occurs, no definition is given. It is there assumed that such a curve is one whose reciprocal Plücker characteristics are respectively equal, which is sufficient only so long as no distinction is made between real and imaginary singularities. This distinction necessitates a more precise definition.

The dual curve of a given curve $f(x, y, z) = 0$ is the curve whose equation is $\varphi(x, y, z) = 0$, where $\varphi(\lambda, \mu, \nu) = 0$ is the equation of $f(x, y, z) = 0$ in line coördinates. Geometrically, φ is the polar reciprocal of f with respect to the conic $x^2 + y^2 + z^2 = 0$. A definite self dual curve f is, then, a curve whose equation φ is such that $\varphi \equiv f$.

By a curve type is meant a curve of given order, class and genus with a given number of each kind of double points and lines, that is, a sufficient number of these given to determine the rest. If the reality of these double elements is not involved, a curve type shall be considered the totality of curves whose six Plücker characteristics have each a definite value. However, if a distinction is made between real and imaginary cusps and inflections, real, isolated and imaginary nodes and bitangents, then a curve type shall be taken to mean the totality of curves of given order, class and genus for which the number of *each kind* of double element distinguished as to reality is given.

The dual curve type C' of a given curve type C shall mean here the totality of curves C' whose characteristics, (distinguished or not distinguished as to their reality) are those of the dual curve of any curve of type C.

A self dual curve type C shall mean the totality of curves whose characteristics (distinguished or not distinguished as to their reality) are those of any one definite self dual curve.

A self dual curve of order n and genus p and of any given type has $2n - p + 1$ degrees of freedom.

Of the $(2n - p + 1)$-fold infinity of descriptively self dual curves of a given type, comparatively few are metrically self dual, that is, have equations

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† M. W. Haskell, The maximum number of cusps of an algebraic plane curve and enumeration of self dual curves, Bull. Amer. Math. Soc., vol. 23 (1917), pp. 164–165. T. R. Hollcroft, Maximal cuspidal curves, Annals of Math., vol. 26 (1924), pp. 37–46.

such that $f \equiv \varphi$. In order to find the number of these, consider any definite self dual curve type. Write the most general equation $f(x, y, z) = 0$ of this type with $2n-p+1$ independent parameters and consider that the reciprocal equation $\varphi(x, y, z) = 0$ has been found. The function φ will be of degree n and will, in general, contain terms of the same degree in x, y and z respectively as those of f and other terms as well. The coefficient of each term of φ will be a function of degree $2(n-1)$ in the parameters of f. Since for a self dual curve, $\varphi \equiv f$, equate the coefficients of like terms of φ and f. Since f and φ each has at least $2n-p+1$ terms, this will give at least $2n-p+1$ equations involving the $2n-p+1$ independent parameters of f. No more than $2n-p+1$ of these equations are independent, because φ can have no more than $2n-p+1$ independent coefficients. If for a given type, $2n-p+1$ of the equations are independent, then a finite number ($2n-2$ taken $2n-p+1$ times as a factor) of metrically self dual curves of that type exist. If only $2n-p$ of the equations are independent, then a single infinity of metrically self dual curves of that type exist.

The last case holds for the conic. Of the six equations between the homogeneous coefficients of the general conic and those of its dual, only four are independent and a finite number of one parameter families of conics that have the same point and line equation can be found. An example of such a metrically self dual family of conics is

$$x^2 + by^2 - bz^2 \pm 2\sqrt{1-b^2}\,yz = 0.$$

The envelope of this family is

$$x^2 \pm (y^2 + z^2) = 0.$$

Two other similar families exist whose equations contain xy or xz.

For $n > 2$, the algebraic difficulties involved in attempting to find specific examples of metrically self dual curves by this method are practically insurmountable. For example, in the simplest case, that of the cuspidal cubic, seven simultaneous equations, each of order four in the seven independent parameters of f, must be solved.

As a notation for the number of singularities of any algebraic plane curve of order n, class m, and genus p, we shall say that the curve has $\varkappa$ cusps, δ nodes, ι inflections, and τ bitangents; or if the double elements are distinguished as to their reality, K real cusps, k imaginary cusps, D real nodes, d isolated nodes, $\mathfrak{D}$ imaginary nodes, I real inflections, i imaginary inflections, T real bitangents, t isolated bitangents, $\mathfrak{T}$ imaginary bitangents. In all cases:

$$\begin{aligned} \varkappa &= K + k, \\ \delta &= D + d + \mathfrak{D}, \\ \iota &= I + i, \\ \tau &= T + t + \mathfrak{T}. \end{aligned}$$

The definition of a self dual curve type when no distinction is made as to the reality of the singularities is equivalent to the statement that a self dual curve type is that type for which

$$n = m, \quad \varkappa = \iota, \quad \delta = \tau.$$

When a distinction is made as to reality, the definition of a self dual curve type is equivalent to: A self dual curve type is that type for which

$$n = m, \quad K = I, \quad k = i, \quad D = T, \quad d = t, \quad \mathfrak{D} = \mathfrak{T}.$$

Since only curve types and not metrically distinct curves are to be dealt with in this paper, from now on the term "curve" shall be used with the meaning "curve type" and "a self dual curve" shall mean "the totality of curves of a self dual curve type".

2. **Limits for self dual curves.** By the use of Plücker's equations, it has been shown by Haskell* that for any self dual curve

$$\begin{aligned} \varkappa = \iota &= n + 2p - 2, \\ \delta = \tau &= \frac{1}{2}(n-2)(n-3) - 3p, \end{aligned}$$

and that the minimum number of cusps for a self dual curve is $n-2$ when $p = 0$ and the maximum number $\frac{1}{3}n(n-2)$ when $p = \frac{1}{6}(n-2)(n-3)$. Counting one and but one self dual curve for each value of p between the upper limit and 0, including both limits, for a given value of n the number of self dual curves (whose singularities are not distinguished as to reality) is the largest integer contained in $\frac{1}{6}(n^2 - 5n + 12)$.†

For any algebraic plane curve, Lefschetz‡ has proved that a node accounts for one independent absolute invariant and that for $p \geqq 2$ a curve of order n can have at most $\frac{1}{2}n(n+3) - 8$ such invariants. He assumed further that each cusp always accounts for two such invariants, calling

* M. W. Haskell, loc. cit.

† Cf. Hilton's Plane Algebraic Curves, p. 114, Ex. 4.

‡ S. Lefschetz, On the existence of loci with given singularities, Trans. Amer. Math. Soc., vol. 14 (1913), pp. 23–41.

this assumption "The postulate of singularities". This postulate has been rigorously established for a given n only for a genus less than the genus belonging to the least class for that n.*

In order to find an upper limit to the genus for which self dual curves satisfy this postulate for a given n, substitute the above values of δ and $\varkappa$ for a self dual curve in

$$\delta + 2\varkappa = \frac{1}{2}n(n+3) - 8$$

and obtain

$$p = 2n - 7.$$

Since $2n - 7 < \frac{1}{6}(n-2)(n-3)$ for $n \geqq 14$, it results that, in accordance with the Lefschetz postulate, for $n \geqq 14$ the self dual curve of highest genus has the following characteristics:

$$p = 2n - 7,$$
$$\varkappa = \iota = 5n - 16,$$
$$\delta = \tau = \frac{1}{2}(n-4)(n-13) - 2.$$

In accordance with this, for $n \geqq 14$ there are but $2n - 6$ self dual curves for a given n.

For $n \geqq 16$ the only self dual curve that has the greatest number of cusps for its genus is the one of highest genus $2n - 7$. When $n \leqq 13$ there are $\frac{1}{3}(n-3)$, $\frac{1}{3}(n-4)$, $\frac{1}{3}(n-2)$ self dual curves with the maximum number of cusps for their genus and for n of form $3a$, $3a+1$, $3a+2$ respectively, and when $n = 14, 15$ some of these exist.†

In the proofs of the theorems of this paper, the various forms of Plücker's equations will be used and Klein's‡ Theorem

$$n + I + 2t = m + K + 2d$$

when the reality of the singularities is involved.

Plücker was the first to notice that some of the singularities of certain curves must be imaginary. The following general theorem may be proved:

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* Cf. Lefschetz, loc. cit., pp. 25–26, and J. L. Coolidge, On the existence of curves with assigned singularities, Bull. Amer. Math. Soc., vol. 28 (1922), pp. 451–455.

† Cf. T. R. Hollcroft, loc. cit., pp. 44–45.

‡ F. Klein, Eine neue Relation zwischen den Singularitäten einer algebraischen Kurve, Math. Ann., vol. 10 (1876), pp. 199–209 and Gesammelte Mathematische Abhandlungen, vol. 2, pp. 78–88.

THEOREM I. *In the case of any algebraic plane curve whose equation has only real coefficients, when the curve is self dual and then only may all its double points and lines be*

(a) *wholly real, that is real points (lines) with real tangents (points of contact);*

(b) *wholly imaginary, that is imaginary points (lines) with imaginary tangents (points of contact).*

Isolated singularities are neither wholly real nor wholly imaginary; so to prove this theorem we assume that none of the singularities are isolated. Then all must be either wholly real or wholly imaginary or consist of some of each of these two kinds. From Klein's theorem there results, since $t = d = 0$,

$$K - I = n - m$$

and, since by Plücker's equations $\varkappa - \iota = 3(n-m)$, there results

$$k - i = 2(n-m).$$

From these expressions it is evident that when $t = d = 0$ and $m \neq n$, it can happen neither that both K and I are zero nor that both k and i are zero. Therefore, when $m \neq n$, the singularities of a curve can not all be either wholly real or wholly imaginary.

When $m = n$, it may be that $K = I = 0$ or that $k = i = 0$ but not both. In this case, however, the following still more general theorem holds:

THEOREM II. *There are no restrictions as to reality on the singularities of a self dual algebraic plane curve whose equation has real coefficients, except that imaginary singularities must occur in pairs.*

To prove this, it is sufficient to note that, when a curve is self dual, Klein's formula is identically satisfied and it is necessary that only the following relations hold:

$$K + k = I + i = n + 2p - 2,$$

$$D + d + \mathfrak{D} = T + t + \mathfrak{T} = \frac{1}{2}(n-2)(n-3) - 3p,$$

$$D = T, \quad d = t, \quad \mathfrak{D} = \mathfrak{T}, \quad K = I, \quad k = i.$$

3. **Conditions that a plane curve be self dual.** Many conditions for self duality exist among the characteristics of plane curves. Of these, some are both necessary and sufficient while some are only necessary and others only sufficient. From the definition of self dual curves it follows that the equality of each characteristic with its reciprocal is always a necessary condition.

In order to avoid confusion, the theorems giving the conditions for self duality will first be established when no distinction as to the reality of the singularities is made, after which all those conditions that involve such a distinction will be proved.

In section 2 the value of each Plücker characteristic for a self dual curve was found in terms of n and p. These conditions are thus necessary. They are also sufficient. Moreover, the following theorem is readily proved by Plücker's equations:

THEOREM III. *For any algebraic plane curve of order n (class m) and genus p, any one of the following eight conditions is both necessary and sufficient that the curve be self dual.*

$$(1)\ \varkappa = n+2p-2, \qquad (5)\ \delta = \frac{1}{2}(n-2)(n-3)-3p,$$

$$(2)\ \varkappa = m+2p-2, \qquad (6)\ \tau = \frac{1}{2}(n-2)(n-3)-3p,$$

$$(3)\ \iota = n+2p-2, \qquad (7)\ \delta = \frac{1}{2}(m-2)(m-3)-3p,$$

$$(4)\ \iota = m+2p-2, \qquad (8)\ \tau = \frac{1}{2}(m-2)(m-3)-3p.$$

The following theorems will now be established.

THEOREM IV. *For any algebraic plane curve, each of the following equalities is both necessary and sufficient that the curve be self dual:*

(1) $a\delta + b\varkappa + cn = a\tau + b\iota + cm$

for any fixed values of a, b and c when a and $3b+c$ have the same sign except $b = c = 0$ or $a = 3b+c$.

(2) $p+q = 2n-7$

wherein p is the genus and q is the invariant characteristic defined by $q \equiv \frac{1}{2}n(n+3)-\delta-2\varkappa-8$, (See section 2).

(3) *Any set of three independent relations from among the six necessary relations: $n = m$, $D = T$, $d = t$, $\mathfrak{D} = \mathfrak{T}$, $K = I$, $k = i$.*

To prove (1), write it in the form

$$a(\delta-\tau)+b(\varkappa-\iota)+c(n-m) = 0.$$

Substitute in this the values of $\delta-\tau$ and $\varkappa-\iota$ in terms of n and m from Plücker's equations, and there results

$$(n-m)[a(n+m-9)+6b+2c] = 0.$$

Then either $n = m$ and the curve is self dual or

$$n+m = 9-\frac{2}{a}(3b+c).$$

Since for a non self dual curve $n+m \geqq 7$ in the above equality, it follows that $a \geqq 3b+c$. But in order that $\frac{2}{a}(3b+c)$ may be an integer, a must be a factor of $2(3b+c)$. The only values of a, b and c satisfying these conditions are

$$b = c = 0 \quad \text{and} \quad a = 3b+c.$$

The discussion of these two exceptions will be completed in Theorem V under which they come.

To prove (2), substitute in the expression defining q the following expressions for $\varkappa$ and δ

$$\varkappa = 2(n+p-1)-m,$$

$$\delta = \frac{1}{2}(n-1)(n-6)+m-3p,$$

and we obtain

$$n = m.$$

THEOREM V. *For any algebraic plane curve, each of the following equalities is necessary that the curve be self dual, but it is sufficient when, and only when, the additional conditions given with each holds:*

(1) $a\delta + b\varkappa + cn = a\tau + bi + cm.$

(1_1) *For* a *and* $3b+c$ *both of the same sign and* $b = c = 0$ *when* $n+m \neq 9$ *or* $a = 3b+c$ *when* $n+m \neq 7$.

(1_2) *For* a *and* $3b+c$ *opposite in sign when* a *is not a factor of* $2(3b+c)$.

(2) $D = T$, $\mathfrak{D} = \mathfrak{T}$, $K = I$, *when* $n+m \neq 10$, *except* $n = 5$.

(3) $D = T$, $\mathfrak{D} = \mathfrak{T}$, $k = i$, *when* $n+m \neq 7$.

The proof of (1) was begun in the first part of Theorem IV. There it was shown that either $n = m$ or this relation results:

$$n+m = 9-\frac{2}{a}(3b+c).$$

For (1_1), when $b = c = 0$, $\delta = \tau$, and the above relation becomes $n+m = 9$. There are four and only four curves for which $\delta = \tau$ and $n+m = 9$, namely the elliptic cubic, the rational quartic of class 5 and their reciprocals. When $a = 3b+c$ the above relation becomes $n+m = 7$. Two curves, the nodal cubic and its reciprocal, satisfy this.

In the case of (1_2), the term $\frac{2}{a}(3b+c)$ is always negative and is an integer when a is a factor of $2(3b+c)$. Then curves for which $n+m$ equals the sum of 9 and this integer can be found whose characteristics satisfy condition (1) when $n \neq m$.

THEOREM VI. *For any algebraic plane curve, each of the following sets of equalities is sufficient, but not necessary, that the curve be self dual:*

(1) If any linear function of two or more of the six Plücker characteristics $n, m, \delta, \varkappa, \tau, \iota$, vanishes, and if there vanishes also the same linear function of the same number of characteristics such that equal coefficients of the two functions are associated with dual characteristics, then the curve is self dual with the exception only of certain particular relations among the coefficients. As an example, consider two functions involving all the characteristics. (This really includes all cases, as functions involving a smaller number can be found from these by setting the appropriate coefficients equal to zero.)

$$a\delta + b\tau + c\varkappa + d\iota + en + fm + g = 0,$$
$$a\tau + b\delta + c\iota + d\varkappa + em + fn + g = 0.$$

This holds for all values of the coefficients when the expressions $(a-b)$ and $3(c-d)+e-f$ have the same sign, except when $c-d$ and $e-f$ both vanish or when $a-b = 3(c-d)+e-f$.

(2) $K = d,\ I = t,\ D = T,\ \mathfrak{D} = \mathfrak{T}$.

(3) $\mathfrak{D} = d,\ T = t,\ i = D,\ K = I,\ k = \mathfrak{T}$.

To prove (1), subtract the second equation from the first and the statement of (1) Theorem IV results. The proof of (1) of the present theorem therefore reduces to the proof of (1), Theorem IV.

If $d = e = f = g = 0$ and either $a = 0,\ b = -c,\ n+m \neq 15$, or $b = 0,\ a = -c$, the curves that satisfy (1) are such that

$$p = \frac{1}{10}(n-2)(n-5),$$
$$\delta = \varkappa = \tau = \iota = \frac{1}{5}n(n-2).$$

The four singularities can be equal only when $n = 0, 2 \bmod 5$. If we put $5n = n(n-2)$, we find that a curve of order 7 exists for which all six of the Plücker characteristics are equal. For $n > 23$, the number of nodes and cusps of these curves exceed the maximum number allowed by the Lefschetz postulate, so that there are but nine curves for which $\varkappa = \tau = \delta = \iota$ of which the orders are 2, 5, 7, 10, 12, 15, 17, 20, 22.

THEOREM VII. *For any algebraic plane curve, the following sets of conditions, which are not necessary, are sufficient that the curre be self dual except for the particular relations given:*

(1) General statement the same as in Theorem VI (1). The exceptional values of the coefficients come under the present theorem.

(1_1) When the expressions $a-b$ and $3(c-d)+e-f$ have the same sign and either $c-d = e-f = 0$, $n+m \neq 9$ or $a-b = 3(c-d)+e-f$, $n+m \neq 7$.

(1_2) When the expressions $a-b$ and $3(c-d)+e-f$ have opposite signs and $a-b$ is not a factor of $2[3(c-d)+e-f]$.

(2) Of the six Plücker characteristics, if any two not duals of each other are so related that the first equals a linear fractional function of the second and if the dual of the first equals the same function of the dual of the second, the curve is self dual except for certain particular values of n and m. As an example, consider

$$\varkappa = \frac{a\tau+b}{c\tau+d}, \quad \iota = \frac{a\delta+b}{c\delta+d}, \quad ad-bc \neq 0.$$

Under these conditions, either $n = m$ or distinct integral values of n and m must be found satisfying the relation

$$[(ad-bc)(n+m-9)-3(3bc-3ad+2d^2)]$$
$$\left[(n+m-9)\{2(ad-bc)+3c(n-m)[3a-2d-c(n^2-n-m)]\}-6(3bc-3ad+2d^2)\right]$$
$$+3(n+m-9)(ad-bc)[3a-2d-c(n^2-n-m)]^2 = 0.$$

(3) $\varkappa = \frac{1}{3}n(n-2)$ for n of form $3a$ or $3a+2$ and $n \neq 6$,

(4) $K = t$, $I = d$, $T = D$, $\mathfrak{T} = \mathfrak{D}$, $n+m \neq 11$.

The proof of (1) bears the same resemblance to V(1) as VI(1) bears to IV(1).

In (2), eliminate δ, $\varkappa$, τ and ι from the two fractional functions and from Plücker's equations, obtaining the condition given in the theorem that must be satisfied by distinct integral values of n and m if $n \neq m$. In general, the equation in n and m resulting from a given collineation form is not satisfied by integral values of n and m so that there remains only $n = m$.

To prove (3), substitute the given value of $\varkappa$ in the Plücker formula for the class and there results

$$2\delta = n-m.$$

From this it results that $m < n$ when $\delta > 0$. Substitute this value of δ in the Plücker formula for $\delta-\tau$ and we have

$$\tau = \frac{1}{2}(m-n)(m+n-10).$$

Then for $m < n$, $\tau < 0$ except when $m+n \leqq 10$. The only curve for which $\varkappa = \frac{1}{3}n(n-2)$ and $m < n$ is the elliptic sextic of class four.

Then, except for $n = 6$, $\delta = 0$ when $\varkappa = \frac{1}{3}n(n-2)$ and therefore $n = m$ and the curve is self dual. In accordance with the Lefschetz postulate, these curves do not exist for $n \geqq 14$.

THEOREM VIII. *The following conditions are only necessary that a curve be self dual. Conditions* (1) *to* (4) *inclusive are not sufficient when taken in any combination or all together, nor can any or all of them be combined with any set of necessary but not sufficient conditions to make a set of sufficient conditions.*

(1) $\varkappa \geqq n-2$.

(2) $\delta \leqq \frac{1}{2}(n-2)(n-3)$.

(3) $p \leqq \frac{1}{6}(n-2)(n-3)$ for $n \leqq 13$.

(4) $p \leqq 2n-7$ for $n \geqq 14$.

(5) *When the characteristics are distinguished as to their reality, any set of conditions from among the following six if such a set is equivalent to no more than two independent conditions*:

$$n = m, \quad D = T, \quad d = t, \quad \mathfrak{D} = \mathfrak{T}, \quad K = I, \quad k = i.$$

The first four conditions are merely limits, and the fact that a curve has characteristics within these limits does not at all imply that it is self dual. In (3) is implied the limit $\varkappa \leqq \frac{1}{3}n(n-2)$, $\delta \geqq 0$ and in (4), $\varkappa \leqq 5n-16$, $\delta \geqq \frac{1}{2}(n-4)(n-13)-2$.

The conditions for self duality in the preceding theorems that involve the reality of the double elements will now be established. These are put together because the term "self dual curve", when the reality of the singularities is considered, is used in a more restricted sense than when only the Plücker characteristics are involved and it would be confusing to turn often from the more restricted to the general meaning and vice versa. (The term "more restricted" does not imply that a self dual curve has fewer degrees of freedom when a distinction is made as to the reality of its double elements.)

When the reality of the double elements is taken into consideration, those conditions that were sufficient for the curve to be self dual when no distinction was made as to reality, are no longer sufficient. However, if they were then necessary, they still remain so. Any condition sufficient to make $n = m$ may be considered as equivalent to the condition $n = m$.

Proof of Theorem IV, (3). The condition $n = m$ leads to the three equalities, the first two from Plücker's equations and the third from Klein's theorem:

$$K+k = I+i,$$
$$D+d+\mathfrak{D} = T+t+\mathfrak{T},$$
$$I+2t = K+2d.$$

In this case, in addition to $n = m$, two intependent necessary conditions are needed to cause the curve to be self dual, that is, any one of the seven sets obtained by taking $D = T$ or $\mathfrak{D} = \mathfrak{T}$ with each other or with any other of the three remaining necessary conditions. In all there are sixteen such sets, each containing three independent, necessary conditions and each set is sufficient (with the exception of two which are sufficient except for particular values of n and m and thus come under Theorem V) for the curve to be self dual. Any such set must contain either $D = T$ or $\mathfrak{D} = \mathfrak{T}$. The proof for any set that does not contain $n = m$, is similar to that given in the next theorem.

Proof of Theorem V, (2). When $K = I$, from Klein's theorem and Plücker's equations, respectively, there results

$$d-t = \frac{1}{2}(n-m),$$
$$k-i = 3(n-m).$$

Subtracting from

$$p = \frac{1}{2}(m-1)(m-2)-(T+t+\mathfrak{T})-(I+i)$$

the similar expression for p in terms of n and the point singularities, we obtain, using the conditions $D = T$, $\mathfrak{D} = \mathfrak{T}$, $K = I$,

$$(m-1)(m-2)-(n-1)(n-2)+2(d-t)+2(k-i) = 0.$$

Substitute in this the values of $d-t$ and $k-i$ in terms of n and m found above, and there results

$$(n-m)(n+m-10) = 0.$$

Then, when conditions (2) hold, either $n = m$ or $n+m = 10$. Two and only two curves, duals of each other, satisfy the latter case.

For (3), when $k = i$, $K-I = 3(n-m)$. Substituting this in Klein's theorem, we obtain $d-t = m-n$. Substitute these values of $K-I$ and $d-t$ in the difference of the two expressions for p as in (2) and there results:

$$(n-m)(n+m-7) = 0.$$

Then, when conditions (3) hold, either $n = m$ or $n + m = 7$. Two curve types, duals of each other, satisfy the latter relation.

Proof of Theorem VI, (2). Using the conditions $K = d$, $I = t$ and Klein's theorem, we obtain

$$3(d - t) = n - m.$$

When $D = T$ and $\mathfrak{D} = \mathfrak{T}$ from Plücker's equations

$$2(d - t) = (n - m)(n + m - 9).$$

Equating these two values of $d - t$, we obtain

$$2(n - m) = 3(n - m)(n + m - 9).$$

Then $n = m$, since otherwise $n + m$ is not an integer.

Part (3) can be proved in a similar manner. If the additional condition $K = k$ is put on this curve, a self dual curve is obtained for which

$$D = d = \mathfrak{D} = K = k = T = t = \mathfrak{T} = I = i = \frac{1}{12} n(n-2)$$

and $p = \frac{1}{12}(n-2)(n-6)$. Only curves of orders 2, 12, 14, 24, 26 can satisfy these conditions and the Lefschetz postulate. For $n = 14$ the self dual curve satisfying these conditions has the order, class and all ten singularities equal.

Other sets of relations similar to (3) can be found. Some of these may be obtained from (3) by permuting the characteristics.

Under the conditions given in VII, (4), Klein's theorem and Plücker's equations give

$$(n - m)(n + m - 11) = 0.$$

When $n = m$, the two independent, necessary conditions of (4) make the curve self dual. When $n \neq m$, there are two quintic curves, one rational and one elliptic, which together with their duals satisfy conditions (4) and $n + m = 11$.

Other sets of relations of this kind may also be found.

In Theorem VIII, (5), the four necessary conditions $n = m$, $d = t$, $K = I$, $k = i$ taken simultaneously are not sufficient for the curve to be self dual. As stated in the proof of Theorem IV, (3), any such set of conditions in order to be sufficient must contain either $D = T$ or $\mathfrak{D} = \mathfrak{T}$.

4. Conditions under which two or more of Plücker's numbers vanish. For $n > 1$ no more than two of the four Plücker numbers δ, $\varkappa$, τ, ι may vanish simultaneously with these exceptions: for the conic, all four may be zero and for the elliptic cubic and its reciprocal, three.

Under the condition, $\delta = \tau = 0$, we obtain either the elliptic cubic and its reciprocal or the self dual curve for which $\varkappa = \iota = \frac{1}{3}n(n-2)$. As we have seen, this is the curve with the maximum number of cusps for a given n and it does not exist for $n \geqq 14$. Then for $n \geqq 14$, δ and τ can not vanish simultaneously.

For $n > 2$ no curve exists for which $\varkappa = \iota = 0$.

To find the curves for which $\delta = \iota = 0$, take these conditions simultaneously with Plücker's equations and obtain

$$m = \frac{1}{8}n(10-n).$$

This is satisfied by two curves, the sextic with nine cusps and the tricuspidal quartic. The conditions $\tau = \varkappa = 0$ lead to a relation reciprocal to that above which is satified only by the elliptic and nodal cubics, the reciprocals of the above curves respectively.

The necessary and sufficient condition for $\tau = \iota = 0$ is $n = m(m-1)$ and dually, for $\delta = \varkappa = 0$, $m = n(n-1)$. For $n \geqq 14$, these are the only curves for which two of Plücker's numbers are simultaneously zero.

WELLS COLLEGE.

SEMI-PARALLEL MAPS OF SURFACES.

By W. C. Graustein.

1. Let S and $\bar{S}$ be two surfaces in one-to-one point correspondence. If the tangent planes at corresponding points, P and $\bar{P}$, are parallel, S and $\bar{S}$ correspond by a parallel map. If, on the other hand, the tangent planes at P and $\bar{P}$ are not parallel, but there exist two families of curves, one on each surface, which have the property that their curves correspond and have at corresponding points parallel tangents, the map of S on $\bar{S}$ shall be called a semi-parallel map.

In the case of an arbitrary non-parallel map, there are just two directions tangential respectively to S and $\bar{S}$ at P and $\bar{P}$ which are mutually parallel. It is only when the curves C and $\bar{C}$ generated by these directions correspond that the map of S on $\bar{S}$ is semi-parallel.

We proceed to discuss a number of interesting types of semi-parallel maps.

2. Let S and $\bar{S}$ be represented parametrically so that corresponding points $x:(x_1, x_2, x_3)$, $\bar{x}:(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ have the same curvilinear coördinates (u, v). Choose, in particular, as the u-curves, the families of curves C and $\bar{C}$ whose corresponding tangents are parallel. Then,

$$(1) \qquad \begin{aligned} \bar{x}_u &= \lambda x_u, \\ \bar{x}_v &= \alpha x_u + \beta x_v + \gamma \zeta, \qquad \gamma \not\equiv 0, \end{aligned}$$

where $\alpha, \beta, \gamma, \lambda$ are scalar functions of u, v, and $\zeta_1, \zeta_2, \zeta_3$ are the direction cosines of the oriented normal to S.

We proceed to ascertain when the curves C and $\bar{C}$ belong to corresponding conjugate systems on S and $\bar{S}$, or, when these fail to exist, when C and $\bar{C}$ constitute corresponding families of asymptotic lines.

Consider the latter case first and assume that the curves C are asymptotic lines: $e = 0$.* From (1), we find:

$$\bar{D}\bar{e} = -D\lambda^2\gamma \begin{Bmatrix} 1 & 1 \\ 2 & \end{Bmatrix}.$$

Hence, if the curves $\bar{C}$ are asymptotic lines, if follows that the curves C are straight lines. Then this must be true also of the curves $\bar{C}$, since the roles of S and $\bar{S}$ may be interchanged.

* We denote the first and second fundamental quadratic forms and their discriminants as follows:

$$\begin{aligned} E du^2 + 2F du dv + G dv^2, \qquad & D^2 = EG - F^2, \\ e du^2 + 2f du dv + g dv^2, \qquad & d^2 = eg - f^2. \end{aligned}$$

If the curves C and $\bar{C}$ which correspond by parallel tangents in a semi-parallel map are asymptotic lines, the surfaces are ruled and the curves C and $\bar{C}$ are the rulings.

It is evident that any two ruled surfaces whose director cones are congruent can be related in infinitely many ways by a semi-parallel map in which the rulings are the curves which correspond by parallel tangents.

3. We consider now the general case in which there exists a conjugate system of curves on each surface corresponding to a conjugate system on the other and investigate under what conditions the families of curves C and $\bar{C}$ belong to these systems.

Take as the v-curves on the surface S the family of curves conjugate to the family of curves C. The parametric curves on S then form a conjugate system: $f = 0$. It follows from (1) that, if this is true also of the parametric curves on $\bar{S}$, then

$$\begin{Bmatrix} 1 & 2 \\ 2 & \end{Bmatrix} = 0.$$

Consequently,

$$x_{uv} = A x_u,$$

and hence

$$x_u = aU, \tag{2}$$

where a is a scalar function of u, v, and $U: (U_1, U_2, U_3)$ is a triple of functions depending on u alone.

Since the surfaces S and $\bar{S}$ may be interchanged without affecting the argument, we have also:

$$\bar{x}_u = \bar{a}U. \tag{3}$$

Conversely, if equations (2) and (3) subsist, S and $\bar{S}$ are related as desired.

A necessary and sufficient condition that two surfaces S and $\bar{S}$ can be made to correspond by a semi-parallel map in which the families of curves which correspond by parallel tangents belong to corresponding conjugate systems is that parameters, common to S and $\bar{S}$, can be so chosen that

$$x_u = aU, \qquad \bar{x}_u = \bar{a}U, \tag{4}$$

where a and $\bar{a}$ are scalar functions of u, v and U a triple of functions of u alone.

It is evident that the existence of parameters rendering the relation (2) valid is equivalent to the existence on the surface of a family of curves each two of which can be related by a transformation of Combescure,

i. e., can be put into a one-to-one point correspondence so that the tangents at corresponding points are parallel.

A necessary and sufficient condition that a surface S admit a semi-parallel map in which the curves C which are mapped by parallel tangents belong to a conjugate system which remains conjugate in the map is that there exist on the surface S a family of curves each two of which can be related by a transformation of Combescure. The curves of this family are, then, to be taken as the curves C.

The necessity of the condition has already been established. To prove the condition sufficient, we have to show that when curvilinear coördinates on S are so chosen that

$$x_u = aU,$$

scalar functions, $\bar{a}$, α, β, $\gamma(\neq 0)$, can be found so that the equations

$$\bar{x}_u = \bar{a}U, \qquad \bar{x}_v = \alpha x_u + \beta x_v + \gamma\zeta$$

are compatible.

The conditions of compatibility are readily found to be:

$$\begin{aligned}
&\alpha_u + \left\{\begin{matrix}1 & 1\\ 1 & \end{matrix}\right\}\alpha + \frac{a_v}{a}\beta - \frac{eG}{D^2}\gamma = \frac{\bar{a}_v}{a},\\
&\beta_u + \left\{\begin{matrix}1 & 1\\ 2 & \end{matrix}\right\}\alpha \qquad\quad + \frac{eF}{D^2}\gamma = 0,\\
&\gamma_u + \qquad e\alpha \qquad\qquad\quad = 0.
\end{aligned}$$

It is evident that $\bar{a}$ can be chosen at pleasure; v can then be considered as a parameter and the three equations thought of as a system of ordinary differential equations linear and of the first order in α, β, γ as functions of u.

4. When the developables of the congruence of lines joining corresponding points of the two surfaces S and $\bar{S}$ cut the surfaces in conjugate systems, the transformation of the one surface into the other is known as a fundamental transformation or a transformation F.* It is understood that neither surface is a focal surface of the congruence.

If S and $\bar{S}$ are related by a semi-parallel map, the ruled surfaces of the congruence which intersect S and $\bar{S}$ in the curves C and $\bar{C}$ corresponding by parallel tangents form one family of developables of the congruence. Consequently, if the transformation of S into $\bar{S}$ is a transformation F, it follows, first, that the families of curves C' and $\bar{C}'$ conjugate respectively to the families of curves C and $\bar{C}$ correspond, and, secondly, that the surfaces of the congruence which intersect S and $\bar{S}$ in these new

* Cf., e. g., Eisenhart, Transformations of Surfaces, Princeton (1923), Ch. 2.

families of curves are developables, or, what amounts to the same thing, that the tangents to corresponding curves C' and $\bar{C}'$ at corresponding points always intersect.

In light of the first theorem of the preceding paragraph we can express our conclusions analytically, as follows:

A necessary and sufficient condition that two surfaces S and $\bar{S}$ can be related by a semi-parallel, fundamental transformation is that parameters, common to both, can be so chosen that

$$x_u = aU, \qquad \bar{x}_u = \bar{a}U, \tag{4}$$

$$x = \mu x_v = \bar{x} + \bar{\mu}\bar{x}_v, \tag{5}$$

where a, $\bar{a}$, μ, $\bar{\mu}$ are non-vanishing scalar functions of u, v and U_1, U_2, U_3 are functions of u alone.

Differentiating both sides of equation (5) with respect to u, we obtain a relation which can be reduced, by means of (4), to the form:

$$(a + \mu a_v)U + \mu_u x_v = (\bar{a} + \bar{\mu}\bar{a}_v)U + \bar{\mu}_u \bar{x}_v.$$

Since the transformation of S into $\bar{S}$ is not a parallel transformation, we have

$$\mu_u = 0, \qquad \bar{\mu}_u = 0,$$
$$a + \mu a_v = \bar{a} + \bar{\mu}\bar{a}_v.$$

In the equations (4) *and* (5) *representing a semi-parallel, fundamental transformation, μ and $\bar{\mu}$ are functions of v alone and*

$$a + \mu a_v = \bar{a} + \bar{\mu}\bar{a}_v. \tag{6}$$

The transformation of S into $\bar{S}$ establishes between the pencils of tangent lines at corresponding points, P and $\bar{P}$, of S and $\bar{S}$ a projective correspondence. The two pencils cut the line of intersection, L, of the tangent planes at P and $\bar{P}$ in two projective ranges of points. The invariant cross-ratio, I, of the projectivity on L is an invariant of the transformation of S into $\bar{S}$. Its value, in the present case, is*

$$I = \frac{\bar{a}\bar{\mu}}{a\mu}. \tag{7}$$

5. If $I \equiv 1$, the projectivity on L is the identity, all the ruled surfaces of the congruence of lines joining corresponding points S and $\bar{S}$ are develop-

* Cf. Author, An invariant of a general transformation of surfaces, to appear shortly in the Bull. Amer. Math. Soc.

ables, and hence the lines of the congruence all pass through a point, finite or at infinity. In this case, the transformation of S into $\bar{S}$ is known as a radial transformation.

It is evident that the general radial, semi-parallel, fundamental transformation can be written as

$$\bar{x} = vx, \quad \text{or} \quad \bar{x} = x + vc,$$

according as the point common to all the lines of the associated congruence is finite or at infinity, it being understood that in both cases $x_u = aU$.

An important case is that in which $I = I(v) \neq 1$, for then the invariant is constant along each pair of the curves which are mapped by parallel tangents. In this case, since μ and $\bar{\mu}$ are functions of v alone, $\bar{a}/a$ is a function of v alone and we can replace a and $\bar{a}$ respectively by $a(v)b(u,v)$ and $\bar{a}(v)b(u,v)$. Equation (6) then becomes

$$a(v) + \mu(v)a'(v) = \bar{a}(v) + \bar{\mu}(v)\bar{a}'(v) + (\bar{\mu}\,\bar{a} - \mu a)\frac{b_v}{b}.$$

Since $I \neq 1$, b_v/b is a function of v alone and b is of the form $k(v)l(u)$. But $k(v)$ can be absorbed in $a(v)$ and $\bar{a}(v)$, and $l(u)$ in U. In other words, we can assume without loss of generality that $b = 1$, or that the original a and $\bar{a}$ are functions of v alone.

We have, then, from (4),

$$x = a(v)U + V, \qquad \bar{x} = \bar{a}(v)U + \bar{V}, \tag{8}$$

where the new triple U is an integral of the original one. Equation (5) is then satisfied, provided that

$$\begin{aligned} a + \mu a' &= \bar{a} + \bar{\mu}\bar{a}', \\ V + \mu V' &= \bar{V} + \bar{\mu}\bar{V}'. \end{aligned} \tag{9}$$

A necessary and sufficient condition that a surface S admit a semi-parallel fundamental transformation whose invariant is constant ($\neq 1$) *along each of the curves mapped by parallel tangents is that S contain a family of curves C each two of which are homothetic, that is, that S admit a parametric representation of the form $x = a(v)U + V$, where $a(v)$ is a scalar function and $U(u)$, $V(v)$ are triples.**

The surface $\bar{S}$ which is the general transform of S of the type in question is then $\bar{x} = \bar{a}(v)U + \bar{V}$, where the triple $\bar{V}(v)$ is first chosen so that the curve $\bar{y} = \bar{V}(v)$ is so related to the curve $y = V(v)$ that corresponding

* A surface of revolution constitutes a case in point.

tangents always intersect.* Then μ and $\bar{\mu}$ are determined and the first equation in (9) becomes a linear differential equation of the first order for the determination of $\bar{a}(v)$.

Characteristic of this type of map is that each two curves C and $\bar{C}$ which are mapped by parallel tangents are homothetic.

Suppose now that we demand that each two curves C and $\bar{C}$ be congruent. Then $\bar{a} = a$, and equation (6) becomes

$$a_v(\mu - \bar{\mu}) = 0.$$

If we assume that $I \neq 1$, then $\mu \neq \bar{\mu}$, and hence $a_v = 0$. We can, then, take $a = \bar{a} = 1$. Equations (4) and (5) give rise to

$$x = U + V, \qquad \bar{x} = U + \bar{V}, \tag{10}$$

as the forms of S and $\bar{S}$, where V and $\bar{V}$ are connected by the relation

$$V + \mu V' = \bar{V} + \bar{\mu}\bar{V}'. \tag{11}$$

Each two curves C and $\bar{C}$ which correspond by parallel tangents in a non-radial, semi-parallel, fundamental transformation of a surface S into a surface $\bar{S}$ are congruent if and only if S and $\bar{S}$ are translation surfaces whose parametric representations can be reduced simultaneously to the forms $x = U + V$, $\bar{x} = U + \bar{V}$, where the curves $y = V(v)$, $\bar{y} = \bar{V}(v)$ are so related that the tangents at corresponding points intersect.

6. As a final problem, we propose to isolate the cases in which the correspondence established between two surfaces S and $\bar{S}$ by a non-radial, semi-parallel, fundamental transformation is isometric. It is evident from (4) that a necessary condition is that $\bar{a}^2 = a^2$. Thus two cases arise.

Case 1. $\bar{a} = a$. Here, S and $\bar{S}$ are the translation surfaces (10) subject to the relation (11). It is readily found that the map of S on $\bar{S}$ is isometric if and only if

$$(U' \mid V' - \bar{V}') = 0, \qquad (V' \mid V') = (\bar{V}' \mid \bar{V}'). \tag{12}$$†

The first condition demands that either U' or $V' - \bar{V}'$ have always the same direction. But U' cannot have a fixed direction, since otherwise

* For example, $\bar{y} = \bar{V}(v)$ might be taken as an evolute of an involute of the curve $y = V(v)$.

† We denote by $(a \mid b)$ the inner, or scalar, product of two ordered triples of numbers (a_1, a_2, a_3), (b_1, b_2, b_3).

the surfaces S and $\bar{S}$ would be cylinders and the transformation not fundamental. Consequently,

$$\bar{V}' = V' + f(v)c,$$

where $f(v)$ is a scalar and c a triple of constant direction cosines.

Substituting this value of $\bar{V}'$ into the second condition of (12), we find that $f(v) = -2(V'|c)$ and hence that

$$\bar{V} = V - 2(V|c)c + d,$$

where d is a triple of constants.

On the other hand, $(U'|c) = 0$. Consequently, we can write:

$$(U|c) = 0.$$

It remains to take account of the relation (11). For this purpose, we choose c in particular as 0, 0, 1, and write out the three equations for which (11) stands. The conditions that these equations, considered as equations in $\mu, \bar{\mu}$, be compatible lead readily to the final forms of S and $\bar{S}$:

$$\begin{aligned} &x_1 = f_1(u) + d_1 v, \quad && x_2 = f_2(u) + d_2 v, \quad && x_3 = f_3(v), \\ &\bar{x}_1 = x_1 + d, && \bar{x}_2 = x_2 + d_2, && \bar{x}_3 = -x_3 + d_3. \end{aligned}$$

The surface S is an arbitrary translation surface having the property that its generators are plane curves lying in two pencils of parallel planes, the planes of the one pencil being orthogonal to those of the other; $\bar{S}$ is symmetric to S and can be obtained from S by a reflection in a plane of the one pencil followed by a translation in a direction parallel to the planes of the second pencil.

Case 2. $\bar{a} = -a$. In this case, (7) becomes

$$(\mu + \bar{\mu})a_v + 2a = 0.$$

Since $I \not\equiv 1$, $\mu + \bar{\mu} \not\equiv 0$ and therefore a_v/a is a function of v alone. But we can then assume that $a = a(v)$.

Consequently, we find, from (4) and (5), as the general forms of S and $\bar{S}$:

$$x = vU + V, \qquad \bar{x} = -vU + \bar{V},$$

where

$$\mu + \bar{\mu} + 2v = 0, \qquad V + \mu V' = \bar{V} + \bar{\mu}\bar{V}'. \tag{13}$$

The conditions that the map of S on $\bar{S}$ be isometric are

$$(U'|V' + \bar{V}') = 0, \qquad (V'|V') + 2(U|V' + \bar{V}') = (\bar{V}'|\bar{V}'). \tag{14}$$

From the first of these conditions we have

$$V'+\overline{V}' = 2f'(v)c, \qquad (U'\,|\,c) = 0,$$

where $f(v)$ is a scalar and c a triple of constant direction cosines. We can, then, without loss of generality, write

$$V+\overline{V} = 2f(v)c, \qquad (U\,|\,c) = 0.$$

It follows that V and $\overline{V}$ are of the forms

$$V = f(v)c+\beta(v), \qquad \overline{V} = f(v)c-\beta(v),$$

where $\beta(v)$ is a triple. Substituting these values in the second of conditions (14), we find that $\beta'(v)$ must be perpendicular to c and hence that

$$(\beta\,|\,c) = k, \tag{15}$$

where k is a constant.

When we substitute the values of V and $\overline{V}$ in the second of the relations (13), we readily find, by applying the first of these relations and (15), the following form for β:

$$\beta = kc+dv,$$

where d is a triple of constants such that $(c\,|\,d) = 0$.

In the forms for S and $\overline{S}$ which we have thus obtained:

$$x = v(U+d)+(f(v)+k)c, \qquad \overline{x} = -v(U+d)+(f(v)-k)c,$$

we can take the triple d as 0, 0, 0. Hence

$$x = vU+(f(v)+k)c, \qquad \overline{x} = -vU+(f(v)-k)c,$$

where c is a triple of constant direction cosines, $y = U(u)$ is a plane curve in the plane $(y\,|\,c) = 0$, $f(v)$ a scalar function of v and k a constant.

If we choose c in particular as 0, 0, 1, we have

$$x_1 = vU_1(u), \qquad x_2 = vU_2(u), \qquad x_3 = \psi(v),$$
$$\overline{x}_1 = -x_1, \qquad \overline{x}_2 = -x_2, \qquad \overline{x}_3 = x_3+l.$$

The surface S is an arbitrary surface possessing two conjugate families of plane curves such that each two curves of the one family are homothetic and hence lie in a pencil of parallel planes, whereas the curves of the second family lie in a pencil of intersecting planes whose axis is perpendicular to the planes of the first pencil. The surface $\overline{S}$ is congruent to the surface S and can be obtained from S by a screw motion through two right angles about the axis of the second pencil of planes.

PROJECTIVE INVARIANTS OF AFFINE GEOMETRY OF PATHS.

By Oswald Veblen and Joseph Miller Thomas.*

1. **Introduction.** The present article sets forth a complete set of projective invariants† for the affine geometry of paths and in terms of them gives an equivalence theorem analogous to that of Christoffel for the Riemann geometry. In solving the problem we employ direct elimination processes like those of Christoffel, and incidentally prove a theorem on partial differential equations which is of interest in itself.

An invariant for the purposes of the present paper is an entity with definite determining components in any coördinate system, such that the transformations of the components from one coördinate system to another form a group isomorphic with the group of analytic transformations of the coördinates. In § 8 we introduce the idea of the invariant extension of such an invariant.

The projective invariants in terms of which we state our equivalence theorem are not in general tensors, but they are a subclass of the general class of invariants defined above. Their laws of transformation are rational and integral in the first derivatives of the old variables with respect to the new variables and also in certain combinations of the second derivatives. It is shown that the set of invariants can be generated by a particular type of invariant extension which we call projective differentiation.

In the preliminary part of the paper we have included a few general properties of the components of projective connection in addition to those

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† This problem has already been considered by E. Cartan, *Sur les variétés à connexion projective*, Bull. Soc. math. de France, vol. 52 (1924), pp. 205–241, who indicates a method of obtaining a *geometrical* invariant theory of systems of differential equations of the type (3.2) based on the idea of associating a projective space with each point of the given manifold. J. A. Schouten's paper *On the place of conformal and projective geometry in the theory of linear displacements*, Proc. Kon. Akad. Amsterdam, vol. 27 (1924), pp. 417–424, attempts to render Cartan's work analytically intelligible and incidentally shows that the one invariant obtained by Cartan is the projective curvature tensor of Weyl. T. Y. Thomas has kindly let us read the manuscript of his paper, *A projective theory of affinely connected manifolds* which is to appear in the Mathematische Zeitschrift. His method is an ingenious correlation of the projective theory of a manifold of n dimensions with the affine theory of an auxiliary manifold of $(n+1)$ dimensions. It generates a different type of invariants from ours, though the components of his ${}^*B^i_{jkl}$ for which $i=0$ are the same as the components of our $(n+1)\,r_{jkl}$. His process of extension is different from ours.

required for the equivalence theorem, and toward the end of the paper we have given a geometrical interpretation of the contracted components of affine connection which will perhaps throw some light on the relation between affine and projective geometry. In the last section we describe a type of path geometry which has no affine connection.

2. **Affine geometry of paths.** The geometry of paths, in so far as it has yet been effectively studied, is the theory of a set of differential equations of the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{2.1}$$

where $\Gamma^i_{jk} = \Gamma^i_{kj}$. In these equations the indices take the integral values* from 1 to n and repeated indices are summed over this range of values. Any curve whose coördinates given in the form $x^i = \psi^i(s)$ satisfy (2.1) is a *path.*†

The quantities Γ in (2.1) determine a definition of infinitesimal parallelism. If a vector ξ at a point x becomes $\xi + \delta\xi$ by parallel displacement to the point $x + dx$, then we have as far as terms of the first order

$$\delta\xi^i = -\Gamma^i_{jk}\xi^j dx^k.$$

Writing

$$a^i_j = \delta^i_j - \Gamma^i_{jk}dx^k,$$

we find

$$\xi^i + \delta\xi^i = a^i_j\xi^j,$$

a relation which shows that the vectors at the point $x+dx$ are connected with those at x by an affine transformation. The quantities Γ^i_{jk} are therefore called *components of affine connection.*‡

If the coördinates x are changed to $\bar{x}$ by an arbitrary analytic transformation, the equations (2.1) will maintain their form if the components of affine connection in the new coördinate system are given by the formulas

$$\Gamma^l_{jk}\frac{\partial\bar{x}^i}{\partial x^l} = \bar{\Gamma}^i_{lp}\frac{\partial\bar{x}^l}{\partial x^j}\frac{\partial\bar{x}^p}{\partial x^k} + \frac{\partial^2\bar{x}^i}{\partial x^j\,\partial x^k}. \tag{2.2}$$

Hence the affine geometry of paths can be regarded as the theory of an invariant, the affine connection, whose components in a coördinate system x are the functions Γ^i_{jk}, and which has the property that its com-

* This convention will be followed throughout the paper except where the contrary is expressly stated.

† For a systematic account of the geometry of paths cf. O. Veblen and T. Y. Thomas, *The Geometry of Paths*, Trans. Amer. Math. Soc., vol. 25 (1923), pp. 551–580.

‡ H. Weyl, *Raum, Zeit, Materie*, 4th ed., p. 100.

ponents in any two coördinate systems x and $\bar{x}$ are related by the equations (2.2).

3. **Projective change of affine connection.** If the parameter is changed along each path by an analytic transformation from s to t, equations (2.1) become

$$\left(\frac{d^2x^i}{dt^2}+\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right)\left(\frac{dt}{ds}\right)^2+\frac{dx^i}{dt}\frac{d^2t}{ds^2}=0. \tag{3.1}$$

From this result it is obvious that the form of the equations (2.1) is preserved only under a change of parameter of the type $t=as+b$, where a and b are constants.

To find a set of equations of the paths invariant under an arbitrary change of the parameter, we multiply equations (3.1) by dx^j/dt, interchange i and j, and subtract, getting

$$\delta^{ij}_{kl}\left(\frac{d^2x^k}{dt^2}+\Gamma^k_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\right)\frac{dx^l}{dt}=0, \tag{3.2}$$

where δ^{ij}_{kl} is zero unless $i=k$ and $j=l$, or $i=l$ and $j=k$, and is $+1$ in the first case of exception and -1 in the second.* From the manner in which they were derived it is clear that equations (3.2) are invariant under an arbitrary change of parameter. These equations, which were first suggested for the geometry of paths by J. L. Synge, define exactly the same curves† as equations (2.1).

Suppose now that the components of the affine connection are changed to Γ'^i_{jk} without the paths being altered. Then equations like (3.2) hold along any path for both affine connections. Since the form of these equations is independent of the parameter, we can assume that any point on a given path corresponds to the same parameter value for both affine connections. A comparison of the two sets of equations of the form (3.2) gives

$$\delta^{ij}_{kl}\Gamma^k_{pq}\,dx^l\,dx^p\,dx^q=\delta^{ij}_{kl}\Gamma'^k_{pq}\,dx^l\,dx^p\,dx^q.$$

These relations must be identities in the differentials, since it follows from (2.1) that the direction of a path through a given point can be chosen arbitrarily. Equating the coefficients, we get

$$\delta^{ij}_{kl}\Gamma^k_{pq}+\delta^{ij}_{kp}\Gamma^k_{ql}+\delta^{ij}_{kq}\Gamma^k_{lp}=\delta^{ij}_{kl}\Gamma'^k_{pq}+\delta^{ij}_{kp}\Gamma'^k_{ql}+\delta^{ij}_{kq}\Gamma'^k_{lp}. \tag{3.3}$$

* These generalized Kronecker symbols were discovered independently by J. W. Alexander and F. D. Murnaghan. Cf. a paper by the latter, *The generalized Kronecker symbol and its application to the theory of determinants*, Amer. Math. Monthly, vol. 32 (1925), p. 233.

† Cf. O. Veblen and T. Y. Thomas, loc. cit., p. 556.

Contraction for j and p in the last equations gives

$$(3.4)\qquad (n+1)\,\Gamma^i_{jk} - \delta^i_j\,\Gamma^l_{lk} - \delta^i_k\,\Gamma^l_{lj} = (n+1)\,\Gamma'^i_{jk} - \delta^i_j\,\Gamma'^l_{lk} - \delta^i_k\,\Gamma'^l_{lj}.$$

Conversely, if equations (3.4) are satisfied, (3.3) are also. Hence equations (3.4) are necessary and sufficient conditions that the Γ's and the Γ''s be associated with the same system of paths. By contracting the equations (2.2) we find

$$(3.5)\qquad \Gamma^l_{lj} = \bar{\Gamma}^l_{lk}\frac{\partial \bar{x}^k}{\partial x^j} - \frac{\partial \log \Delta\,(x,\bar{x})}{\partial x^j},$$

where Δ is the jacobian

$$\Delta(x,\bar{x}) = \left|\frac{\partial x}{\partial \bar{x}}\right|.$$

From these equations it follows that the functions φ_j defined by

$$(3.6)\qquad \Gamma^l_{lj} - \Gamma'^l_{lj} = (n+1)\varphi_j$$

are the components of a covariant vector. Equations (3.4) can therefore be written

$$(3.7)\qquad \Gamma^i_{jk} = \Gamma'^i_{jk} + \delta^i_j\,\varphi_k + \delta^i_k\,\varphi_j,$$

and they define a *projective change of affine connection** for an arbitrary vector φ_j.

Properties of the system of differential equations (2.1) which persist under the change of affine connection (3.7) will be called projective properties.

4. **The projective connection.** The equations (3.4) state that the quantities

$$(4.1)\qquad \Pi^i_{jk} = \Gamma^i_{jk} - \frac{\delta^i_j}{n+1}\Gamma^l_{lk} - \frac{\delta^i_k}{n+1}\Gamma^l_{lj}$$

are the components of a projective invariant, the *projective connection*.† These quantities also arise by the elimination of φ_j from equations (3.7).

By the use of equations (2.2) and (3.5) we find that the law of transformation of the Π's is

$$(4.2)\qquad \Pi^l_{jk}\frac{\partial \bar{x}^i}{\partial x^l} = \bar{\Pi}^i_{lp}\frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^p}{\partial x^k} + \frac{\partial^2 \bar{x}^i}{\partial x^j\,\partial x^k} + \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial \theta}{\partial x^k} + \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \theta}{\partial x^j},$$

* H. Weyl, *Zur Infinitesimalgeometrie*, Gött. Nachr., 1921, p. 99. Cf. also L. P. Eisenhart, *Spaces with corresponding paths*, Proc. Nat. Acad. Sci., vol. 8 (1922), p. 233, and O. Veblen, ibid., p. 347.

† T. Y. Thomas, *On the projective and equi-projective geometries of paths*, Proc. Nat. Acad. Sci., vol. 11 (1925), p. 199.

where

$$\theta = \frac{\log \Delta(x, \bar{x})}{n+1}.$$

The equations (4.2) will reduce to the law of transformation of a tensor if, and only if, the equations of transformation satisfy the differential equations

$$\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} = -\frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial \theta}{\partial x^k} - \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \theta}{\partial x^j}. \tag{4.3}$$

The integrability conditions

$$\frac{\partial^3 \bar{x}^i}{\partial x^j \partial x^k \partial x^l} = \frac{\partial^3 \bar{x}^i}{\partial x^j \partial x^l \partial x^k}$$

are reducible to

$$\frac{\partial^2 \theta}{\partial x^j \partial x^k} + \frac{\partial \theta}{\partial x^j}\frac{\partial \theta}{\partial x^k} = 0.$$

These last equations are equivalent to

$$\frac{\partial^2 \Delta^{\frac{1}{n+1}}}{\partial x^j \partial x^k} = 0,$$

of which the solution is

$$\Delta^{\frac{1}{n+1}} = k + h_i x^i,$$

k and h_i denoting constants of integration. We therefore have

$$\frac{\partial \theta}{\partial x^j} = \frac{h_j}{k + h_i x^i} = \frac{a_j}{1 + a_i x^i}.$$

Equations (4.3) are therefore to be replaced by

$$\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} = -\frac{\partial \bar{x}^i}{\partial x^j}\frac{a_k}{1 + a_l x^l} - \frac{\partial \bar{x}^i}{\partial x^k}\frac{a_j}{1 + a_l x^l}. \tag{4.4}$$

The integrability conditions of (4.4) are identically satisfied. The equations therefore possess a unique system of solutions satisfying the initial conditions $\bar{x}^i = c^i$, $\partial \bar{x}^i / \partial x^j = b^i_j$ for $x^i = 0$. But the functions

$$\bar{x}^i = \frac{b^i_j x^j}{1 + a_k x^k} + c^i$$

satisfy both the differential equations and the initial conditions. Hence we have

THEOREM 1. *The components of projective connection transform like the components of a tensor under the transformations of the projective group and under them alone.*

5. **Projective parameter.** When the substitution

$$\Gamma^i_{jk} = \Pi^i_{jk} + \frac{\delta^i_j}{n+1}\Gamma^l_{lk} + \frac{\delta^i_k}{n+1}\Gamma^l_{lj}$$

is made in them, equations (3.1) become

$$(5.1)\quad \left(\frac{d^2x^i}{dt^2} + \Pi^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right)\left(\frac{dt}{ds}\right)^2 + \frac{dx^i}{dt}\left(\frac{d^2t}{ds^2} + \frac{2}{n+1}\Gamma^l_{lj}\frac{dx^j}{ds}\frac{dt}{ds}\right) = 0.$$

If we define a particular parameter p by means of the equation

$$(5.2)\quad \frac{d^2p}{ds^2} + \frac{2}{n+1}\Gamma^l_{lj}\frac{dx^j}{ds}\frac{dp}{ds} = 0,$$

the equations (5.1) of the paths become

$$(5.3)\quad \frac{d^2x^i}{dp^2} + \Pi^i_{jk}\frac{dx^j}{dp}\frac{dx^k}{dp} = 0.$$

A parameter p satisfying (5.2) is called a *projective parameter** for the following reason. The equations corresponding to (2.1) for the affine connection Γ' are

$$\frac{d^2x^i}{dt^2} + \Gamma'^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

Inserting in these equations the values of Γ' from (3.7) and comparing the resulting equations with (3.1) we get

$$(5.4)\quad \frac{d^2t}{ds^2} = -2\varphi_i\frac{dx^i}{ds}\frac{dt}{ds}.$$

If we substitute from (3.6) and (5.4) in (5.2), we find that the original p also satisfies the differential equation,

$$\frac{d^2p}{dt^2} + \frac{2}{n+1}\Gamma'^l_{li}\frac{dx^i}{dt}\frac{dp}{dt} = 0,$$

that is, the equation satisfied by a parameter p is invariant under a projective change of affine connection.

* Cf. T. Y. Thomas, Proc. Nat. Acad. Sci., vol. 11 (1925), p. 200.

On the other hand, if we subject the coördinates to an arbitrary transformation, we readily see from (3.5) that equation (5.2) is not invariant. In order that the form of equations (5.3) be preserved, it is therefore necessary that such a transformation of coördinates be accompanied by a transformation of projective parameter, the formula for which we proceed to determine. The projective parameter $\bar{p}$ for a coördinate system $\bar{x}$ satisfies the equations

$$\frac{d^2\bar{p}}{ds^2} + \frac{2}{n+1}\,\bar{\Gamma}^l_{li}\,\frac{d\bar{x}^i}{ds}\,\frac{d\bar{p}}{ds} = 0,$$

or

$$d\left(\log\frac{d\bar{p}}{ds}\right) + \frac{2}{n+1}\,\bar{\Gamma}^l_{li}\,d\bar{x}^i = 0.$$

Subtracting these equations from the corresponding form of (5.2), we get

$$d\left[\log\left(\frac{dp}{d\bar{p}}\right) - \frac{2}{n+1}\log\Delta\right] = 0,$$

whence

$$\frac{dp}{d\bar{p}} = A\Delta^{\frac{2}{n+1}},$$

or

$$p = A\int\Delta^{\frac{2}{n+1}}\,d\bar{p} + B, \tag{5.5}$$

where A and B are constants and the integral is evaluated along the path in question. The formula (5.5) gives the transformation of projective parameter which accompanies a transformation of coördinates from x to $\bar{x}$.

In a similar fashion, we may write

$$p = A\int e^{-\frac{2}{n+1}\int\Gamma^i_{li}dx^i}\,ds + B, \tag{5.6}$$

a formula which gives a projective parameter in terms of the parameter s associated with an arbitrary affine connection Γ. We may render the projective parameter definite by assigning A and B the values 1 and 0, but defining any solution of (5.2) as a projective parameter is entirely satisfactory since equations (5.3) preserve their form under linear transformation of the parameter.

6. **The projective equivalence theorem.** It is now clear that the problem of the projective equivalence of two geometries of paths may be reduced to a study of the integrability conditions of the equations of transformation (4.2) of the projective connection.* For if a space of paths

* Cf. T. Y. Thomas, loc. cit. p. 201.

given by components of affine connection Γ in a coördinate system x is in a point-to-point and path-to-path correspondence with a space of paths given by $\overline{\Gamma}$ in a coördinate system $\overline{x}$, the correspondence determines equations of transformation $x^i = f^i(\overline{x})$ which transform the differential equations (2.1) of the first system of paths into equations of the form

$$\frac{d^2\overline{x}^i}{ds^2} + \Gamma'^i_{jk}\frac{d\overline{x}^j}{ds}\frac{d\overline{x}^k}{ds} = 0$$

which must be satisfied by the second system of paths. Since these equations represent the same paths as the corresponding equations in the functions $\overline{\Gamma}$ it follows by §§ 3 and 4 that the components of projective connection Π must be related to the components $\overline{\Pi}$ by (4.2). Conversely if a correspondence $x^i = f^i(\overline{x})$ exists in which Π and $\overline{\Pi}$ are related by (4.2) this correspondence together with the transformation of parameters (5.5) carries the differential equations (5.3) of the first system of paths into the corresponding differential equations in the $\overline{\Pi}$'s for the second system of paths. Hence the question of the path-to-path correspondence of two geometries of paths resolves itself into the following: when is it possible to find a transformation $x^i = f^i(x)$ of the points x of one manifold into the points $\overline{x}$ of the other which relates the projective connections of the two spaces by formulas (4.2)?

It is convenient to rewrite equations (4.2) in the form

$$\frac{\partial u^i_j}{\partial \overline{x}^k} = \overline{\Pi}^l_{jk}u^i_l - \Pi^i_{lp}u^l_j u^p_k + \theta_j u^i_k + \theta_k u^i_j, \tag{6.1}$$

where

$$\frac{\partial x^i}{\partial \overline{x}^j} = u^i_j, \tag{6.2}$$

and

$$\theta_i = \frac{1}{n+1}\frac{\partial \log \Delta}{\partial \overline{x}^i}. \tag{6.3}$$

If we differentiate equations (6.1) with respect to $\overline{x}^l$, eliminate the second derivatives of the x's which occur explicitly by means of (6.1), interchange k and l, and subtract, we find equations which are equivalent to

$$\begin{aligned}\overline{\mathfrak{B}}^i_{jkl} = \mathfrak{B}^p_{qst}v^i_p u^q_j u^s_k u^t_l + \delta^i_l\left(\frac{\partial\theta_j}{\partial\overline{x}^k} - \theta_p\overline{\Pi}^p_{jk} - \theta_j\theta_k\right)\\ - \delta^i_k\left(\frac{\partial\theta_j}{\partial\overline{x}^l} - \theta_p\overline{\Pi}^p_{jl} - \theta_j\theta_l\right),\end{aligned} \tag{6.4}$$

where

$$\mathfrak{B}^i_{jkl} = \frac{\partial \Pi^i_{jk}}{\partial x^l} - \frac{\partial \Pi^i_{jl}}{\partial x^k} + \Pi^p_{jk}\,\Pi^i_{pl} - \Pi^p_{jl}\,\Pi^i_{pk},$$

$$v^i_j = \frac{\partial \bar{x}^i}{\partial x^j}.$$

Contraction for i and l in (6.4) gives

$$\frac{\partial \theta_j}{\partial \bar{x}^k} = \bar{r}_{jk} - r_{pq}\,u^p_j\,u^q_k + \theta_p\,\bar{\Pi}^p_{jk} + \theta_j\,\theta_k, \tag{6.5}$$

where

$$r_{jk} = \frac{1}{n-1}\,\mathfrak{B}^i_{jki}.$$

If $\partial\theta_j/\partial\bar{x}^k$ is eliminated from (6.4) by means of (6.5), there results a system of equations equivalent to

$$\bar{W}^p_{jkl}\,u^i_p = W^i_{pqs}\,u^p_j\,u^q_k\,u^s_l, \tag{6.6}$$

where

$$W^i_{jkl} = \mathfrak{B}^i_{jkl} + \delta^i_k\,r_{jl} - \delta^i_l\,r_{jk}. \tag{6.7}$$

The invariant W is the projective curvature tensor discovered by Weyl.*

If we differentiate (6.5) and eliminate in a similar manner, we find

$$\bar{r}_{ijk} = r_{pqs}\,u^p_i\,u^q_j\,u^s_k - \theta_l\bar{W}^l_{ijk}, \tag{6.8}$$

where

$$r_{ijk} = \frac{\partial r_{ij}}{\partial x^k} - \frac{\partial r_{ik}}{\partial x^j} + r_{lk}\,\Pi^l_{ij} - r_{lj}\,\Pi^l_{ik}. \tag{6.9}$$

The invariant r_{ijk} will be called the *projective covariant.*

If we differentiate equations (6.6) and (6.8) successively with respect to $\bar{x}$, eliminating each time the second derivatives which occur explicitly by means of (6.1) and (6.5), we get a set of projective invariants together with their laws of transformation. The latter involve only the first derivatives u^i_j of the old variables with respect to the new and the particular combinations of second derivatives denoted by θ_i.

Let us denote the system of equations (6.6) and (6.8) together by $F_1 = 0$. Also let $F_2 = 0$ denote the equations obtained from $F_1 = 0$ by differentiating once and eliminating explicit second derivatives by means (6.1) and (6.5).

* H. Weyl, loc. cit., p. 101. For the process of elimination employed above to obtain it, cf. J. M. Thomas, *Note on the projective geometry of paths,* Proc. Nat. Acad. Sci., vol. 11 (1925), p. 207–209.

Let $F_3 = 0$ denote the set of equations obtained similarly from $F_2 = 0$, and so on. We thus find a sequence of sets of equations*

$$F_\lambda = 0, \qquad (\lambda = 1, 2, \cdots) \tag{6.10}$$

which may be regarded as equations for the determination of x^i, u^i_j, θ_j as functions of $\bar{x}$. By a general theorem on partial differential equations which we shall prove in the next section there now follows:

THEOREM 2. *A necessary and sufficient condition for the projective equivalence of two affine geometries of paths is the existence of a positive integer N such that $F_1 = 0$, $F_2 = 0$, $\cdots$, $F_N = 0$ are consistent equations for the determination of x^i, u^i_j, θ_i as functions of $\bar{x}$ and that every solution of them satisfies $F_{N+1} = 0$.*

7. A theorem on partial differential equations. Let us consider a set of $n \cdot R$ partial differential equations,

$$\frac{\partial Z^\varrho}{\partial \bar{x}^i} - \psi^\varrho_i(Z, \bar{x}) = 0. \tag{7.1}$$

The integrability conditions of these differential equations are obtained by eliminating the derivatives $\partial Z^\varrho / \partial \bar{x}^i$ from

$$\frac{\partial \psi^\varrho_i}{\partial \bar{x}^j} + \frac{\partial \psi^\varrho_i}{\partial Z^\tau} \frac{\partial Z^\tau}{\partial \bar{x}^j} = \frac{\partial \psi^\varrho_j}{\partial \bar{x}^i} + \frac{\partial \psi^\varrho_j}{\partial Z^\tau} \frac{\partial Z^\tau}{\partial \bar{x}^i} \tag{7.2}$$

by means of (7.1). If these relations are identically satisfied, (7.1) are completely integrable.† If not, let the relations obtained be denoted by $F_1(Z, \bar{x}) = 0$. Differentiating these equations and eliminating the derivatives $\partial Z^\varrho / \partial \bar{x}^i$ by means of (7.1), another set of relations, $F_2(Z, \bar{x}) = 0$ is obtained, and by continuing the process an infinite sequence of such sets of relations,

$$F_\lambda(Z, \bar{x}) = 0, \qquad (\lambda = 1, 2, \cdots). \tag{7.3}$$

We shall assume that the equations represented by (7.3) impose no relation on the variables $\bar{x}$, for if such a relation were imposed we could reduce the problem to one involving a smaller number of independent variables $\bar{x}$. The equations (7.3) therefore impose a number of relations on the functions Z, and if they are compatible there must exist certain

* Greek letters will be used from this point onward for indices whose range of values is not 1, 2, $\cdots$, n.

† M. Bouquet, Bull. Sci. math. et astron. (1), vol. 3 (1872), p. 265.

functions $Z^\varrho(\bar{x})$ which satisfy (7.3) so long as the point $\bar{x}$ remains in a sufficiently restricted n-cell. If we proceed through this sequence of sets of equations (7.3) and form from it a set of independent relations by adding to our set each new relation encountered which is independent of those already found, we must finally reach a system of equations, say

$$F_{N+1} = 0, \tag{7.4}$$

which are identically satisfied by any values of Z which satisfy the independent relations obtained from the preceding systems, namely,

$$F_1 = 0, \quad F_2 = 0, \quad \cdots, \quad F_N = 0, \tag{7.5}$$

that is, the system (7.4) contributes no new independent relation to our set. Hence if the differential equations (7.1) are to be compatible it is necessary that a number N shall exist such that the functional relations (7.5) shall be consistent and that all functions $Z(\bar{x})$ satisfying them shall also satisfy (7.4). This condition is also sufficient, as we now proceed to show.

Suppose that M is the number of the independent relations in (7.5) and let us denote them by

$$G_\mu(Z, \bar{x}) = 0, \qquad (\mu = 1, 2, \cdots, M \leqq R). \tag{7.6}$$

We may further suppose without loss of generality that the Z's are so numbered that the determinant

$$\left| \frac{\partial G_\mu}{\partial Z_\nu} \right| \neq 0, \qquad (\mu, \nu = 1, 2, \cdots, M), \tag{7.7}$$

that is, the first M Z's may be regarded as functions of the remaining Z's and the $\bar{x}$'s,

$$Z^\sigma = Z^\sigma(Z^{M+1}, \cdots, Z^R, \bar{x}), \quad (\sigma = 1, 2, \cdots, M). \tag{7.8}$$

The equations (7.6) imply

$$\frac{\partial G_\mu}{\partial Z^\sigma} \frac{\partial Z^\sigma}{\partial \bar{x}^i} + \frac{\partial G_\mu}{\partial \bar{x}^i} = 0.$$

Also the process of formation of the sequence (7.3) is such that the relations

$$\frac{\partial G_\mu}{\partial Z^\sigma} \psi_i^\sigma + \frac{\partial G_\mu}{\partial \bar{x}^i} = 0$$

are all included in the set of relations $F_1 = 0, \cdots, F_N = 0, F_{N+1} = 0$ whenever the relations $G_\mu = 0$ are included in (7.5). Hence on our hypotheses we have

$$\frac{\partial G_\mu}{\partial Z^\sigma}\left(\frac{\partial Z^\sigma}{\partial \bar{x}^i} - \psi_i^\sigma\right) = 0. \tag{7.9}$$

For an arbitrary but fixed value of i this yields a set of M linear homogeneous equations among the quantities $\partial Z^\sigma/\partial \bar{x}^i - \psi_i^\sigma$. In view of the non-vanishing of the determinant (7.7), the first M of these quantities are expressible linearly in terms of the remaining ones. Hence if the relations (7.8) are substituted in (7.1) all the resulting equations are consequences of the differential equations,

$$\frac{\partial Z^\beta}{\partial \bar{x}^i} = \bar{\psi}_i^\beta, \qquad (\beta = M+1, \cdots, R) \tag{7.10}$$

the right hand members of which are the functions of $\bar{x}$ and $Z^{M+1}, \cdots, Z^R$ obtained by substituting (7.8) in the corresponding equations in the set (7.1).

The integrability conditions of (7.10) are formed by eliminating the first derivatives $\partial Z^\rho/\partial x^i$ between (7.2), (7.9) and (7.10). But from the method of formation of the conditions $F_1 = 0$ it is seen that the conditions so obtained are satisfied identically because they are certain of the relations $F_1 = 0$ with (7.8) substituted in them.

Hence we have the theorem: *For the compatibility of the differential equations* (7.1) *it is necessary and sufficient that there exist a number* N *such that the functional relations* $F_1(Z, \bar{x}) = 0, \cdots, F_N(Z, \bar{x}) = 0$ *be compatible for all points* $\bar{x}$ *in a given region* (*n-cell*) *and that all functions* $Z(\bar{x})$ *which satisfy them also satisfy* $F_{N+1}(Z, \bar{x}) = 0$. The functions Z (unless $M = R$) fall into two classes, one class including $R - M$ functions obtained by solving a completely integrable system of differential equations of the type (7.10) and the other class including M functions found by substituting the other $R - M$ functions in the solutions of a set of implicit functional relations. In case $R = M$ all the Z's are found by solving functional relations.

To apply this theorem to the proof of Theorem 2 we let $Z^1, Z^2, \cdots, Z^R$ denote the $n^2 + 2n$ variables, x^i, u_j^i and θ_i and let (6.2), (6.1) and (6.5) be the equations (7.1). Since (6.3) is a consequence of (6.2) and (6.1) any solution of (7.1) will determine a transformation of Π into $\bar{\Pi}$. By reference to § 6 one sees that $F_1 = 0$ is the system of equations (6.6) and (6.8). Hence the general theorem reduces to Theorem 2.

The theorem of the present section may also be applied to the proof of the equivalence theorem for quadratic differential forms, for affine path geometries, for conformal geometries, and so on. The method used to obtain it resembles that outlined by Wright* to prove an equivalence theorem for quadratic forms which is somewhat more general than that of Christoffel.

8. **Invariant derivatives.** In order to describe the invariants which appear in the equations $F_\lambda = 0$ more explicitly we introduce the idea of the *invariant derivative* of an invariant. Such a derivative is the ordinary derivative evaluated in an invariant coördinate system which is determined at the point of the manifold under consideration as a simultaneous invariant of the coördinate system to which the manifold is referred and some fundamental invariant.

If the invariant coördinate system chosen is the system of geodesic coördinates and the above process is applied to a tensor, the invariant derivative is the covariant derivative with respect to the affine connection which, in this case, is the fundamental invariant.

We also define the *r-th invariant extension* of an invariant as its r-th derivative evaluated as above. The invariant derivative is, of course, the first invariant extension.

As different choices of a basic invariant coördinate system are possible, there will be different types of invariant extension. We shall, however, designate as *affine extension* the one obtained by choosing the affine normal coördinate system as the basis for differentiation. Likewise *projective extension* will be defined in terms of projective normal coördinates† and *conformal extension* in terms of conformal normal coördinates.‡

With the definitions adopted, the affine extensions of the affine connection are the affine normal tensors.§ Of the projective extensions of the projective connection, only the first is a tensor, and a similar remark applies to the conformal extensions of the conformal connection.

A projectively invariant coördinate system y associated with a given coördinate system x and a point x_0 is defined by

$$x^i = x_0^i + y^i - \frac{1}{2!}(\Pi^i_{jk})_0\, y^j y^k - \frac{1}{3!}(\Pi^i_{jkl})_0\, y^j y^k y^l, \tag{8.1}$$

* J. E. Wright, *Invariants of quadratic differential forms*, Cambridge, 1908, pp. 15–18.

† O. Veblen and J. M. Thomas, *Projective normal coördinates for the geometry of paths*, Proc. Nat. Acad. Sci., vol. 11 (1925), pp. 204–207.

‡ The definition of such coördinates is entirely analogous to that of projective normal coördinates.

§ Cf. O. Veblen, *Normal coördinates for the geometry of paths*, Proc. Nat. Acad. Sci., vol. 8 (1922), pp. 192–197.

where

$$\Pi^i_{jkl} = \frac{1}{3} P\left(\frac{\partial \Pi^i_{jk}}{\partial x^l} - 2\Pi^p_{jk}\Pi^i_{pl} + 2\delta^i_j r_{kl}\right),$$

and the symbol P denotes the sum of the terms obtained from those within the parentheses by cyclic permutation of the subscripts.* If we adopt the coördinate system (8.1) as our basis for invariant differentiation, *the invariants arising by repeated differentiation of* (6.6) *and* (6.8) *in the manner described in* § 6 *are the successive invariant derivatives of* W^i_{jkl} *and* r_{ijk}. To prove this for a given one of the invariants, we differentiate the law of transformation of the invariant immediately preceding it in the sequence, then we regard $\bar{x}$ as the coördinates y of (8.1) and evaluate at the origin where the following conditions obtain

$$(8.2)\quad \frac{\partial^2 x^i}{\partial y^j \partial y^k} = -\Pi^i_{jk}, \quad \bar{\Pi}^i_{jk} = 0, \quad \frac{\partial x^i}{\partial y^j} = \delta^i_j, \quad \theta_i = 0, \quad \bar{r}_{jk} = 0.$$

Had we applied the projective normal coördinates as our basis, we should have reached the same result. For the conditions (8.2) are also true at the origin of the projective normal coördinates and the invariants differentiated have laws of transformation involving no derivatives higher than the second, so that in differentiating and evaluating we should not have to go beyond third derivatives. Hence the invariant derivatives involved are actually the successive projective derivatives.

9. **The two fundamental projective invariants.** From the law of transformation (6.8) we readily infer

THEOREM 3. *The projective covariant is a tensor if, and only if, the projective curvature tensor vanishes.*

The first invariant arising from the differentiation of (6.6) is the projective derivative of W^i_{jkl} and can be formed from its (affine) covariant derivative by replacing each Γ by the corresponding Π. Let us denote it by $W^i_{jkl/p}$, and let us use a similar notation for the *formal expression* for the covariant derivative of $\mathfrak{B}^i_{jkl}$ and r_{jk} with respect to the basis Π^i_{jk}, e. g.,

$$r_{ij/k} = \frac{\partial r_{ij}}{\partial x^k} - r_{lj}\Pi^l_{ik} - r_{il}\Pi^l_{jk}.$$

From (6.7) we have

$$(9.1)\qquad W^i_{jkl/p} = \mathfrak{B}^i_{jkl/p} - \delta^i_l r_{jk/p} + \delta^i_k r_{jl/p}.$$

* The expressions (8.1) are those obtained by stopping at the terms of the third order in the expansions of x in terms of the projective normal coördinates previously given by us in the paper, *Projective normal coördinates for the geometry of paths*, Proc. Nat. Acad. Sci., vol. 11 (1925), pp. 204–207.

The so-called Bianchi identity holds for $\mathfrak{B}^i_{jkl/p}$ because this invariant can be formed from the covariant derivative of the ordinary curvature tensor by replacing each Γ by the corresponding H. Hence

$$\mathfrak{B}^i_{jkl/p} + \mathfrak{B}^i_{jlp/k} + \mathfrak{B}_{jpk/l} = 0. \tag{9.2}$$

Using (9.1), (9.2) and the relations

$$r_{jkl} = r_{jk/l} - r_{jl/k}$$

we find

$$W^i_{jkl/p} + W^i_{jlp/k} + W^i_{jpk/l} = \delta^i_k r_{jlp} + \delta^i_l r_{jpk} + \delta^i_p r_{jkl},$$

whence by contraction for i and p

$$W^p_{jkl/p} = (n-2) r_{jkl}. \tag{9.3}$$

Equations (9.3) show that for $n > 2$, equations (6.8) and those arising from them are not independent of those coming from (6.6) and therefore need not be considered in applying the equivalence theorem. If $n = 2$, the projective curvature tensor vanishes so that we need only consider the equations arising from (6.8).

10. **Completeness theorems.** We call the sets of invariants obtained in § 6 *complete* because an equivalence theorem can be stated in terms of them. From the results contained in § 8 and § 9 we can state the following theorems.

THEOREM 4. *A complete set of projective invariants for an affine geometry of paths consists of the projective curvature tensor and the projective covariant together with their successive projective derivatives.*

THEOREM 5. *A complete set of projective invariants for an affine geometry of paths of more than two dimensions consists of the projective curvature tensor and its successive projective derivatives.*

THEOREM 6. *A complete set of projective invariants for an affine geometry of paths of two dimensions consists of the projective covariant, which is in this case a tensor, and its successive projective derivatives.*

11. **Projectively flat spaces.** By a projectively flat space is meant one in which it is possible to choose a preferred coördinate system in terms of which the equations of all the paths are linear. From (5.3) it is seen that such preferred coördinate systems are characterized by the identical vanishing of the projective connection. The analytical work done in proving Theorem 1 shows that all such coördinate systems can be obtained from any one of them by linear fractional transformations.*

* Cf. H. Weyl, *Mathematische Analyse des Raumproblems*, Berlin, 1923, p. 73. Although Weyl does not employ the projective connection, he proves substantially the statement just made above.

If x is such a preferred coördinate system, it is obvious from the way in which the invariants W^i_{jkl} and r_{ijk} are formed from the projective connection that they must vanish for the coördinate system x. Equations (6.6) and (6.8) then show that these invariants vanish in all coördinate systems $\bar{x}$. Conversely, if $\bar{W}^i_{jkl}$ and $\bar{r}_{ijk}$ vanish in a particular coördinate system $\bar{x}$, the integrability conditions of the equations obtained by making $\Pi^i_{jk} = 0$ in (6.1) are identically satisfied. Those equations consequently define a system x such that $\Pi^i_{jk} = 0$.

We now state in terms of our invariants three theorems due to Weyl.*

THEOREM 7. *A necessary aud sufficient condition for a projectively flat space is the vanishing of the projective curvature tensor and the projective covariant.*

If $n > 2$, we see from (9.3) that the vanishing of W^i_{jkl} implies that of r_{ijk}.

THEOREM 8. *A necessary and sufficient condition that a space of more than two dimensions be projectively flat is the vanishing of the projective curvature tensor.*

If $n = 2$, the projective curvature tensor vanishes identically, equations (9.3) are identically satisfied, and r_{ijk} is a tensor by theorem 3.

THEOREM 9. *A necessary and sufficient condition that a surface be projectively flat is the vanishing of the projective covariant.*

12. **The significance of the contracted affine connection.** The projective connection is completely determined by the formulas (4.1) when the affine connection is given, but not conversely. The Γ's can be determined from (4.1) only when the values of Γ^l_{lk} have been assigned. We may choose arbitrary functions for these n sums, but their law of transformation is of course determined by that of the Γ's. It is given by (3.5).

With each point of the x manifold we can associate an ordinary projective space and can refer the paths to a system of projective normal coördinates such that the paths through the point are represented in the ordinary projective space by the straight lines through the origin. The projective normal coördinates are uniquely determined by the paths and are independent

* H. Weyl, *Zur Infinitesimalgeometrie*, Gött. Nach., 1921, pp. 102–105. His conditions are

(i) $$W^i_{jkl} = 0,$$

(ii) $$G_{i,jlk-klj} = 0,$$

where in general

$$G_{i,jlk-klj} = r_{ijk} + \frac{\Gamma^l_{ip}}{n+1} W^p_{jkl}.$$

His conditions are, therefore, equivalent to ours though his point of view and analysis are different. It is to be noted that the left hand members of (ii) are the components of a projective invariant only by virtue of (i) and that their projective character is not obvious from their form.

of the particular affine connection chosen in representing the paths by differential equations.

A general analytic transformation of the variables x is represented in the projective normal coördinates y by a linear fractional transformation.* If we introduce homogeneous variables by the relations $y^i = z^i/z^0$, the transformation of these new coördinates is linear and homogeneous,† $\bar{z}^\alpha = b^\alpha_\beta z^\beta$, the coefficients being

$$b^i_j = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_0, \quad b^0_i = \left(\frac{\partial \theta}{\partial x^i}\right)_0, \quad b^i_0 = 0, \quad b^0_0 = 1.$$

The symbol $(\ )_0$ denotes evaluation at the origin of projective normal coördinates.

If we put $(\Gamma^i_{ij})_0 = -(n+1)\,v_j/v_0$, equations (3.5) become $v_\alpha = \bar{v}_\beta\, b^\beta_\alpha$. Hence the $n+1$ quantities v_α are transformed contragrediently to the z's. The hyperplane $v_\alpha z^\alpha = 0$, therefore, has for its new equation $\bar{v}_\alpha \bar{z}^\alpha = 0$ when the x's undergo an arbitrary analytic transformation. The assignment of values to the contracted Γ's, therefore, converts the projective space associated with any point into a Euclidean space by singling out one of its hyperplanes.

If we choose the contracted Γ's equal to zero, we get an affine connection which is said to be normalized for the given coördinate system. The corresponding hyperplane is $z^0 = 0$, that is, the hyperplane at infinity.

13. **A non-affine geometry of paths.** From a broad point of view, the geometry of paths is the theory of a system of curves having the local property that two points are joined by one and only one curve of the system.‡ In general such a system of paths will not have an affine connection and therefore will not have the sort of a projective theory which is described in the pages above.

The case next in simplicity after the affine case appears to be the system of paths which satisfy a set of differential equations of the form

$$\begin{vmatrix} \dfrac{d^2x^i}{ds^2} & \dfrac{d^2x^j}{ds^2} \\ \dfrac{dx^i}{ds} & \dfrac{dx^j}{ds} \end{vmatrix} + \Pi^{ij}_{pqr} \frac{dx^p}{ds} \frac{dx^q}{ds} \frac{dx^r}{ds} = 0 \tag{13.1}$$

* Cf. O. Veblen and J. M. Thomas, loc. cit., p. 206.

† In the formulas of this section, Greek indices are to take the values $0, 1, \cdots, n$, and Roman indices the values $1, 2, \cdots n$.

‡ Cf. O. Veblen, *Remarks on the foundations of geometry*, Bull. Amer. Math. Soc, vol. 31 (1925), p. 127.

in which the Π's are functions of x which are symmetric in the indices p, q, r and alternating in the indices i and j.

These differential equations are direct generalizations of (3.2) and are themselves capable of further generalization of an interesting sort. They are invariant under arbitrary analytic transformations of the independent variable s and are invariant in form under analytic transformation of the dependent variables if the functions Π are transformed according to the equations

$$(13.2)\quad \bar{\Pi}^{ab}_{cdf}\, u^i_a\, u^j_b = \Pi^{ij}_{pqr}\, u^p_c\, u^q_d\, u^r_f + \frac{1}{3}\delta^{ij}_{pq}\,(u^p_{cd}\, u^q_f + u^p_{df}\, u^q_c + u^p_{fc}\, u^q_d),$$

$$u^i_{jk} = \frac{\partial^2 x^i}{\partial \bar{x}^j\, \partial \bar{x}^k}.$$

The functions Π^{ij}_{pqr} are unlike the functions Γ^i_{jk} which appear in the affine theory in that they are uniquely determined by the paths and a specified coördinate system. In this respect as well as in others they generalize the functions which appear in (3.3).

Let us now define a set of functions by the equations

$$\Pi^j_{qr} = \frac{-3}{n+1}\Pi^{ij}_{iqr}.$$

The law of transformation for these functions is the same as (4.2). It now follows by a short computation that the functions

$$\Pi^{ij}_{pqr} - \frac{1}{3}(\delta^{ij}_{kp}\,\Pi^k_{qr} + \delta^{ij}_{kq}\,\Pi^k_{rp} + \delta^{ij}_{kr}\,\Pi^k_{pq})$$

are the components of a tensor. Moreover the vanishing of this tensor is a necessary and sufficient condition that the system of paths defined by (13.1) have an affine connection. The theory of these differential equations would seem to be a fruitful new field for study.

PRINCETON UNIVERSITY.

ON CLASSES OF IDEALS IN A QUADRATIC FIELD.

By Howard H. Mitchell.

1. **Introduction.** It is the purpose of this paper to give an exposition of the determination of the number of classes of ideals in a quadratic number field, and of the class to which any particular ideal belongs, by methods that follow closely those used by Lagrange and Gauss in the case of quadratic forms. The results obtained seem simpler than those arrived at by the use of the Minkowski theorems on linear forms, which are ordinarily employed for this purpose. The methods used here moreover make it possible to bring out clearly the relationship between classes of ideals and classes of quadratic forms. The author has tried to present the material in such a form as to be intelligible to a reader with only a slight acquaintance with the subject.

It will be convenient, first of all, to recall some well known results concerning integers and ideals in a quadratic field. Any *integer* in the quadratic field determined by $\sqrt{m}$, where m is a positive or negative rational integer without a square factor, can be represented in the form $x+y\omega$, where $\omega = \sqrt{m}$ if $m \equiv 2, 3$, mod. 4, $\omega = (1+\sqrt{m})/2$ if $m \equiv 1$, mod. 4, and x, y are rational integers. Conversely, every such number is an integer in the field. By the *norm* of an integer is meant the product of the integer and its conjugate, i. e. the norm of $x+y\omega$ is $x^2 - my^2$ if $m \equiv 2, 3$, mod. 4, and $x^2+xy+(1-m)y^2/4$ if $m \equiv 1$, mod. 4.

An *ideal* in the field may be defined as a set of integers such that the sum of any two integers of the set is in the set and such that the product of any integer of the set by any integer in the field is in the set. It may be shown that any ideal in the field may be represented in the form $c[ax+(b+\omega)y]$, where a, b, c, denote particular rational integers such that the norm of $b+\omega$ is divisible by a, and x, y, assume all the rational integral values. Conversely, every such set of numbers is an ideal. The set of integers obtained by multiplying a particular integer by each of the integers of the field forms a particular kind of ideal called a *principal ideal.*

A pair of integers ξ, η in an ideal that have the property that every integer in the ideal may be represented in the form $\xi x+\eta y$, where x, y are rational integers, are said to form a *basis* of the ideal. Thus, in particular, every ideal possesses a basis (called a *canonical basis*) of the form ac, $(b+\omega)c$, where the norm of $b+\omega$ is divisible by a. If ξ, η

form a basis, a necessary and sufficient condition that $\xi^* = \delta\xi + \gamma\eta$, $\eta^* = \beta\xi + \alpha\eta$, where $\alpha, \beta, \gamma, \delta$ are rational integers, should also form a basis of the ideal is that $\alpha\delta - \beta\gamma = \pm 1$.

If we set $\theta = \eta/\xi$, it follows that a necessary and sufficient condition that θ^* should also be the quotient of two basis numbers of the ideal is that θ^* should be connected with θ by means of a relation of the type $\theta^* = (\alpha\theta + \beta)/(\gamma\theta + \delta)$, where $\alpha, \beta, \gamma, \delta$ are rational integers such that $\alpha\delta - \beta\gamma = \pm 1$. Associated with each ideal there are therefore an infinity of such numbers θ.

If two ideals are thus associated with the same set of numbers θ, we may suppose that their basis numbers $[\xi_1, \eta_1]$, $[\xi_2, \eta_2]$ have been so selected that $\eta_2/\xi_2 = \eta_1/\xi_1$. Hence by multiplying the integers $\xi_1 x + \eta_1 y$ of one ideal by $\xi_2/\xi_1 = \eta_2/\eta_1$ the integers of the other ideal are obtained. Conversely, if, by multiplying the integers of an ideal $\xi x + \eta y$ by a number $\varkappa$ of the field, the resulting system of numbers are found to be all integers, this system is then an ideal with the basis $[\xi\varkappa, \eta\varkappa]$ and is thus associated with the same set of numbers θ as the given ideal. Two ideals that are related in this way are said to be *equivalent.*

For example, if we multiply the integers of the ideal $[2, 1+\sqrt{-5}]$ by $(1-\sqrt{-5})/2$ we obtain the equivalent ideal $[3, 1-\sqrt{-5}]$.

From the definition of equivalence, it follows that two ideals that are equivalent to the same ideal are equivalent to each other. All ideals that are equivalent each to each are said to form an *ideal class.* In particular, all the principal ideals in a field form a class called the *principal class.*

Any number $\theta = \eta/\xi$ that is the quotient of two basis numbers of an ideal $[\xi, \eta]$ may be reduced to the form $(b + c\omega)/a$, where a, b, c, are rational integers having no common divisor. Hence by multiplying $[\xi, \eta]$ by a/ξ we obtain the equivalent ideal $[a, b + c\omega]$. It follows then that a and b must each be divisible by c, whence $c = \pm 1$, and that the norm of $b + c\omega$ must be divisible by a. Conversely, any number $(b \pm \omega)/a$, such that the norm of $b \pm \omega$ is divisible by a, is the quotient of two basis numbers of an ideal.

A slightly different representation of the integers of a field will be found more convenient in the discussions that follow. The *discriminant* D of a field defined by $\sqrt{m}$, is equal to $4m$ if $m \equiv 2, 3$, mod. 4, and to m, if $m \equiv 1$, mod 4. Hence any integer in the field having the discriminant D may be expressed in the form $(x + y\sqrt{D})/2$, where x is even if $D \equiv 8, 12$, mod. 16, and x, y are either both even or both odd if $D \equiv 1$, mod. 4. Any number that is the quotient of the two basis numbers of an ideal may be represented in the form $(b \pm \sqrt{D})/2a$, where b is even or odd according as D is even or odd, and where $b^2 - D$ is divisible by $4a$.

2. The imaginary quadratic field. In discussing the separation of ideals into classes, we shall distinguish between the cases where the field is real or imaginary, and shall consider the latter case first. Since the discriminant in this case is negative, we shall find it convenient to represent it by $-\Delta$.

Those integers in an ideal $[\xi, \eta]$ that have the minimum norm all have the form $\delta\xi+\gamma\eta$, where γ, δ are rational integers having no common divisor. Any such integer may be chosen as one of the two basis numbers of the ideal, since the other could then be selected as $\beta\xi+\alpha\eta$, where α, β are so chosen that $\alpha\delta-\beta\gamma = \pm 1$. We shall suppose that ξ has the minimum norm.

Since the ratio η/ξ is equal to a number of the form $\pm(b+\sqrt{-\Delta})/2a$, where $b^2 \equiv -\Delta$, mod. $4a$, we see that the ideal $[\xi, \eta]$ is equivalent to $[a, (b+\sqrt{-\Delta})/2]$, in which a has the minimum norm. *An ideal (in an imaginary field) that contains no rational integral factor, and that contains a rational integer with minimum norm will be referred to as a reduced ideal.*

By a proper modification of the second basis number we may choose b in the reduced ideal $[a, (b+\sqrt{-\Delta})/2]$ so that $-a<b\leqq a$. Since a has the minimum norm, $b^2+\Delta \geqq 4a^2$, whence $a \leqq \sqrt{\Delta/3}$.

Conversely, if $-a<b\leqq a$, $b^2+\Delta \geqq 4a^2$, the ideal is reduced. For the norm of any integer $ax+(b+\sqrt{-\Delta})\,y/2$ will exceed $\Delta/3$ if $|y|>1$, and hence no irrational integer in the ideal can have a norm less than that of $(b+\sqrt{-\Delta})/2$. Hence a has the minimum norm. We have therefore the

THEOREM. *In order that an ideal be reduced, it is necessary and sufficient that it possess a canonical basis* $[a, (b+\sqrt{-\Delta})/2]$ *such that* $b^2+\Delta \equiv 0$, *mod.* $4a$, $-a<b\leqq a$; $b^2+\Delta \geqq 4a^2$.

For a given Δ it is evident that there are only a finite number of pairs of integers a, b, satisfying these conditions, and hence only a finite number of reduced ideals. Since, as was shown above, every ideal is equivalent to at least one reduced ideal, the number of ideal classes is less than or equal to the number of reduced ideals.

In order to determine the number of classes, we must investigate the conditions under which two reduced ideals are equivalent. If, by multiplying the reduced ideal $[a,(b+\sqrt{-\Delta})/2]$ by an irrational number, another reduced ideal is obtained, there must be in the first an irrational integer whose norm is equal to the norm of a. Since, as we have seen above, no irrational integer in a reduced ideal can have a norm less than that of $(b+\sqrt{-\Delta})/2$, it follows that $b^2+\Delta = 4a^2$. In this case by multiplying the given reduced ideal by $(b-\sqrt{-\Delta})/2$ we obtain the equivalent ideal $[a, (-b+\sqrt{-\Delta})]/2$, which is reduced and is distinct from the first except when $b=0$, a.

If $b = a$, $b^2+\Delta = 4a^2$, the norm of $(-b+\sqrt{-\Delta})/2$, which is then in the ideal, is also equal to the norm of a, but in this case $\Delta = 3$, $a = 1$, and there is but one reduced ideal.

We may summarize the results obtained in the

THEOREM. *In an imaginary quadratic field with discriminant $-\Delta$ every ideal is equivalent to one and only one of the ideals $[a, (b+\sqrt{-\Delta})/2]$ such that $b^2+\Delta$ is divisible by $4a$, and either $-a<b\leqq a$, $b^2+\Delta>4a^2$, or $0\leqq b\leqq a$, $b^2+\Delta = 4a^2$.*

In writing down these representative ideals for a particular Δ it is probably most convenient to assign to b the odd or even values numerically less than $\sqrt{\Delta/3}$, and then for each value of b to assign to a the integral values that are divisors of $(b^2+\Delta)/4$ and such that $|b|\leqq a\leqq(b^2+\Delta)/4a$. Finally, any pairs of values for which $b = -a$ or for which b is negative and $b^2+\Delta = 4a^2$ are to be excluded.

Thus, if we indicate the ideal $[a, (b+\sqrt{-\Delta})/2]$ by the symbol (a, b), we find that, for $\Delta = 143$, every ideal is equivalent to one and only one of the following ten: $(1,1)$ $(2,\pm1)$ $(3,\pm1)$ $(4,\pm1)$ $(6,1)$ $(6,\pm5)$.

The most convenient process for finding a reduced ideal equivalent to any given ideal is probably as follows. We may first express it in terms of a canonical basis, and discard any rational factor, if such be present. In this ideal $[a, (b+\sqrt{-\Delta})/2]$ we may then by a suitable modification of the second basis number choose b so that $-a<b\leqq a$.

If then $b^2+\Delta\geqq 4a^2$, the ideal is reduced. On the other hand, if $b^2+\Delta<4a^2$, say $b^2+\Delta = 4ac$, where $c<a$, the ideal is equivalent to

$$\frac{b-\sqrt{-\Delta}}{2a}[a, (b+\sqrt{-\Delta})/2] = [c, (-b+\sqrt{-\Delta})/2],$$

which contains a rational integer less than a. If then, by modifying the second basis number as described above, we find that c has not the minimum norm, we may repeat the process and thus obtain an equivalent ideal containing a still smaller rational integer. Since this process cannot be continued indefinitely, we must finally obtain a reduced ideal.

Taking, as before, $\Delta = 143$, and using the symbol $\sim$ to denote equivalence, we find, for example, that

$$[53, (49+\sqrt{-143})/2] \sim \frac{49-\sqrt{-143}}{106}[53, (49+\sqrt{-143})/2]$$

$$= [12, (-49+\sqrt{-143})/2] = [12, (-1+\sqrt{-143})/2]$$

$$\sim \frac{1+\sqrt{-143}}{24}[12, (-1+\sqrt{-143})/2] = [3, (1+\sqrt{-143})/2].$$

This process is convenient in constructing the abelian group of the products of classes. Thus, if, as above, we indicate the ideal $[a, (b+\sqrt{-143})/2]$ by the symbol (a, b) and set $\mathfrak{j} = (2, 1)$, we find $\mathfrak{j}^2 = (4, 1)$, $\mathfrak{j}^3 = (8, -7) \sim (6, -5)$, $\mathfrak{j}^4 \sim (3, 1)$, $\mathfrak{j}^5 \sim (6, 1)$, $\mathfrak{j}^6 \sim (3, -1)$, $\mathfrak{j}^7 \sim (6, 5)$, $\mathfrak{j}^8 \sim (4, -1)$, $\mathfrak{j}^9 \sim (2, -1)$, $\mathfrak{j}^{10} \sim (1, 1)$.

3. Conditions for a single class of ideals in an imaginary field. It is of interest to inquire under what conditions there is but one class of ideals in an imaginary field. This can be true only when $[1, \omega]$ is the only reduced ideal. If $-\Delta \equiv 8$, mod. 16, the ideal $[2, \sqrt{-\Delta}/2]$ is reduced except for $\Delta = 8$, and if $-\Delta \equiv 12$, mod. 16, the ideal $[2, (2+\sqrt{-\Delta})/2]$ is reduced except for $\Delta = 4$. In each of these two fields there is but one class.

If $-\Delta \equiv 1$, mod. 4, the attempt to determine the fields where there is but one class leads to the consideration of certain sequences of prime numbers. Let a represent the smallest prime factor contained in any of the numbers $(x^2+\Delta)/4$ for any odd integral value of x, so that, if $\Delta \neq 3$, $a \leqq (\Delta+1)/4$. Let b represent the smallest positive value of x such that $x^2+\Delta \equiv 0$, mod. $4a$. Since $b-2a$, $2a-b$ also satisfy this congruence, it follows that $b \leqq a$.

If $a<(\Delta+1)/4$, then $(b^2+\Delta)/4 \geqq a^2$, since otherwise $(b^2+\Delta)/4$ would contain a smaller factor than a. Since $b \leqq a$, it follows that $a \leqq \sqrt{\Delta/3}$, $b \leqq \sqrt{\Delta/3}$. In this case the ideal $[a, (b+\sqrt{-\Delta})]/2$ is reduced, and there exists more than one class.

On the other hand, if $a = (\Delta+1)/4$, the numbers $(x^2+\Delta)/4$ must be prime for all odd values of x such that $(x^2+\Delta)/4<(\Delta+1)^2/16$, i. e. for $|x|<(\Delta-1)/2$. In this case there is but one class, since, except for $\Delta = 3$, the value of a in a reduced ideal $[a, (b+\sqrt{-\Delta})/2]$ cannot be as large as $(\Delta+1)/4$.

We see, therefore, that if there is but one class, then $a = (\Delta+1)/4$, and the numbers $(x^2+\Delta)/4$ are prime for all odd values of x less than $(\Delta-1)/2$. On the other hand, if these numbers are prime for all odd values of x less than $\sqrt{\Delta/3}$, then a has the same value, and there is but one class. We have therefore the

THEOREM. *A necessary and sufficient condition that there should be but one class of ideals in the imaginary quadratic field with discriminant* $-\Delta$, *where* $-\Delta \equiv 1$, *mod.* 4, *is that the numbers* $(x^2+\Delta)/4$ *be prime for all odd values of* x *less than* $\sqrt{\Delta/3}$. *If this is the case, they are also prime for all odd values of* x *less than* $(\Delta-1)/2$.*

* A number of writers have called attention to the relationship between these series of prime numbers and discriminants for which there is but one class of positive quadratic forms. This is essentially the same problem. Cf. § 6.

Thus, since the numbers $(x^2+67)/4$ are prime for $x = 1, 3$, they are also prime for all odd values of x less than 33, and there is but one class of ideals in the field with discriminant -67.

It is easy to show that, when this case occurs, Δ must itself be prime. For, if we set $x = \Delta$, then $(x^2+\Delta)/4 = \Delta(\Delta+1)/4$, whereas Δ cannot be expressed as a product of primes, none of which is less than $(\Delta+1)/4$. There are only seven known values of Δ of this sort, i. e. $\Delta = 3, 7, 11, 19, 43, 67, 163$. While it has been believed for a long time that there are no others, it does not seem to have been proved.†

4. The real quadratic field. We consider now the real quadratic field with discriminant D. Any ideal in this field that does not possess a rational integral factor may be expressed in terms of a canonical basis $[a, (b+\sqrt{D})/2]$, where b is even if $D \equiv 8,12$, mod. 16, and odd if $D \equiv 1$, mod. 4, and where $b^2 - D \equiv 0$, mod. $4a$.

We shall define a reduced ideal as one that possesses a canonical basis $[a, (b+\sqrt{D})/2]$, *where* $b^2 - D \equiv 0$, *mod.* $4a$, *and* $0 < \sqrt{D} - b < 2a < \sqrt{D} + b$.

That there are only a finite number of reduced ideals in a field is easily seen. For, from these relations, it follows that b must be positive and less than $\sqrt{D}$, and, for each such value of b, that a must be a divisor of $(D-b^2)/4$ that is included between the limits $(\sqrt{D}-b)/2$, $(\sqrt{D}+b)/2$.

If an ideal be indicated merely by the symbol (a, b), we find, for example, that, for $D = 124$, the reduced ideals are (1,10), (2,10), (3,10), (6,10), (3,8), (5,8), (5,2), (6,2).

We may show that any ideal is equivalent to one or more reduced ideals. To do this we first suppose that in the canonical basis $[a, (b+\sqrt{D})]/2$ of the given ideal (or one equivalent to it obtained by discarding a rational integral factor) b is so chosen that $0 < \sqrt{D} - b < 2a$. If then $\sqrt{D} + b > 2a$, the ideal is reduced.

On the other hand, if $\sqrt{D} + b < 2a$, it follows from the relation $b + \sqrt{D} > b - \sqrt{D} > -2a$ that $|D - b^2| = |\sqrt{D} - b| \cdot |\sqrt{D} + b| < 4a^2$. We have thus the following equivalence

$$[a, (b+\sqrt{D})/2] \sim \frac{\sqrt{D}-b}{2a}[a, (b+\sqrt{D})/2] = [a^*, (-b+\sqrt{D})/2],$$

where a^* is defined by the equation $|D - b^2| = 4aa^*$, and is therefore less than a.

We then express the new ideal in terms of the basis $[a^*, (b^*+\sqrt{D})/2]$, where b^* is determined by the conditions: $b^* \equiv -b$, mod. $2a^*$;

† There are certainly no others less than 1,500,000. Cf. L. E. Dickson, Bull. Am. Math. Soc., vol. 17 (1910–11), p. 534.

$0<\sqrt{D}-b^*<2a^*$. If then we should find that it is not reduced, i. e. if $\sqrt{D}+b^*<2a^*$, we may proceed as above to find an equivalent ideal containing a smaller rational integer. Since this process cannot be continued indefinitely, we must ultimately obtain a reduced ideal equivalent to the given ideal.

To illustrate this process in a particular case, we have, for example, in the field in which $D=124$,

$$[54, -25+\sqrt{31}] \sim \frac{25+\sqrt{31}}{54}[54, -25+\sqrt{31}]$$

$$= [11, 25+\sqrt{31}] = [11, 3+\sqrt{31}] \sim \frac{\sqrt{31}-3}{11}[11, 3+\sqrt{31}]$$

$$= [2, -3+\sqrt{31}] = [2, 5+\sqrt{31}],$$

which is included in the set of reduced ideals for this field given above.

Certain equivalences exist between the reduced ideals in a given field, which we proceed to establish. Starting with any reduced ideal $[a, (b+\sqrt{D})/2]$, where $0<\sqrt{D}-b<2a<\sqrt{D}+b$, we now express it in the form $[a, (-b^*+\sqrt{D})/2]$, where b^* is determined by the conditions, $b^*+b\equiv 0$, mod. $2a$; $0<-b^*+\sqrt{D}<2a$. We notice that $b^*+\sqrt{D}>2a$, since $\sqrt{D}-b>0$, and $b^*=-b+2ka$, where k is a positive integer. Hence, if we set $(\sqrt{D}-b^*)(\sqrt{D}+b^*)=4aa^*$, we see that $\sqrt{D}+b^*>2a^*$, $\sqrt{D}-b^*<2a^*$. The equivalent ideal

$$\frac{b^*+\sqrt{D}}{2a}[a, (-b^*+\sqrt{D})/2] = [a^*, (b^*+\sqrt{D})/2]$$

is therefore reduced.

Following this procedure, we obtain in general a series of reduced ideals equivalent to a given one. Since there are only a finite number of reduced ideals, we must finally obtain one of them a second time, whence the same cycle will be repeated indefinitely.

We must moreover obtain the reduced ideal we started with. For, if not, the ideal that first appeared a second time would be obtained by such a step as that described above from two different reduced ideals. But this is impossible, for a reduced ideal $[a^*, (b^*+\sqrt{D})/2]$ could be obtained only from the reduced ideal $[a, (b+\sqrt{D})/2]$, where a, b satisfy the conditions, $(\sqrt{D}-b^*)(\sqrt{D}+b^*)=4aa^*$; $b+b^*\equiv 0$, mod. $2a$.

To illustrate this grouping of reduced ideals in cycles, we have, for example,

$$[1, 5+\sqrt{31}] = [1, -5+\sqrt{31}] \sim \frac{\sqrt{31}+5}{1}[1, -5+\sqrt{31}]$$
$$= [6, 5+\sqrt{31}] = [6, -1+\sqrt{31}] \sim \frac{\sqrt{31}+1}{6}[6, -1+\sqrt{31}]$$
$$= [5, 1+\sqrt{31}] \sim [3, 4+\sqrt{31}] \sim [2, 5+\sqrt{31}] \sim [3, 5+\sqrt{31}]$$
$$\sim [5, 4+\sqrt{31}] \sim [6, 1+\sqrt{31}] \sim [1, 5+\sqrt{31}], \text{ etc.}$$

Since this cycle contains all the reduced ideals in this field, the latter therefore contains but one class of ideals.†

We shall now show that, if two reduced ideals are equivalent, they belong to the same cycle. We observe first that the ratios, $\theta = (b+\sqrt{D})/2a$, $\theta^* = (b^*+\sqrt{D})/2a^*$, of the basis numbers of two "adjacent" reduced ideals satisfy the conditions, $\theta > 1$, $\theta^* > 1$, $\theta = k + 1/\theta^*$, where k is a positive integer. Hence, if we expand θ in a continued fraction, choosing the partial quotients in the usual way, the ratios of the basis numbers of the other reduced ideals in the same cycle will appear in it and the expansion will be periodic. We shall show that, if two reduced ideals are equivalent, the expansions in continued fractions of the ratios of their basis numbers contain the same periods of partial quotients. Each of these ratios must therefore appear in the expansion of the other, and since each ratio corresponds to but one reduced ideal, the two corresponding ideals must belong to the same cycle.

To prove this, we observe that, if two reduced ideals are equivalent, the ratios θ, θ^*, of their canonical basis numbers must satisfy the conditions

$$(1) \qquad \theta = \frac{\alpha\theta^*+\beta}{\gamma\theta^*+\delta}, \qquad \theta > 1, \qquad \theta^* > 1;$$

$$(2) \qquad \bar{\theta} = \frac{\alpha\bar{\theta}^*+\beta}{\gamma\bar{\theta}^*+\delta}, \qquad -1 < \bar{\theta} < 0, \qquad -1 < \bar{\theta}^* < 0;$$

$$(3) \qquad \alpha\delta - \beta\gamma = \pm 1;$$

where $\bar{\theta}$, $\bar{\theta}^*$ denote the conjugates of θ, θ^* respectively.

We suppose first that none of the four coefficients α, β, γ, δ is 0. We may suppose α positive, since, if it were negative, the signs of all four coefficients could be changed without affecting the relations (1), (2), (3). From (3) we see that the products $\alpha\delta$, $\beta\gamma$ must have the same sign. Of

† J. Sommer gives the number of classes in this field as three (Vorlesungen über Zahlentheorie, Leipzig and Berlin 1907, pp. 75–77, 356). The error remains uncorrected in the French translation (Introduction à la Théorie des Nombres Algébriques, Paris, 1911).

the four combinations of signs that satisfy these conditions $(+\,+\,-\,-)$ is impossible in view of (1) and $(+\,-\,+\,-)$ in view of (2). If the signs were $(+\,-\,-\,+)$, the inverse of the transformation would have the signs $(+\,+\,+\,+)$. Hence without loss of generality we may assume that, if none is 0, all four coefficients are positive.

We see moreover that $\alpha \geqq \gamma$, for, if $\alpha < \gamma$, then, by (3), $\beta \leqq \delta$, which is impossible by (1). Similarly we find that $\gamma \geqq \delta$, $\alpha \geqq \beta$, $\beta \geqq \delta$.

If $\gamma > 1$, we expand α/γ in a continued fraction,

$$\frac{\alpha}{\gamma} = (k_0, k_1, \cdots, k_{n-1}),$$

in which the partial quotients k_i are all positive, and denote the next to the last convergent by β'/δ'. We may choose n odd or even at pleasure since k_{n-1} may be replaced by $(k_{n-1}-1, 1)$. We may thus suppose that

$$\alpha\delta' - \gamma\beta' = (-1)^n = \alpha\delta - \beta\gamma,$$

whence

$$\alpha(\delta' - \delta) = \gamma(\beta' - \beta).$$

Since, by (3), α, γ can have no common factor, and since δ, δ' are each positive and $\leqq \gamma$, it follows that $\delta' = \delta$, $\beta' = \beta$. Hence, if none of the coefficients is 0, and, if $\gamma > 1$,

$$(k_0, k_1, \cdots, k_{n-1}, \theta^*) = \frac{\alpha\theta^* + \beta}{\gamma\theta^* + \delta} = \theta.$$

If $\gamma = 1$, then $\delta = 1$, $\beta = \alpha - 1$, $\alpha > 1$, and

$$(\alpha - 1, 1, \theta^*) = \frac{\alpha\theta^* + \alpha - 1}{\theta^* + 1} = \theta.$$

The inverse of a transformation with $\alpha = 0$ has $\delta = 0$. If $\delta = 0$, α may be taken positive, and, by (1), (2), (3), $\beta = +1$, $\gamma = +1$.

Hence

$$(\alpha, \theta^*) = \frac{\alpha\theta^* + 1}{\theta^*} = \theta.$$

The cases $\beta = 0$ or $\gamma = 0$ are found to be impossible by (1), (2), (3).

Hence in every case the expansions in continued fractions of θ, θ^* have the same periods of partial quotients and thus the corresponding ideals belong to the same cycle.

We have thus proved the

THEOREM. *Every ideal in the real quadratic field with discriminant D is equivalent to one or more (reduced) ideals in the set defined by $[a, (b+\sqrt{D})/2]$, where*

$$b^2 \equiv D, \mod 4a; \qquad 0<\sqrt{D}-b<2a<\sqrt{D}+b.$$

When these ideals are arranged in cycles as described above, a necessary and sufficient condition that any two of them be equivalent is that they should belong to the same cycle.

5. Units in a real quadratic field. The theory as developed above enables us in particular to determine whether or not any given ideal is a principal ideal, and, if so, to find an integer of the field in terms of which it may be expressed. For, by multiplying the given ideal by the properly chosen numbers, we may find a reduced ideal equivalent to it. If the given ideal is principal, this reduced ideal will belong to the cycle containing the unit ideal, and we may obtain the latter by multiplying the reduced ideal we have found by properly chosen numbers.

Thus, from the two examples given above, we observe that by multiplying the ideal $[54, -25+\sqrt{31}]$ by the product

$$\left(\frac{25+\sqrt{31}}{54}\right)\left(\frac{-3+\sqrt{31}}{11}\right)$$

the reduced ideal $[2, 5+\sqrt{31}]$ is obtained, and that the product of the latter by

$$\left(\frac{5+\sqrt{31}}{2}\right)\left(\frac{4+\sqrt{31}}{3}\right)\left(\frac{1+\sqrt{31}}{5}\right)\left(\frac{5+\sqrt{31}}{6}\right)$$

is the unit ideal $[1, 5+\sqrt{31}]$. Hence by calculating the product of the reciprocals of these numbers (preferably in the reverse order) the desired integer is obtained. We find that

$$[54, -25+\sqrt{31}] = (-139+25\sqrt{31}).$$

We notice also that by completing a cycle of reduced ideals any reduced ideal can be expressed as a number of the field times itself, e. g.

$$[1, 5+\sqrt{31}] = (1520+273\sqrt{31})[1, 5+\sqrt{31}].$$

It is obvious that, if we are dealing with the cycle containing the unit ideal, such a number must be an integer and its reciprocal must be an integer, i. e. it is a *unit*. While it follows from the theory of factori-

zation of ideals that the same is true for any cycle, we may prove it independently as follows:

If the integers of an ideal, $ax+(b+\sqrt{D})y/2$, are permuted among themselves when multiplied by a number $(t+u\sqrt{D})/2$, where we assume t, u rational, but not necessarily integral, then

$$a(t+u\sqrt{D})/2, \qquad (b+\sqrt{D})(t+u\sqrt{D})/4$$

must form a basis of the same ideal, and hence there must exist four rational integers α, β, γ, δ such that

$$\begin{aligned} a(t+u\sqrt{D})/2 &= a\delta+(b+\sqrt{D})\gamma/2, \\ (b+\sqrt{D})(t+u\sqrt{D})/4 &= a\beta+(b+\sqrt{D})\alpha/2, \\ \alpha\delta-\beta\gamma &= \pm 1. \end{aligned} \tag{4}$$

By equating rational and irrational parts we obtain from the first two equations

$$\begin{aligned} \alpha &= (t+bu)/2, \quad \beta = (D-b^2)u/4a, \\ \gamma &= au, \qquad\quad \delta = (t-bu)/2, \end{aligned} \tag{5}$$

whence by substitution in the third

$$t^2-Du^2 = \pm 4. \tag{6}$$

We notice that $t=\alpha+\delta$ is an integer, whence Du^2 is an integer. Since D cannot contain any squared factor except 4, u is either an integer or an odd integer divided by 2. In the latter case we must have $D\equiv 8, 12$, mod. 16, $Du^2\equiv 2, 3$, mod. 4, and hence $t^2\equiv 2, 3$, mod. 4, which is impossible. Hence u is an integer.

We see also that, if $D\equiv 8, 12$, mod. 16, then t is even, whereas, if $D\equiv 1$, mod. 4, then t, u must be both even or both odd. Hence in every case $(t+u\sqrt{D})/2$ is an integer with norm ± 1 and is therefore a unit.

We have thus proved the

THEOREM. *If the integers of an ideal are permuted among themselves when multiplied by a number of the field, this number is a unit.*

Conversely, if $(t+u\sqrt{D})/2$ is a unit, then equations (4)-(6) hold for properly chosen values of α, β, γ, δ and hence also

$$\theta = \frac{\alpha\theta+\beta}{\gamma\theta+\delta}, \quad \text{where} \quad \theta = \frac{b+\sqrt{D}}{2a}, \tag{7}$$

i. e., θ has an automorph of determinant $+1$ or -1 according as the norm of the unit is $+1$ or -1.

On the other hand, if θ has an automorph given by (7), then there must exist a number of the field $(t+u\sqrt{D})/2$, where t, u are rational, such that equations (4) hold, whence by the proof of the preceeding theorem this number is a unit. Moreover, if S is an automorph of θ of determinant -1, and T is any transformation under which θ goes into θ^*, then $T^{-1}ST$ is an automorph of θ^* of determinant -1. We have thus the

THEOREM. *A necessary and sufficient condition that the ratio of two basis numbers of an ideal should have an automorph of determinant* -1 *is that there should exist a unit of norm* -1.

The products of the numbers obtained in forming a cycle of reduced ideals may be expressed in terms of the convergents in the expansion of the ratio of the basis numbers of the initial ideal in a continued fraction. To show this, we observe first that the number by which the initial ideal is multiplied in order to obtain the one adjacent to it may be expressed in the form $k_0-\bar{\theta}$, where k_0 is the first partial quotient in the expansion of $\theta=(b+\sqrt{D})/2a$ in a continued fraction and $\bar{\theta}=(b-\sqrt{D})/2a$. This multiplier may also be expressed as $p_1-q_1\bar{\theta}$, where p_1/q_1 denotes the first convergent.

We proceed by induction and assume that the product of the first n numbers obtained in forming the cycle is $p_n-q_n\bar{\theta}$, where p_n/q_n denotes the n th convergent. The next multiplier is $k_n-\bar{\theta}_n$, where k_n is the $(n+1)$ th partial quotient in the expansion of θ, and $\bar{\theta}_n$ is the conjugate of the ratio of the basis numbers of the n th ideal from the initial one. Hence the product of the first $n+1$ multipliers is $(p_n-q_n\bar{\theta})(k_n-\bar{\theta}_n)$. By use of the relation, $\bar{\theta}=(p_n\bar{\theta}_n+p_{n-1})/(q_n\bar{\theta}_n+q_{n-1})$, this is readily found to reduce to

$$p_nk_n+p_{n-1}-(q_nk_n+q_{n-1})\bar{\theta}=p_{n+1}-q_{n+1}\bar{\theta},$$

thus completing the induction.

In particular, if n is a multiple of the number of ideals in the cycle, $p_n-q_n\bar{\theta}$ is a unit.

We have thus the

THEOREM. *If* $[a, (b+\sqrt{D})/2]$ *is a reduced ideal in a real quadratic field with discriminant D, the product of the first n multipliers arising in forming the cycle to which this ideal belongs may be expressed in the form,* $p_n+q_n(\sqrt{D}-b)/2a$, *where* p_n/q_n *is the nth convergent in the expansion of* $(b+\sqrt{D})/2a$ *in a continued fraction. If n is a multiple of the number of ideals in the cycle, this product is a unit.*

For example, if we expand the ratio of the two basis numbers of the reduced ideal $[6, 5+\sqrt{31}]$, we obtain

$$\frac{5+\sqrt{31}}{6} = \left(1, 1, 3, 5, 3, 1, 1, 10, \frac{5+\sqrt{31}}{6}\right),$$

whence we find $p_8/q_8 = 2885/1638$, and hence the corresponding unit is $2885+1638\ (\sqrt{31}-5)/6 = 1520+273\sqrt{31}$, which agrees with the result already arrived at by forming the product of the eight multipliers in the ordinary way.

If ε denotes any irrational unit in a real quadratic field, then one of the four units, $\varepsilon, -\varepsilon, -1/\varepsilon, 1/\varepsilon$ is positive and greater than one. We shall now show that any unit that is positive and greater than one may be obtained as the product of the multipliers arising in forming any given cycle of reduced ideals a sufficient number of times, and is therefore a power of the unit obtained in forming the cycle once. The latter is called the *fundamental unit.*

We suppose that $(t+u\sqrt{D})/2$ is any positive unit greater than one (whence t, u are each positive) and that $[a, (b+\sqrt{D})/2]$ is any reduced ideal. As above, we conclude that there exist four rational integers α, β, γ, δ such that equations (4)–(7) are true.

From (5) we see that α, β, γ are all positive and hence from the last of (4) that $\delta \geqq 0$. From the proof in the latter part of § 4 it therefore follows that, if α/γ be expanded in a continued fraction,

$$\frac{\alpha}{\gamma} = (k_0, k_1, \cdots, k_{n-1}),$$

with positive partial quotients, and n be so chosen that $\alpha\delta - \beta\gamma = (-1)^n$, then

$$\theta = (k_0, k_1, \cdots, k_{n-1}, \theta)$$

i. e. α/γ is the nth convergent p_n/q_n in the expansion of θ, where now n is a multiple of the period. Hence, from (5),

$$(t+u\sqrt{D})/2 = \alpha+\gamma(\sqrt{D}-b)/2a = p_n - q_n\bar{\theta}.$$

Since the norm of each of the numbers by which a reduced ideal is multiplied in order to obtain the one adjacent to it is negative, the norm of this unit is positive or negative according as n is even or odd.

We have thus proved the

THEOREM. *Every unit in a real quadratic field that is positive and greater than one is expressible as the product of the numbers by which the reduced ideals of a cycle are multiplied in order to obtain those adjacent to them, and is a power of the (fundamental) unit obtained in forming*

this cycle once. The number of ideals in each cycle is even or odd according as the norm of the fundamental unit is $+1$ *or* -1.

6. Relations between classes of ideals and classes of quadratic forms. We consider here quadratic forms, $ax^2+bxy+cy^2$, with rational integral coefficients, that have for their *discriminant** $b^2-4ac = D$, where D is the discriminant of a quadratic field. We observe that a, b, c can have no common factor. For, if they had a common odd factor d, then D would be divisible by d^2, whereas, if they were each even, then $D \equiv 0, 4$, mod. 16, conditions which are impossible for the discriminant of a quadratic field.

By the *principal root* of such a form we shall mean the number $(-b+\sqrt{D})/2a$, this being one of the values of the ratio x/y for which the form vanishes. Here $\sqrt{D}$ denotes the positive square root of D if D is positive, and the square root with amplitude $\pi/2$ if D is negative. Conversely, any number of the form $(-b+\sqrt{D})/2a$, where b^2-D is divisible by $4a$, is the principal root of a unique form with discriminant D.

Two quadratic forms $ax^2+bxy+cy^2$, $a'x'^2+b'x'y'+c'y'^2$, are said to be *equivalent* provided there exists a substitution of the type, $x = \alpha x'+\beta y'$, $y = \gamma x'+\delta y'$, where α, β, γ, δ are rational integers such that $\alpha\delta-\beta\gamma = 1$, that transforms one into the other. The coefficients of the second form are therefore

$$a' = a\alpha^2+b\alpha\gamma+c\gamma^2, \quad b' = 2a\alpha\beta+b\alpha\delta+b\beta\gamma+2c\gamma\delta,$$
$$c' = a\beta^2+b\beta\delta+c\delta^2.$$

By writing this substitution in the form, $x' = \delta x-\beta y$, $y' = -\gamma x+\alpha y$, we see that the value of x'/y' that corresponds to $x/y = (-b+\sqrt{D})/2a$ is

$$\frac{\delta(-b+\sqrt{D})-2\beta a}{-\gamma(-b+\sqrt{D})+2\alpha a},$$

and by rationalizing the denominator this is found to reduce to $(-b'+\sqrt{D})/2a'$, i. e. the principal root of one form is transformed by the substitution into the principal root of the other. Conversely, if two principal roots can be transformed one into the other by means of a substitution with determinant $+1$, then the corresponding forms are equivalent.

If two forms are each equivalent to a third form, they are equivalent to each other. All the forms that are equivalent each to each are said

* This definition of the discriminant of a quadratic form is somewhat different from that used by Lagrange and Gauss. It agrees, however, with that employed by some of the later writers, e. g. Kronecker and Weber.

to form a *class* of forms. Associated with each class of forms there are thus an infinity of numbers θ of the type $(-b+\sqrt{D})/2a$, any two of which, θ, θ', are connected by a relation of the type, $\theta' = (\alpha\theta+\beta)/(\gamma\theta+\delta)$, where α, β, γ, δ are rational integers such that $\alpha\delta-\beta\gamma=+1$.

We have seen that associated with each class of ideals there are an infinity of numbers θ of the same type, any two of which are connected by a relation of the same sort, except that $\alpha\delta-\beta\gamma=\pm 1$. A set of numbers θ of this latter sort will constitute either one or two sets when the determinant of the substitution is restricted to $+1$. For, if θ', θ'' are any two numbers connected with θ by substitutions of determinant -1, say

$$\theta' = (\alpha\theta+\beta)/(\gamma\theta+\delta), \qquad \theta'' = (\alpha'\theta+\beta')/(\gamma'\theta+\delta'),$$

where $\alpha\delta-\beta\gamma=\alpha'\delta'-\beta'\gamma'=-1$, we find by eliminating θ from these two equations that θ' is connected with θ'' by means of a substitution of determinant $+1$.

If a set of numbers θ connected by substitutions of determinant ± 1 forms a single set under substitutions of determinant $+1$, any number in the set will have an "automorph" of determinant -1. For, if $\theta' = (\alpha\theta+\beta)/(\gamma\theta+\delta)$, $\theta = (\alpha'\theta'+\beta')/(\gamma'\theta'+\delta')$, where $\alpha\delta-\beta\gamma=-1$, $\alpha'\delta'-\beta'\gamma'=+1$, we find by forming the "product" that θ is connected with itself by a substitution of determinant -1.

Conversely, if θ has an automorph of determinant -1, and if θ' is connected with θ by a substitution of determinant -1, then θ' will also be connected with θ by a substitution of determinant $+1$, and hence a set of numbers θ connected by substitutions of determinant ± 1 will form a single set under substitutions of determinant $+1$.

If $D=-\Delta$ is negative, and hence θ is imaginary, there can be no automorph of determinant -1, since, if $\theta = (\alpha\theta+\beta)/(\gamma\theta+\delta)$, where $\alpha\delta-\beta\gamma=-1$, the resulting quadratic equation, $\gamma\theta^2+(\delta-\alpha)\theta-\beta=0$, has the positive discriminant $(\alpha+\delta)^2+4$. Hence there are two classes of forms corresponding to each class of ideals.

Since θ and $-\theta$ are obviously equivalent under a substitution of determinant -1, they are not equivalent under a substitution of determinant $+1$, and hence the forms, $ax^2+bxy+cy^2$, $-ax^2+bxy-cy^2$, of which $\theta = (-b+\sqrt{-\Delta})/2a$, $-\theta$ are principle roots, belong to different classes. These two classes correspond to the single class of ideals containing $[a, (-b+\sqrt{-\Delta})/2]$. If however we consider only *positive* forms, i. e. those that are positive for real values of x, y, there is then a one to one correspondence between classes of forms and classes of ideals.

We have thus the

THEOREM. *The number of classes of ideals in an imaginary quadratic field with discriminant $-\Delta$ is equal to the number of classes of positive quadratic forms, $ax^2+bxy+cy^2$, with discriminant, $b^2-4ac=-\Delta$.*

Turning our attention now to the case of a real quadratic field, we recall that the ratio of the two basis numbers of an ideal has an automorph of determinant -1 if and only if the norm of the fundamental unit is -1. We conclude, therefore, that each set of numbers θ that are connected by substitutions of determinant ± 1 forms a single set under substitutions of determinant $+1$ or divides into two sets according as the norm of the fundamental unit is -1 or $+1$. In the former case the class of ideals containing $[a, (b+\sqrt{D})]/2$ corresponds to the single class of forms containing $ax^2-bxy+cy^2$, where $b^2-4ac=D$; in the latter case to the two classes of forms containing $ax^2-bxy+cy^2$, $-ax^2-bxy-cy^2$.

We have thus the

THEOREM. *The number of classes of ideals in the real quadratic field with discriminant D is equal to the number of classes of forms, $ax^2+bxy+cy^2$, with discriminant $D=b^2-4ac$, or to one-half of the latter according as the norm of the fundamental unit is -1 or $+1$.*

Lagrange and Gauss considered only quadratic forms in which the middle coefficient is even, and defined the discriminant of such a form $ax^2+bxy+cy^2$ as $(b/2)^2-ac$. Hence, if $D \equiv 8, 12$, mod. 16, the discriminant of the forms is, under this definition, one-quarter of the discriminant of the corresponding quadratic field. On the other hand, if $D \equiv 1$, mod. 4, the form $2ax^2+2bxy+2cy^2$, where $b^2-4ac=D$, is, in their terminology, "improperly primitive" with discriminant D. The two theorems stated above would then read as follows:

The number of classes of ideals in an imaginary quadratic field with discriminant $-\Delta$ is equal to the number of classes of positive quadratic forms with discriminant $-\Delta/4$ if $-\Delta \equiv 8, 12$, mod. 16, and to the number of classes of improperly primitive positive forms with discriminant $-\Delta$ if $-\Delta \equiv 1$, mod. 4.

The number of classes of ideals in the real quadratic field with discriminant $D \equiv 8, 12$, mod. 16, is equal to the number of classes of quadratic forms with discriminant $D/4$ or to one-half of the latter according as the norm of the fundamental unit is -1 or $+1$; if $D \equiv 1$, mod. 4, the number of ideal classes is equal to the number of classes of improperly primitive forms with discriminant D or to one-half of the latter according as the norm of the fundamental unit is -1 or $+1$.

It might seem at first sight as if a closer relation between classes of ideals and classes of forms could be established if a class of forms were defined as a set of all forms connected by substitutions with determinant ± 1. The

relation between them in this case, however, turns out to be less simple. Any substitution of determinant -1 can be compounded of a properly chosen one of determinant $+1$ and the particular substitution $x' = x, y' = -y$. The latter transforms the principal root of any form, $ax^2+bxy+cy^2$, into the nonprincipal root of the form, $ax'^2-bx'y'+cy'^2$, and these two forms may or may not be connected by substitutions of determinant $+1$.

For example, as we have noted above, there are in the field with discriminant -143 ten classes of ideals, and hence, under the original definition, ten classes of positive forms, $ax^2+bxy+cy^2$, with discriminant $b^2-4ac = -143$. Under the substitution $x' = x, y' = -y$, the two classes that contain the forms, $x^2-xy+36y^2$, $6x^2-xy+6y^2$, each remain invariant, while the other eight are interchanged in pairs, giving therefore six classes of positive forms under substitutions of determinant ± 1.

On the other hand, if we retain the original definition of a class of forms, but modify somewhat the definiton of a class of ideals, the relationship between the two kinds of classes may be made still simpler. We suppose now that two ideals belong to the same class if and only if the integers belonging to one of them can be obtained by multiplying the integers belonging to the other by a number of *positive norm.*

This modification of the definition does not affect classes in an imaginary quadratic field, since there all numbers have a positive norm. It also does not affect the classes in a real quadratic field provided there is a unit of norm -1, for, if a multiplier with negative norm transforms one ideal into another, the product of this multiplier by a unit of norm -1 would be a number with positive norm having the same property.

If, however, there is no unit with norm -1 in a real quadratic field, each class of ideals under the former definition divides into two classes under the new definition. For, if two ideals are connected by a multiplier with negative norm, they cannot be connected by a multiplier with positive norm, since, if so, the ratio of the two multipliers would be a multiplier leaving each ideal invariant and hence a unit with norm -1 (Cf. the first theorem in § 5). Consequently under the new definition the number of ideal classes in a real quadratic field is equal to the number of corresponding classes of forms whether or not there is a unit with norm -1.

7. **Literature.** The literature in this field is too extensive to be cited here in detail. A reader is advised to consult the *Report of the Committee on Algebraic Numbers,* Bulletin of the National Research Council, vol. 5, part 3, Feb., 1923, Washington, D. C.

An expository paper dealing with different aspects of algebraic numbers by L. E. Dickson, *Fermat's Last Theorem and the Origin and Nature of*

Algebraic Numbers, these Annals, vol. 18, 1917–18, pp. 161–187, may be read to advantage.

Texts on algebraic numbers of a fairly elementary character are:

L. W. Reid, *The Elements of the Theory of Algebraic Numbers,* New York, 1910;

J. Sommer, *Vorlesungen über Zahlentheorie,* Leipzig and Berlin, 1907; *Introduction à la Théorie des Nombres Algebriques,* Paris, 1911 (French translation).

The discussions in this paper have run closely parallel to the corresponding portions of the theory of quadratic forms. Expositions of this subject are given by

Dirichlet-Dedekind, *Zahlentheorie,* 4th edition, Braunschweig, 1894, pp. 128–212;

G. B. Mathews, *Theory of Numbers,* Cambridge, 1892, chapters III, IV.

The literature on quadratic forms is cited by

L. E. Dickson, *History of the Theory of Numbers,* Washington, D. C., 1922, vol. 3, chapters 1–8.

The corresponding theory of quadratic irrational numbers is treated by

H. Weber, *Algebra,* vol. 1, 129–136, 2nd. edition, Braunschweig, 1898,

and in three recent papers in these Annals by

A. Arwin, *Periodically closed chains of reduced fractions,* vol. 24, 1922–23, pp. 39–68; *A contribution to the theory of closed chains,* vol. 25, 1923–24, pp. 91–117; *On continued fractions in the theory of binary forms,* vol. 26, 1924–25, pp. 247–272. These papers deal to some extent also with the associated ideals and quadratic forms.

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DEVELOPMENTS IN LEGENDRE POLYNOMIALS.*

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Expansions in terms of Legendre polynomials have been discussed by many authors†; yet, so far as the writer is informed, there exists no treatment which can fairly be described as "elementary". The papers to which reference has been made are designed to be more or less exhaustive, and, in some instances, are based upon complicated theories. For one thoroughly familiar with the type of analysis employed in these researches, the problem of representing an "arbitrary" function by a series of Legendre polynomials has been fully handled, and, in a certain sense, solved.‡ For those who have not specialised to this extent there is no available material on the convergence of Legendre series. Because of the importance of these developments in mathematical physics, a discussion of certain interesting cases with a minimum of analytical detail seems highly desirable. The present paper is intended as a step in this direction.

I. Definition and properties of the Legendre polynomials.

The function

$$H(x, r) \equiv (1-2xr+r^2)^{-\frac{1}{2}}$$

is analytic in the variable r throughout the finite r-plane save at the points $r = e^{\pm i \arccos x}$, the two roots of the equation $1-xr+r^2 = 0$. Thus we find§

$$H(x, r) = \sum_{n=0}^{n=\infty} P_n(x) r^n, \qquad -1 \leqq x \leqq +1, \qquad |r|<1$$

where the coefficient $P_n(x)$ is a polynomial of degree n in x. These polynomials are called the Legendre polynomials. In particular, $P_0 = 1$, $P_1 = x$. If we develop the identity

$$(1-2xr+r^2)\frac{\partial H}{\partial r} \equiv (x-r)H$$

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* Presented to the American Mathematical Society, January 2, 1926.

† Hobson, Proc. Lond. Math. Soc. (2), vol. 6 (1908), pp. 349–395; (2), vol. 7 (1909), pp. 24–39. Jackson, Trans. Amer. Math. Soc., vol. 13 (1912), pp. 305–318. Burkhardt, Münch. Sitz., 1909, 10. Abh. Haar, Math. Ann., vol. 78 (1917–1918), pp. 121–136. W. H. Young, Proc. Lond. Math. Soc. (2), vol. 18 (1920), pp. 141–162.

‡ The fullest results are those of Young's paper.

§ For this and the succeeding developments of the present paragraph, reference may be made to Whittaker and Watson, *Modern Analysis*, 3d ed., chap. 15.

in power series about $r = 0$ and equate the coefficients of like powers of r, we obtain

$$(1) \quad (n+2)P_{n+2} - (2n+3)xP_{n+1} + (n+1)P_n = 0, \quad n = 0, 1, 2, \cdots.$$

From this recurrence formula we can construct all the Legendre polynomials, starting with $P_0(x)$ and $P_1(x)$. In similar fashion the identities

$$r\frac{\partial}{\partial r}(rH) \equiv (1-rx)\frac{\partial H}{\partial x},$$

$$r\frac{\partial H}{\partial r} \equiv (x-r)\frac{\partial H}{\partial x}$$

yield

$$(2) \quad (n+1)P_n = P'_{n+1} - xP'_n, \quad n = 0, 1, 2, \cdots,$$

$$(3) \quad nP_n = xP'_n - P'_{n-1}, \quad n = 1, 2, 3, \cdots.$$

Adding (2) and (3) we find

$$(4) \quad (2n+1)P_n = P'_{n+1} - P'_{n-1}, \quad n = 1, 2, 3, \cdots.$$

If in (2) we replace n by $n-1$ and eliminate P_{n-1} between the resulting equation and (3), we see that

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] + n(n+1)P_n = 0, \quad n = 0, 1, 2, \cdots.$$

We may likewise, by comparison with the familiar series for $H(+1, r)$ and $H(-1, r)$, show that

$$(5) \quad P_n(+1) = +1, \qquad P_n(-1) = (-1)^n.$$

Finally we deduce from the identity $H(-x, r) \equiv H(x, -r)$ the relation

$$(6) \quad P_n(-x) = (-1)^n P_n(x).$$

We shall now prove*

THEOREM I. *The Legendre polynomials $P_n(x)$ satisfy the equations*

$$\int_{-1}^{+1} P_n P_m \, dx = 0, \qquad m \neq n,$$

$$\int_{-1}^{+1} P_n^2 \, dx = \frac{2}{2n+1}$$

for $m, n = 0, 1, \cdots$.

* Cf. Goursat, *Cours d'analyse*, 1re éd., vol. 1, § 88.

Proof. In the integral $\int_{-1}^{+1} P_n P_m \, dx$, $m \neq n$, we may suppose that $n > m$. The first equation of the theorem can thus be made to depend upon the truth of the equalities

$$\int_{-1}^{+1} x^k P_n \, dx = 0, \qquad k = 0, 1, 2, \cdots, n-2, n-1.$$

For $n = 1$ this set of equations is valid; by induction it can be proved for general n. For if

$$\int_{-1}^{+1} x^k P_{n-1} \, dx = 0, \qquad k = 0, 1, 2, \cdots, n-2,$$

we find from (3) and (5)

$$\begin{aligned} n\int_{-1}^{+1} x^k P_n \, dx &= \int_{-1}^{+1} (x^{k+1} P_n' - x^k P_{n-1}') \, dx \\ &= (x^{k+1} P_n - x^k P_{n-1})_{x=-1}^{x=+1} - (k+1) \int_{-1}^{+1} x^k P_n \, dx \\ &\qquad + k \int_{-1}^{+1} x^{k-1} P_{n-1} \, dx, \end{aligned}$$

so that

$$(n+k+1) \int_{-1}^{+1} x^k P_n \, dx = 0, \quad k = 0, 1, \cdots, n-1.$$

Thus we see that

$$\int_{-1}^{+1} P_n P_m \, dx = 0, \qquad m \neq n.$$

To evaluate the integral $C_n = \int_{-1}^{+1} P_n^2 \, dx$, we write $P_n = A_n x^n + \cdots$, so that

$$C_n = A_n \int_{-1}^{+1} x^n P_n \, dx.$$

If we replace n by $n-1$ in (1), we see that

$$x^n P_n = \frac{(n+1) x^{n-1} P_{n+1} + n x^{n-1} P_{n-1}}{2n+1},$$

$$C_n = \frac{n A_n}{2n+1} \int_{-1}^{+1} x^{n-1} P_{n-1} \, dx = \frac{n A_n}{A_{n-1}} \frac{C_{n-1}}{2n+1}.$$

To determine nA_n/A_{n-1} we write $n-1$ for n in (2) and equate the coefficients of the highest power of x in the resulting identity:

$$\begin{aligned} n P_{n-1} &= P_n' - x P_{n-1}', \\ n A_{n-1} &= n A_n - (n-1) A_{n-1}. \end{aligned}$$

Thus $nA_n/A_{n-1} = 2n-1$; and

$$(2n+1)C_n = (2n-1)C_{n-1} = \cdots = C_0 = 2,$$
$$C_n = \frac{2}{2n+1}.$$

When we come to the investigation of expansions in terms of the polynomials $P_n(x)$ on the interval $-1 \leqq x \leqq +1$, we shall require certain inequalities for these polynomials. In order to establish them, we need two preparatory lemmas.

LEMMA I. *The coefficients a_k in the development*

$$(1-\varrho)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} a_k \varrho^k, \qquad |\varrho| < 1,$$

satisfy the relations

$$0 < a_{k+1} < a_k < \cdots < a_0 = 1,$$

and

$$a_k < \frac{K}{\sqrt{k}},$$

where K is independent of k.

Proof. We observe immediately that

$$a_0 = 1, \qquad a_k = \frac{1}{k!}\frac{d^k}{d\varrho^k}(1-\varrho)^{-\frac{1}{2}}\Big|_{\varrho=0} = \frac{1\cdot 3\cdots(2k-1)}{2^k \cdot k!},$$

whence

$$0 < a_{k+1} = \frac{2k+1}{2k+2} a_k < a_k < \cdots < a_0 = 1.$$

The second inequality is more difficult to prove. It is well known that the integral $I_k = \int_0^{\pi/2} \cos^{2k}\theta\, d\theta$ is a solution of the difference equation $I_{k+1} = \frac{2k+1}{2k+2} I_k$, satisfied by a_k. We may prove this relation from the identity

$$\int_0^{\pi/2} \cos^{2k+2}\theta\, d\theta = \int_0^{\pi/2} \cos^{2k}\theta\, d\theta - \int_0^{\pi/2} \sin^2\theta \cos^{2k}\theta\, d\theta$$

by an integration by parts:

$$\begin{aligned} I_{k+1} &= I_k - \int_0^{\pi/2} \sin^2\theta \cos^{2k}\theta\, d\theta \\ &= I_k + \left(\frac{\sin\theta\cos^{2k+1}\theta}{2k+1}\right)_{\theta=0}^{\theta=\frac{\pi}{2}} - \int_0^{\pi/2} \frac{\cos^{2k+2}\theta}{2k+1}\, d\theta \\ &= I_k - \frac{I_{k+1}}{2k+1}, \\ I_{k+1} &= \frac{2k+1}{2k+2} I_k. \end{aligned}$$

Since a_k also satisfies this equation we find

$$\frac{a_k}{I_k} = \frac{a_0}{I_0} = \frac{2}{\pi}, \qquad a_k = \frac{2I_k}{\pi}.$$

We next consider a possible interpretation of the integral

$$I_k = \int_0^{\pi/2} \cos^{2k}\theta \, d\theta.$$

If we introduce polar coördinates (r, θ), $\frac{1}{2} I_k$ represents the area bounded by the curve $r = \cos^k\theta$ and the two radii vectores $\theta = 0$, $\theta = \frac{\pi}{2}$. This area is less than that of a properly determined rectangle with sides of length 1 and Y_k along the radii $\theta = 0$, $\theta = \frac{\pi}{2}$ respectively. We choose Y_k as the maximum of $y = r \sin\theta = \cos^k\theta \sin\theta$ for $0 \leqq \theta \leqq \frac{\pi}{2}$. To determine the magnitude of Y_k we locate the value of θ at which the maximum occurs as that root of the equation

$$\frac{d}{d\theta} \cos^k\theta \sin\theta = \cos^{k-1}\theta \, (\cos^2\theta - k \sin^2\theta) = 0$$

which is interior to the interval $\left(0, \frac{\pi}{2}\right)$. Clearly this value θ_k satisfies the equation

$$\tan \theta_k = \frac{1}{\sqrt{k}}.$$

Thus

$$0 < Y_k = \cos^k\theta_k \sin \theta_k < \sin\theta_k < \tan \theta_k = \frac{1}{\sqrt{k}}.$$

We conclude that

$$a_k = \frac{2I_k}{\pi} < \frac{4Y_k}{\pi} < \frac{K}{\sqrt{k}} \qquad k = 1, 2, 3, \cdots$$

as we desired to show.

LEMMA II. *If* $\alpha_0 > \alpha_1 > \cdots > \alpha_N > 0$, $0 < \beta_0 < \beta_1 < \cdots < \beta_N$, *and if the sums* $s_{ik} = \sum_{l=k}^{l=i} u_l$, $i \geqq k$, $i, k = 0, \cdots, N$, *are such that* $|s_{ik}| < s$, *then* $\sigma = \sum_{l=0}^{l=N} \alpha_l \beta_l u_l$ *satisfies the inequality*

$$|\sigma| \leqq \alpha_0 \beta_N s.$$

Proof. We have on writing $\sigma_i = \sum_{k=0}^{k=i} \beta_k u_k$

$$\sigma = \alpha_0 \sigma_0 + \sum_{i=1}^{i=N} \alpha_i(\sigma_i - \sigma_{i-1}) = \sum_{i=0}^{i=N-1} (\alpha_i - \alpha_{i+1})\sigma_i + \alpha_N \sigma_N,$$

$$|\sigma| \leqq \sum_{i=0}^{i=N-1} (\alpha_i - \alpha_{i+1})|\sigma_i| + \alpha_N|\sigma_N| \leqq \alpha_0 \operatorname*{Max}_{i=0,\cdots,N} |\sigma_i|.$$

Now

$$\sigma_i = \sum_{k=0}^{k=i} \beta_k u_k = \sum_{k=0}^{k=i-1} \beta_k(s_{ik} - s_{ik+1}) + \beta_i s_{ii} = \beta_0 s_{i0} + \sum_{k=1}^{k=i} (\beta_k - \beta_{k-1}) s_{ik},$$

$$|\sigma_i| \leqq \sum_{k=i}^{k=1} (\beta_k - \beta_{k-1})|s_{ik}| + \beta_0|s_{i0}| \leqq \beta_i s \leqq \beta_N s, \qquad i = 0, \cdots, N.$$

In consequence $|\sigma| \leqq \alpha_0 \beta_N s$, which was to be proved.

With the aid of these two lemmas we proceed to prove the following theorem.

THEOREM II. *The Legendre polynomial $P_n(x)$ is such that*

$$|P_n(\cos\theta)| \leqq 1 \qquad 0 \leqq \theta \leqq \pi$$

and

$$|P_n(\cos\theta)| < \frac{M}{\sqrt{n}\sin\theta} \qquad 0 < \theta < \pi,$$

where M is a constant independent of n and θ.

Proof. Since $P_n(\cos\theta)$ is the coefficient of r^n in the power series for

$$H(\cos\theta, r) = (1 - re^{i\theta})^{-\frac{1}{2}}(1 - re^{-i\theta})^{-\frac{1}{2}},$$

we find, by developing the functions on the right in power series and multiplying out,

$$P_n(\cos\theta) = \sum_{k=0}^{k=n} a_k a_{n-k} e^{(2k-n)i\theta}.$$

Thus

$$|P_n(\cos\theta)| \leqq \sum_{k=0}^{k=n} a_k a_{n-k} = P_n(+1) = +1.$$

To prove the second part of the theorem we combine terms in the expression for $P_n(\cos\theta)$, obtaining

$$P_{2m}(\cos\theta) = a_m^2 + 2\sum_{k=0}^{k=m-1} a_k a_{2m-k} \cos 2(m-k)\theta,$$

$$P_{2m+1}(\cos\theta) = 2\sum_{k=0}^{k=m} a_k a_{2m+1-k} \cos(2m - 2k + 1)\theta.$$

For the case $n = 2m$ we apply Lemma II, taking

$$N = m-1, \quad \alpha_l = a_l, \quad \beta_l = a_{2m-l}, \quad u_l = \cos 2(m-l)\theta.$$

The sums s_{ik} are therefore given by

$$s_{ik} = \sum_{l=k}^{l=i} \cos 2(m-l)\theta = \sum_{l=m-k}^{l=m-i} \cos 2l\theta = \sum_{l=m-k}^{l=m-i} \frac{\sin(2l+1)\theta - \sin(2l-1)\theta}{2\sin\theta}$$
$$= \frac{\sin(2m-2i+1)\theta - \sin(2m-2k+1)\theta}{2\sin\theta}$$

so that $s = 1/\sin\theta$, $0 < \theta < \pi$. We have $P_{2m}(\cos\theta) = a_m^2 + 2\sigma$. According to Lemma I the conditions of Lemma II are fulfilled so that we find

$$|\sigma| \leqq \alpha_0 \beta_N s = \frac{a_{m+1}}{\sin\theta},$$

$$|P_{2m}(\cos\theta)| \leqq a_m^2 + 2|\sigma| \leqq a_m^2 + \frac{2a_{m+1}}{\sin\theta} \leqq \frac{M}{\sqrt{2m}\sin\theta}$$

for $0 < \theta < \pi$. Again, for the case $n = 2m+1$, we set

$$N = m, \quad \alpha_l = a_l, \quad \beta_l = a_{2m+1-l}, \quad u_l = \cos(2m-2l+1)\theta.$$

We find

$$s_{ik} = \sum_{l=k}^{l=i} \cos(2m-2l+1)\theta = \sum_{l=m-k}^{l=m-i} \cos(2l+1)\theta = \sum_{l=m-k}^{l=m-i} \frac{\sin 2(l+1)\theta - \sin 2l\theta}{2\sin\theta}$$
$$= \frac{\sin 2(m-i+1)\theta - \sin 2(m-k)\theta}{2\sin\theta},$$

$$s = \frac{1}{\sin\theta}, \qquad 0 < \theta < \pi,$$

$$P_{2m+1}(\cos\theta) = 2\sigma.$$

Thus by Lemmas I and II

$$|P_{2m+1}(\cos\theta)| = 2|\sigma| \leqq 2\alpha_0 \beta_N s = \frac{2a_{m+1}}{\sin\theta} \leqq \frac{M}{\sqrt{2m+1}\sin\theta}.$$

The inequality

$$|P_n(\cos\theta)| \leqq \frac{M}{\sqrt{n}\sin\theta}, \qquad 0 < \theta < \pi,$$

is proved for $n = 1, 2, \cdots$.

II. Expansions in Legendre polynomials.

We shall now investigate the possibility of representing an "arbitrary" function by the series

$$f(x) \sim c_0 P_0(x) + c_1 P_1(x) + \cdots + c_n P_n(x) + \cdots$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{+1} f(t) P_n(t)\, dt.$$

For these coefficients we prove the following uniqueness theorem.

THEOREM III.* *If $\varphi(x)$ is a function of the form*

$$\varphi(x) = \frac{\psi(x)}{\sqrt{1-x^2}}, \qquad \psi(x) \text{ continuous}, \quad -1 \leqq x \leqq +1$$

and if

$$\int_{-1}^{+1} \varphi P_n\, dx = 0, \qquad n = 0, 1, 2, \cdots$$

then

$$\varphi(x) \equiv 0, \qquad -1 \leqq x \leqq +1$$

Proof. From the equations $\int_{-1}^{+1} \varphi P_n\, dx = 0$, $n = 0, 1, 2, \cdots$, we deduce the fact that $\int_{-1}^{+1} \varphi x^n\, dx = 0$, $n = 0, 1, 2, \cdots$; for x^n can be represented as a linear combination of the polynomials $P_0(x), \cdots, P_n(x)$.

If $\varphi(x) \not\equiv 0$ for $x = \xi$, $-1 < \xi < +1$, we may assume $\varphi(\xi) > 0$ and may determine a neighborhood of ξ throughout which $\varphi(x) \geqq m > 0$. Now for $-1 \leqq x \leqq +1$ the polynomial $p(x) = A_\xi - \frac{1}{8} A_\xi (x-\xi)^2 (x^2+1)$, $A_\xi > 0$, is not negative and has a single maximum at $x = \xi$. We can choose $A_\xi > 0$ so that there exist in the neighborhood of ξ two distinct roots of the equation $p(x) = 1$, which we denote by x_1, x_2. Thus we have

$$\begin{aligned} 0 \leqq p(x) &\leqq 1, & -1 \leqq x \leqq x_1 < x_2 \leqq x \leqq 1, \\ p(x) &\geqq 1, & x_1 \leqq x \leqq x_2, \\ \varphi(x) &\geqq m, & x_1 \leqq x \leqq x_2, \\ \psi(x) &\geqq -M, & -1 \leqq x \leqq +1. \end{aligned}$$

For the polynomials $p_n(x) = p^n(x)$ we find $\int_{-1}^{+1} \varphi p_n\, dx = 0$, $n = 1, 2, \cdots$. On the other hand

$$\int_{x_1}^{x_2} \varphi p_n\, dx \geqq m \int_{x_1}^{x_2} p_n\, dx, \qquad \int_{-1}^{x_1} \varphi p_n\, dx \geqq -M \int_{-1}^{x_1} \frac{dx}{\sqrt{1-x^2}},$$

$$\int_{x_2}^{+1} \varphi p_n\, dx \geqq -M \int_{x_2}^{+1} \frac{dx}{\sqrt{1-x^2}}.$$

* Lebesgue, *Leçons sur les séries trigonométriques*, pp. 37–38. Lerch, Acta Math. vol. 27 (1903), pp. 346–347.

When we add these three inequalities we find

$$\int_{-1}^{+1} \varphi p_n \, dx \geqq m \int_{x_1}^{x_2} p_n \, dx - M \int_{-1}^{x_1} \frac{dx}{\sqrt{1-x^2}} - M \int_{x_2}^{1} \frac{dx}{\sqrt{1-x^2}}$$

$$\geqq m \int_{x_1}^{x_2} p_n dx - \pi M.$$

Since $\int_{x_1}^{x_2} p_n \, dx$ becomes infinite with n, we find $\int_{-1}^{+1} \varphi p_n dx > 0$, $n \geqq N$. The contradiction shows that $\varphi(x) = 0$, $-1 < x < +1$. Consequently $\psi(x)$ is identically zero; and finally

$$\varphi(x) \equiv 0, \qquad -1 \leqq x \leqq +1.$$

Concerning the magnitude of the coefficients c_n we can obtain certain information in case that the integral $\int_{-1}^{+1} f^2 dx$ exists in the sense of either Riemann or Lebesgue.

THEOREM IV. *If the integral* $\int_{-1}^{+1} f^2 \, dx$ *exists, the quantities*

$$c_n = \frac{2n+1}{2} \int_{-1}^{+1} f P_n \, dx$$

are such that the infinite series

$$c_0^2 + \frac{c_1^2}{3} + \cdots + \frac{c_n^2}{2n+1} + \cdots$$

converges.

Proof. We see that $\int_{-1}^{+1} (f - c_0 P_0 - \cdots - c_N P_N)^2 \, dx \geqq 0$. On expanding the integral on the left we find

$$\int_{-1}^{+1} f^2 \, dx - 2\left(c_0^2 + \frac{c_1^2}{3} + \cdots + \frac{c_N^2}{2N+1}\right) \geqq 0,$$

$$\frac{1}{2} \int_{-1}^{+1} f^2 \, dx \geqq c_0^2 + \frac{c_1^2}{3} + \cdots + \frac{c_N^2}{2N+1}.$$

If we allow N to become infinite, the theorem is established.

It is now possible to prove the first convergence theorem.

THEOREM V. *If* $f(x)$ *can be written in the form*

$$f(x) = f(-1) + x f'(-1) + \int_{-1}^{x} \int_{-1}^{x} f''(x) \, dx \, dx,$$

where $\int_{-1}^{+1} (f'')^2\,dx$ *exists, then the series*

$$c_0 P_0(x) + c_1 P_1(x) + \cdots + c_n P_n(x) + \cdots, \qquad c_n = \frac{2n+1}{2}\int_{-1}^{+1} f P_n\,dx,$$

converges uniformly to $f(x)$, $-1 \leqq x \leqq +1$.

Proof. From formulas (4) and (5) we see that

$$c_n = \frac{2n+1}{2}\int_{-1}^{+1} f P_n\,dx = \frac{1}{2}\int_{-1}^{+1} f(P'_{n+1} - P'_{n-1})\,dx$$
$$= -\frac{1}{2}\int_{-1}^{+1} f'(P_{n+1} - P_{n-1})\,dx.$$

With a notation whose meaning is obvious we write this equation as

$$c_n = \frac{c'_{n-1}}{2n-1} - \frac{c'_{n+1}}{2n+3}, \qquad n = 1, 2, \cdots.$$

Similarly

$$c'_n = \frac{c''_{n-1}}{2n-1} - \frac{c''_{n+1}}{2n+3}, \qquad n = 1, 2, \cdots.$$

Thus

$$c_n = \frac{c''_{n+2}}{(2n+5)(2n+3)} - \frac{(4n+2)\,c''_n}{(2n+3)(2n+1)(2n-1)} + \frac{c''_{n-2}}{(2n-1)(2n-3)}, \qquad n = 2, 3, \cdots.$$

Next, by using the first inequality of Theorem II and the relation $|ab| \leqq \frac{a^2+b^2}{2}$, we find

$$|c_n P_n(x)| \leqq |c_n| \leqq \frac{|c''_{n+2}|}{(2n+5)(2n+3)} + \frac{(4n+2)\,|c''_n|}{(2n+3)(2n+1)(2n-1)} + \frac{|c''_{n-2}|}{(2n-1)(2n-3)}$$
$$\leqq \frac{c''^2_{n+2}}{2(2n+5)} + \frac{1}{2(2n+5)(2n+3)^2} + \frac{c''^2_n}{2(2n+1)} + \frac{(4n+2)^2}{2(2n+1)(2n+3)^2(2n-1)^2} + \frac{c''^2_{n-2}}{2(2n-3)} + \frac{1}{2(2n-3)(2n-1)^2} = T_n, \qquad n = 2, 3, \cdots.$$

Theorem IV shows that $\sum_{n=2}^{\infty} T_n$ is convergent. Consequently the series $c_0 P_0 + \cdots + c_n P_n + \cdots$ converges uniformly to a continuous function $f^*(x)$,

$-1 \leqq x \leqq +1$. Since it is allowable to multiply the series by $P_n(x)$ and integrate term by term, we see that

$$\frac{2n+1}{2}\int_{-1}^{+1} f^* P_n\, dx = c_n.$$

In other words, if we set $\varphi = f - f^*$, we have

$$\int_{-1}^{+1} \varphi P_n\, dx = 0, \qquad n = 0, 1, 2, \cdots.$$

From Theorem III it is apparent that

$$\varphi \equiv 0, \quad f^* \equiv f, \qquad -1 \leqq x \leqq +1.$$

This completes the proof.

THEOREM VI. *If $f(x)$ can be written in the form*

$$f(x) = f(-1) + \int_{-1}^{x} f'(x)\, dx,$$

where $\int_{-1}^{+1} f'^2\, dx$ exists, then the series $c_0 P_0(x) + \cdots + c_n P_n(x) + \cdots$ converges to $f(x)$, $-1 < x < +1$, the convergence being uniform on any closed interval completely interior to $(-1, +1)$.

Proof. As in Theorem IV we have

$$c_n = \frac{c'_{n-1}}{2n-1} - \frac{c'_{n+1}}{2n+3}.$$

We write $x = \cos\theta$ in the second inequality of Theorem II. Then we obtain the inequalities

$$|c_n P_n(x)| \leqq \frac{|c_n|\, M}{\sqrt{n}\sqrt{1-x^2}} \leqq \frac{M}{\sqrt{1-x^2}}\left(\frac{|c'_{n-1}|}{\sqrt{n}\,(2n-1)} + \frac{|c'_{n+1}|}{\sqrt{n}\,(2n+3)}\right)$$

$$\leqq \frac{M}{\sqrt{1-x^2}}\left(\frac{c'^2_{n-1}}{2(2n-1)} + \frac{1}{2n(2n-1)} + \frac{c'^2_{n+1}}{2(2n+3)} + \frac{1}{2n(2n+3)}\right).$$

Consequently the infinite series

$$c_0\sqrt{1-x^2}\, P_0(x) + c_1\sqrt{1-x^2}\, P_1(x) + \cdots + c_n\sqrt{1-x^2}\, P_n(x) + \cdots$$

converges uniformly to a continuous function $\psi^*(x)$, $-1 \leqq x \leqq +1$; and the infinite series

$$c_0 P_0(x) + c_1 P_1(x) + \cdots + c_n P_n(x)$$

converges to a function $f^*(x) = \dfrac{\psi^*(x)}{\sqrt{1-x^2}}$, $-1 < x < +1$, the convergence being uniform on any closed interval completely interior to the interval $(-1, +1)$. On multiplying the first of these series by $\dfrac{P_n(x)}{\sqrt{1-x^2}}$ and integrating term by term, as we may, we find

$$c_n = \frac{2n+1}{2} \int_{-1}^{+1} \psi^* \frac{P_n}{\sqrt{1-x^2}}\, dx = \frac{2n+1}{2} \int_{-1}^{+1} f^* P_n\, dx.$$

On setting $\varphi = f - f^*$ we see that $\int_{-1}^{+1} \varphi P_n\, dx = 0$, $n = 0, 1, 2, \cdots$. Since Theorem III is applicable, we obtain the desired result

$$\varphi \equiv 0, \qquad f^* \equiv f, \qquad -1 \leqq x \leqq +1.$$

The theorem is established.

We shall now turn to the expansion of a function which has a finite discontinuity at an interior point of the interval $(-1, +1)$. To carry through the discussion we need two lemmas.

LEMMA III. *With the notation* $K_n(x,t) = \dfrac{n+1}{2}[P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)]$ *we have*

(a) $\dfrac{1}{2} P_0(x)P_0(t) + \dfrac{3}{2} P_1(x)P_1(t) + \cdots + \dfrac{2n+1}{2} P_n(x)P_n(t) = \dfrac{K_n(x,t)}{x-t}$,

(b) $\int_{-1}^{+1} \dfrac{K_n(x,t)}{x-t}\, dt = 1$,

(c) $\lim_{n\to\infty} \int_a^b K_n(x,t)\, dt = 0$, $\quad -1 \leqq a < b \leqq +1$, *uniformly* $-1+\varepsilon \leqq x \leqq +1-\varepsilon$, $\varepsilon > 0$,

(d) $\int_a^b |K_n(x,t)|\, dt \leqq G$, *independent of* a, b, n, *and* x *for* $-1+\varepsilon \leqq x \leqq +1-\varepsilon$, $\varepsilon > 0$.

Proof. The identity (*a*) is due to Christoffel.† To prove it we write $k-1$ for n in (1) and solve for $(2k+1)x P_k(x) = (k+1)P_{k+1}(x) - k P_{k-1}(x)$. We see that

† Christoffel, Journ. für Math., vol. 55 (1858), pp. 61–82.

$$(x-t)\sum_{k=0}^{k=n}\frac{2k+1}{2}P_k(x)P_k(t)=\frac{1}{2}P_1(x)P_0(t)-\frac{1}{2}P_0(x)P_1(t)$$
$$+\sum_{k=1}^{k=n}\left\{\frac{k+1}{2}[P_{k+1}(x)P_k(t)-P_k(x)P_{k+1}(t)]\right.$$
$$\left.-\frac{k}{2}[P_k(x)P_{k-1}(t)-P_{k-1}(x)P_k(t)]\right\}=K_n(x,t).$$

We prove (b) at once by integrating term-wise in (a). To show the truth of (c) we use a special case of Theorem IV, namely that in which $f(x)\equiv 1$, $a\leqq x\leqq b$, and $f(x)\equiv 0$ elsewhere. We have from Theorem II and Theorem III

$$\left|\int_a^b K_n(x,t)\,dt\right|\leqq(n+1)\left\{|P_{n+1}(x)|\frac{|c_n|}{2n+1}+|P_n(x)|\frac{|c_{n+1}|}{2n+3}\right\}$$
$$\leqq\frac{L}{\sqrt{1-x^2}}\left\{\frac{|c_n|}{\sqrt{2n+1}}+\frac{|c_{n+1}|}{\sqrt{2n+3}}\right\}$$
$$\leqq L'\left\{\frac{|c_n|}{\sqrt{2n+1}}+\frac{|c_{n+1}|}{\sqrt{2n+3}}\right\}\to 0,$$
$$-1+\varepsilon\leqq x\leqq 1-\varepsilon,$$

since $\frac{c_n^2}{2n+1}$ and therefore $\frac{|c_n|}{\sqrt{2n+1}}$ approach zero with $\frac{1}{n}$. Lastly, to establish (d) we write, from Theorem II,

$$\int_a^b|K_n(x,t)|\,dt\leqq\int_{-1}^{+1}|K_n(x,t)|\,dt$$
$$\leqq\frac{\sqrt{n+1}}{\sqrt{n}}\frac{M}{\sqrt{1-x^2}}\int_{-1}^{+1}\frac{dt}{\sqrt{1-t^2}}\leqq G,$$
$$-1+\varepsilon\leqq x\leqq+1-\varepsilon.$$

LEMMA IV. *If $-1<a<+1$, then the function $\Phi_\mu(x,t)$ defined for $-1\leqq x<a$, $a\leqq t\leqq+1$ by the equations*

$$\Phi_\mu(x,t)=\frac{1}{x-t_k},\qquad t_{k-1}\leqq t<t_k,\ k=1,2,\cdots,\mu,\ -1\leqq x<a,$$

where $t_0=a$, $t_\mu=+1$, $t_k-t_{k-1}=\frac{1-a}{\mu}$ satisfies the inequality

$$\left|\frac{1}{x-t}-\Phi_\mu(x,t)\right|<\eta,\qquad -1\leqq x\leqq a-\varepsilon,\ a\leqq t\leqq+1,$$

for μ sufficiently large.

Proof. By the law of the mean

$$\frac{1}{x-t}-\frac{1}{x-t_k}=\frac{1}{(x-t')^2}(t-t_k), \qquad t' \text{ on } (t_k,\ t).$$

Consequently

$$\left|\frac{1}{x-t}-\frac{1}{x-t_k}\right| \leqq \frac{1}{\varepsilon^2}\frac{1-a}{\mu}<\eta, \quad t_{k-1}\leqq t\leqq t_k,\ -1\leqq x\leqq a-\varepsilon,$$

or

$$\left|\frac{1}{x-t}-\Phi_\mu(x,t)\right|<\eta, \qquad -1\leqq x\leqq a-\varepsilon,\ a\leqq t\leqq +1,$$

for μ sufficiently large.

We are now in a position to prove Theorem VII.

THEOREM VII. *If $f(x)$ is defined by the equations*

$$\begin{aligned} f(x)&\equiv 0, && -1\leqq x\leqq a,\ -1<a<+1,\\ f(a)&=\frac{1}{2},\\ f(x)&\equiv 1, && a<x\leqq +1, \end{aligned}$$

then the series

$$c_0P_0(x)+c_1P_1(x)+\cdots+c_nP_n(x)+\cdots, \quad c_n=\frac{2n+1}{2}\int_a^1 P_n\,dx$$

converges to $f(x)$ everywhere $-1\leqq x\leqq +1$, the convergence being uniform on any closed interval completely interior to either $(-1,\ a)$ or $(a,\ +1)$.

Proof. We first discuss the convergence of the series on the intervals $-1+\varepsilon\leqq x\leqq a-\varepsilon,\ a+\varepsilon\leqq x\leqq 1-\varepsilon$. Having fixed upon the quantity $\varepsilon>0$ we determine the constant G of Lemma III (d) and require that, for a given positive δ, however small, $G\eta<\frac{\delta}{2}$. We then determine the integer μ of Lemma IV so that

$$\left|\frac{1}{x-t}-\Phi_\mu(x,t)\right|<\eta, \qquad -1\leqq x\leqq a-\varepsilon,\ a\leqq t\leqq +1.$$

Now we see that

$$\begin{aligned} |c_0P_0(x)+\cdots+c_nP_n(x)| &= \left|\int_a^1\frac{K_n(x,t)}{x-t}\,dt\right|\\ &\leqq \left|\int_a^1\left[\frac{1}{x-t}-\Phi_\mu(x,t)\right]K_n(x,t)\,dt\right|+\left|\int_a^1\Phi_\mu(x,t)\,K_n(x,t)\,dt\right|\\ &\leqq \eta G+\left|\sum_{k=1}^{k=\mu}\frac{1}{x-t_k}\int_{t_{k-1}}^{t_k}K_n(x,t)\,dt\right|\\ &\leqq \frac{\delta}{2}+\frac{1}{\varepsilon}\sum_{k=1}^{k=\mu}\left|\int_{t_{k-1}}^{t_k}K_n(x,t)\,dt\right|<\delta, \qquad n>N, \end{aligned}$$

independent of x, $-1+\varepsilon \leqq x \leqq a-\varepsilon$, by Lemma III (c). Next, for x on the interval $(a+\varepsilon,\ +1-\varepsilon)$, we write

$$c_0 P_0(x)+\cdots+c_n P_n(x) = \int_a^1 \frac{K_n(x,t)}{x-t}\,dt = 1-\int_{-1}^{a} \frac{K_n(x,t)}{x-t}\,dt$$

with the help of Lemma III (b). Thus, by (6),

$$c_0 P_0(x)+\cdots+c_n P_n(x)-1 = \int_{-1}^{a} \frac{K_n(x,t)}{x-t}\,dt = \int_{a'}^{1} \frac{K_n(x',t')}{x'-t'}\,dt',$$

where $x'=-x$, $t'=-t$, $a'=-a$. The reasoning applied above can be carried through without difficulty for the integral on the right; we see that

$$|c_0 P_0(x)+\cdots+c_n P_n(x)-1|<\delta, \qquad n \geqq N',$$

independent of x, $a+\varepsilon \leqq x \leqq 1-\varepsilon$. Consequently the series converges to $f(x)$, $-1<x<a$, $a<x<+1$, the convergence being uniform on any closed interval completely interior to $(-1, a)$, $(a, +1)$.

To discuss the convergence at $x=-1$, a, $+1$ we write

$$c_n = \frac{1}{2}\int_a^1 (2n+1)\,P_n\,dx = \frac{1}{2}\{P_{n-1}(a)-P_{n+1}(a)\}, \qquad c_0 = \frac{1-a}{2},$$

by (4). Thus we see that

$$\begin{aligned}
c_0 P_0(-1)+\cdots+c_n P_n(-1) &= \frac{1-a}{2}+\frac{1}{2}\sum_{k=1}^{k=n}(-1)^k\{P_{k-1}(a)-P_{k+1}(a)\} \\
&= \frac{(-1)^n}{2}\{P_n(a)-P_{n+1}(a)\} \to 0, \\
c_0 P_0(a) \quad +\cdots+c_n P_n(a) &= \frac{1-a}{2}+\frac{1}{2}\sum_{k=1}^{k=n}\{P_k(a)P_{k-1}(a)-P_{k+1}(a)P_k(a)\} \\
&= \frac{1}{2}-\frac{P_n(a)P_{n+1}(a)}{2} \to \frac{1}{2}, \\
c_0 P_0(1) \quad +\cdots+c_n P_n(1) &= \frac{1-a}{2}+\frac{1}{2}\sum_{k=1}^{k=n}\{P_{k-1}(a)-P_{k+1}(a)\} \\
&= 1-\frac{1}{2}\{P_k(a)+P_{n+1}(a)\} \to 1,
\end{aligned}$$

by reference to (5) and to Theorem II. This completes the proof.

Finally, we can discuss any function expressible as a linear combination of a finite set of functions of the types described in Theorems V, VI, VII. We thus have a fairly comprehensive treatment of series of Legendre polynomials for functions of this simple character.

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ON CERTAIN TWO-DIMENSIONAL VARIETIES OF ORDER $\frac{1}{2}r(r-1)$ IN r-SPACE AND CERTAIN CURVES TRACED ON THEM.

By B. C. Wong.

1. It is well known that the Jacobian of three quadric surfaces in space of three dimensions is a sextic curve whose points can be set in one-to-one correspondence with the points of a plane quartic curve. Generalizing this proposition, it is not difficult to show that in $(r-1)$-space the Jacobian of three $(r-2)$-dimensional quadric varieties is an $(r-1)$-space curve of order $r(r-1)/2$ whose points can be set in one-to-one correspondence with the points of a plane r-ic curve. Since this plane curve can have at most $(r-1)(r-2)/2$ double points, the Jacobian curve, being of order $r(r-1)/2$, may be said to be of deficiency $(r-1)(r-2)/2$ as it can be projected by repeated projections on a 3-space in a curve with $r(r-2)(r^2-9)/2$ apparent double points, which number is the maximum less $(r-1)(r-2)/2$.

2. Since this curve, which we shall denote by $\Gamma_1^{\frac{1}{2}r(r-1)}$, is of order $r(r-1)/2$ and is an $(r-1)$-space curve, i. e., cannot exist in space of lower dimensions, it is certainly the intersection of an $(r-1)$-space with a certain two-dimensional variety $\Sigma_2^{\frac{1}{2}r(r-1)}$ of order $r(r-1)/2$. We shall next find a two-dimensional variety such that every $(r-1)$-space section of it is a $\Gamma_1^{\frac{1}{2}r(r-1)}$, proper or improper.

3. It is convenient for this purpose to employ the involutorial r-ic transformation in r-space by means of r $(r-1)$-dimensional quadric varieties. To any point there corresponds the point of intersection of its r polar $(r-1)$-spaces with respect to the quadric varieties. The locus of points whose images are lines is $J_{r-2}^{\frac{1}{2}r(r+1)}$, an $(r-2)$-dimensional variety of order $r(r+1)/2$, and the locus of these lines is $J_{r-1}^{r^2-1}$, an $(r-1)$-dimensional variety of order r^2-1. An $(r-1)$-space S_{r-1} has for its image an $(r-1)$-dimensional variety of order r, M_{r-1}^r, and, in general, an S_m^n has for its image an $M_m^{n\cdot {}_rC_m}$ where ${}_rC_m = \dfrac{r(r-1)(r-2)\cdots(r-m+1)}{m(m-1)(m-2)\cdots 2\cdot 1}$. In particular, if $m=1$, the transform of any curve of order n is a curve of order nr, and if $n=1$, $m=2$, the transform of any plane is an $M_2^{\frac{1}{2}r(r-1)}$.

4. Consider a plane σ_2 and let its image be designated be $\Sigma_2^{\frac{1}{2}r(r-1)}$. The plane meets $J_{r-2}^{\frac{1}{2}r(r+1)}$ in $r(r+1)/2$ points $j_0^{\frac{1}{2}r(r+1)}$ and J_{r-1}^{r-1} in a curve j_1^{r-1} with $r(r+1)/2$ r-fold points coincident with $j_0^{\frac{1}{2}r(r+1)}$; it also meets the image M_{r-1}^r of any S_{r-1} in an r-ic curve γ_1^r through the points $j_0^{\frac{1}{2}r(r+1)}$. But S_{r-1} meets $\Sigma_2^{\frac{1}{2}r(r-1)}$ in the curve $\Gamma_1^{\frac{1}{2}r(r-1)}$ whose points correspond to the points of γ_1^r in our transformation. It is to be noted that the order of the complete transformation of γ_1^r is r^2 but this curve is composed of, besides $\Gamma_1^{\frac{1}{2}r(r-1)}$, $r(r+1)/2$ lines lying on $\Sigma_2^{\frac{1}{2}r(r-1)}$ and not in S_{r-1} as they come from the points $j_0^{\frac{1}{2}r(r-1)}$ on γ_1^r.

5. In addition to the $r(r+1)/2$ fixed points $j_0^{\frac{1}{2}r(r+1)}$ in the plane σ_2 it requires r more points to define an r-ic curve in this plane, and this is exactly the number of points required to determine an S_{r-1}. Hence, *corresponding to the ∞^r r-ic curves through the points $j^{\frac{1}{2}r(r+1)}$ in σ_2 are the ∞^r $(r-1)$-spaces in S_r, each intersecting $\Sigma_2^{\frac{1}{2}r(r-1)}$ in a $\Gamma_1^{\frac{1}{2}r(r-1)}$.*

6. When r points are chosen on the image $\Sigma_2^{\frac{1}{2}r(r-1)}$ of σ_2, the S_{r-1} they determine meets $\Sigma_2^{\frac{1}{2}r(r-1)}$ in a curve $\Gamma_1^{\frac{1}{2}r(r-1)}$ passing through them, and the corresponding curve γ_1^r in σ_2 passes through the r corresponding points. But if $r-1$ points are chosen on $\Sigma_2^{\frac{1}{2}r(r-1)}$, then through them pass an infinity of $(r-1)$-spaces giving rise to a pencil of $\frac{1}{2}r(r-1)$-ic curves through the $r-1$ points, while in σ_2 there is a pencil of r-ic curves through the points $j_0^{\frac{1}{2}r(r+1)}$ and $r-1$ other points. Of this pencil of r-ic curves $3(r-1)^2$ are nodal, giving rise to as many $\frac{1}{2}r(r-1)$-ic curves each with an actual node. From this we infer that *the class of $\Sigma_2^{\frac{1}{2}r(r-1)}$ is* $3(r-1)^2$, that is, *through an $(r-2)$-space $3(r-1)^2$ $(r-1)$-spaces can pass tangent to $\Sigma_2^{\frac{1}{2}r(r-1)}$*. We can also say that *of a pencil of $(r-1)$-dimensional r-ic varieties $3(r-1)^2$ are tangent to a given plane.*

7. Now suppose $r-2$ points are chosen on $\Sigma_2^{\frac{1}{2}r(r-1)}$. The bundle of $(r-1)$-spaces they determine cut out from $\Sigma_2^{\frac{1}{2}r(r-1)}$ a bundle of $\frac{1}{2}r(r-1)$-ic curves through them. Of these there are infinitely many each with an

actual node. In σ_2, then, there is a corresponding net of r-ic curves through the points $j_0^{\frac{1}{2}r(r+1)}$ and $r-2$ other points. The Jacobian of this net is of order $3(r-1)$ with $(r+4)(r-1)/2$ nodes of which $r(r+1)/2$ are coincident with the points $j_0^{\frac{1}{2}r(r+1)}$, leaving $r-2$ elsewhere in the plane. The curve in S_r corresponding to this Jacobian in σ_2 is of order $3r(r-1)-2\cdot\frac{r(r+1)}{2}$ or $2r(r-2)$. Hence we infer that *through any* $(r-3)$*-space there are infinitely many tangent* $(r-1)$*-spaces to* $\Sigma_2^{\frac{1}{2}r(r-1)}$ *with the points of contact forming a locus of order* $2r(r-2)$. *This curve itself may have a number of actual nodes and this number is equal to the number of points chosen on* $\Sigma_2^{\frac{1}{2}r(r-1)}$ *to determine the* $(r-3)$*-space and cannot exceed* $r-2$.

8. Of the ∞^r r-ic curves through the points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2 there are many degenerate ones. Consider one such composed of a line $\varkappa$ through one point P and an $(r-1)$-ic curve λ_1^{r-1} through the remaining $(r-1)(r+2)/2$ points of $j_0^{\frac{1}{2}r(r+1)}$. The image of $\varkappa$ is an $(r-1)$*—ic* curve K_1^{r-1} and that of λ_1^{r-1} is an $\frac{1}{2}(r-1)(r-2)$-ic curve $L_1^{\frac{1}{2}(r-1)(r-2)}$, both lying in the same $(r-1)$-space whose image meets σ_2 in the degenerate r*-ic* curve. K_1^{r-1} and $L_1^{\frac{1}{2}(r-1)(r-2)}$ have in common $r-1$ points corresponding to the $r-1$ points of intersection of $\varkappa$ with λ_1^{r-1} in σ_2 and hence the $(r-1)$-space is tangent $r-1$ times to $\Sigma_2^{\frac{1}{2}r(r-1)}$, meeting it in K_1^{r-1} and $L_1^{\frac{1}{2}(r-1)(r-2)}$ composing a degenerate $\Gamma_1^{\frac{1}{2}r(r-1)}$.

9. The curve λ_1^{r-1} going through $(r-1)(r+2)/2$ the points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2 is a definite curve, for the $(r-1)(r+2)/2$ points are exactly sufficient to determine a curve (plane) of order $(r-1)$. But the line $\varkappa$ is one of the lines of the pencil with center at P. Hence, *there is a pencil of* $(r-1)$*-spaces through an* $(r-2)$*-space* S_{r-2} *all meeting* $\Sigma_2^{\frac{1}{2}r(r-1)}$ *in degenerate curves of the type* $\Gamma_1^{\frac{1}{2}r(r-1)}$ *which have for common component the curve* $L_1^{\frac{1}{2}(r-1)(r-2)}$ *lying in* S_{r-2} *and each of which has for the other component a curve of type* K_1^{r-1}. Since $(r-1)(r+2)/2$ points can be taken $r(r+1)/2$ ways from $r(r+1)/2$ points, giving rise to $r(r+1)/2$ $\frac{1}{2}(r-1)(r-2)$-ic curves of the type $L_1^{\frac{1}{2}(r-1)(r-2)}$, each lying in an $(r-2)$-space. The remarkable fact is that, while in general we expect an $(r-2)$-space to intersect a

two-dimensional manifold in only a finite number of points, *there are* $r(r+1)/2$ $(r-2)$*-spaces each intersecting* $\Sigma_2^{\frac{1}{2}r(r-1)}$ *in an infinity of points whose locus is a curve of the* $\frac{1}{2}(r-1)(r-2)$*th degree.* Or, in general, an $(r-2)$-space and a line in S_r do not meet, but *there are* $r(r+1)/2$ $(r-2)$*-spaces each of which is met by* $\frac{1}{2}(r-1)(r+2)$ *of the* $r(r+1)/2$ *lines on* $\Sigma_2^{\frac{1}{2}r(r-1)}$ *corresponding to the points* $j_0^{\frac{1}{2}r(r+1)}$ *in* σ_2.

10. For $r=3$, the 6 curves of the type $L_1^{\frac{1}{2}(r-1)(r-2)}$ are 6 lines and the 6 $(r-2)$-spaces are also 6 lines and these two groups of lines are identical. Hence n applying the above formulas to this case special care is required, but the configuration of the 27 lines on a cubic surface is well known. For $r=4$, there are 10 planes intersecting the image Σ_2^6 of σ_2 in cubic curves and for $r=5$, there are 15 3-spaces meeting the image Σ_2^{10} of σ_2 in sextic curves, etc.

11. Any point of $j_0^{\frac{1}{2}r(r+1)}$ determines $(r-1)(r+2)/2$ lines with the remaining $(r-1)(r+2)/2$ points of $j_0^{\frac{1}{2}r(r+1)}$. Each of these lines transforms into an $(r-1)$-ic curve which degenerates into an $(r-2)$-ic curve and a line in an S_{r-1} which is now tangent to $\Sigma_2^{\frac{1}{2}r(r-1)}$ r times. Hence, *of the pencil of* $(r-1)$*-spaces through an* $(r-2)$*-space containing a curve of the type* $L_1^{\frac{1}{2}(r-1)(r-2)}$ $\frac{1}{2}(r-1)(r+2)$ *are r-uply tangent to* $\Sigma_2^{\frac{1}{2}r(r-1)}$. Since there are $r(r+1)/2$ such pencils of $(r-1)$-spaces, there are in all $r(r^2-1)(r+2)/4$ $(r-1)$-spaces of this character, that is, each intersecting $\Sigma_2^{\frac{1}{2}r(r-1)}$ in a $\Gamma_1^{\frac{1}{2}r(r-1)}$ composed of a K_1^{r-2} in an $(r-2)$-space, an $L_1^{\frac{1}{2}(r-1)(r-2)}$ meeting K_1^{r-2} $r-2$ times and a line meeting K_1^{r-2} once and $L_1^{\frac{1}{2}(r-1)(r-2)}$ once. These r points in which the components of $\Gamma_1^{\frac{1}{2}r(r-1)}$ severally meet are the actual nodes of $\Gamma_1^{\frac{1}{2}r(r-1)}$ and are therefore the points of contact between the $(r-1)$-space and $\Sigma_2^{\frac{1}{2}r(r-1)}$.

12. Now consider another class of degenerate r-ic curves through the points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2. Let one of these r-ic curves be composed of a line μ through two points P and Q of $j_0^{\frac{1}{2}r(r+1)}$ and an $(r-1)$-ic curve γ_1^{r-1} through the remaining $(r^2+r-4)/2$. This degenerate r-ic curve, γ_1^r, is the intersection of σ_2 with the image M_{r-1}^r of an S_{r-1} which meets the image $\Sigma_2^{\frac{1}{2}r(r-1)}$ of σ_2 in an $(r-2)$-ic curve M_1^{r-2} in an S_{r-2} and an

$\frac{1}{2}(r^2-3r+4)$-*ic* curve $N_1^{\frac{1}{2}(r^2-3r+4)}$ having $r-1$ points in common. The two points P and Q once chosen from the points $j_0^{\frac{1}{2}r(r+1)}$, the line μ is fixed, but the remaining $(r^2+r-4)/2$ points determine a pencil of $(r-1)$-ic curves. Correspondingly, through the $(r-2)$-space containing the transform M_1^{r-2} of μ pass ∞ $(r-1)$-spaces each containing an $N_1^{\frac{1}{2}(r^2-3r+4)}$ in addition to the fixed M_1^{r-2}. One of the $(r-1)$-ic curves of the pencil in σ_2 passes through P and another through Q, hence there are two $(r-1)$-spaces of the corresponding pencil of $(r-1)$-spaces r-uply tangent to $\Sigma_2^{\frac{1}{2}r(r-1)}$ each meeting it in an M_1^{r-2}, an $N_1^{\frac{1}{2}(r-1)r-2)}$ and a line. But this situation we already met with in the preceding paragraph, for M_1^{r-2} and $N_1^{\frac{1}{2}(r-1)(r-2)}$ are identical with M_1^{r-2} and $L_1^{\frac{1}{2}(r-1)(r-2)}$ respectively. A result of this paragraph is that *there are* $r(r^2-1)(r+2)/8$ $(r-2)$*-spaces each having an* $(r-2)$*-ic curve of the type* K_1^{r-2} *in common with* $\Sigma_2^{\frac{1}{2}r(r-1)}$ *corresponding to the* $r(r^2-1)(r+2)/8$ *lines determined by the points* $j_0^{\frac{1}{2}r(r-1)}$ in σ_2.

13. But there are other r-uply tangent $(r-1)$-spaces to $\Sigma_2^{\frac{1}{2}r(r-1)}$. The pencil of $(r-1)$-ic curves through $(r^2+r-4)/2$ points of $j_0^{\frac{1}{2}r(r+1)}$ in σ_2 contains $3(r-2)^2$ nodal members and, since there are $r(r^2-1)(r+2)/8$ such pencils of $(r-1)$-ic curves determined by all the points of $j_0^{\frac{1}{2}r(r+1)}$, there are in all $\frac{3}{8}r(r^2-1)(r+2)(r-2)^2$ nodal $(r-1)$-ic curves in σ_2. Each of these nodal $(r-1)$-ic curves goes into an $N_1^{\frac{1}{2}(r^2-3+4)}$ with an actual double point. Since $N_1^{\frac{1}{2}(r^2-3r+4)}$ already has $r-1$ points in common with its companion curve M_1^{r-2}, the degenerate curve $\Gamma_1^{\frac{1}{2}r(r-1)}$ has r actual double points which are the points of tangency between the S_{r-1} in which it lies and $\Sigma_2^{\frac{1}{2}r(r-1)}$. The number of such r-uply tangent $(r-1)$-spaces is $\frac{3}{8}r(r^2-1)(r+2)(r-2)^2$, and *the total number,* including those of § 11, *is* $\frac{r}{r}(r^2-1)(r+2)(3r^2-12r+14)/8$.

14. As has been said, the case $r=3$ requires special attention. For $r=4$, in addition to the 630 quadruply tangent 3-spaces to the image Σ_2^6 of σ_2 given by the formula above, there are 126 others each meeting Σ_2^6 in two twisted cubics having in four points which are the points of contact. Each pair of cubics comes from a pair of conics in σ_2, one through 5 and

the other through the other 5 of the 10 points j_0^{10}. For $r>4$, the formula gives all the possible r-uply tangent $(r-1)$-spaces to $\Sigma_2^{\frac{1}{2}r(r-1)}$, for there are no further degenerate r-ic curves through the points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2 unless σ_2 assumes a special position. To verify this statement, suppose a component of an r-ic curve through the points $j_0^{\frac{1}{2}r(r+1)}$ is of order t where $1<t<r-1$ determined by $t(t+3)/2$ of the points $j_0^{\frac{1}{2}r(r+1)}$, the remaining $\frac{r}{2}(r+1)-\frac{t}{2}(t+3)$ points are too many to define the other component which is of order $r-t$.

15. We have seen that the variety $\Sigma_2^{\frac{1}{2}r(r-1)}$ which is the transform of a plane σ_2 has $r(r+1)/2$ lines $g_1^{\frac{1}{2}r(r+1)}$ corresponding to the $r(r+1)/2$ points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2. There are in S_r $r(r+1)/2$ $(r-2)$-spaces meeting $\Sigma_2^{\frac{1}{2}r(r-1)}$ in $\frac{1}{2}(r-1)(r-2)$-ic curves of the type $L_1^{\frac{1}{2}(r-1)(r-2)}$ corresponding to the $(r-1)$-ic curves of the type λ_1^{r-1} each through $(r-1)(r-2)/2$ of the points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2; there are also $r(r^2-1)(r+2)/8$ other $(r-2)$-spaces meeting $\Sigma_2^{\frac{1}{2}r(r+1)}$ in $(r-2)$-ic curves of the type M_1^{r-2} corresponding to the lines each through two points of $j_0^{\frac{1}{2}r(r+1)}$ in σ_2.

16. Each of the lines $g_1^{\frac{1}{2}r(r+1)}$ meets $(r-1)(r+2)/2$ curves of the type $L_1^{\frac{1}{2}(r-1)(r-2)}$ each once and as many of the type M_1^{r-2} each once. Each $L_2^{\frac{1}{2}(r-1)(r-2)}$ meets $(r-1)(r+2)/2$ of the lines $g_1^{\frac{1}{2}r(r+1)}$ each once; $(r-1)(r+2)/2$ curves of the type M_1^{r-2} each $r-2$ times and the remaining $(r-1)(r+2)(r^2+r-4)/8$ each $r-3$ times; also meets all the other curves of the group to which it belongs each $(r-2)(r-3)/2$ times. Each M_1^{r-2} meets only 2 of the lines $g_1^{\frac{1}{2}r(r+1)}$ each once, $\frac{1}{2}(r^2+r-4)$ curves of the group $L_2^{\frac{1}{2}(r-1)(r-2)}$ each $r-3$ times and the remaining 2 each $r-2$ times; also meets $(r-2)(r+3)(r^2+r-4)/8$ other curves of the group to which it belongs each once.

17. For $r=4$, the variety Σ_2^6 has 10 lines, 10 plane cubics and 45 conics. Each line meets 9 plane cubics and 9 conics all each once; each plane cubic meets the other 9 plane cubics each once, also meets 9 lines each once, 9 conics each twice and the remaining 36 conics each once;

and each conic meets 2 lines each once, and 8 plane cubics each once and the remaining 2 plane cubics each twice, and also 28 other conics each once.

18. There is on $\Sigma_2^{\frac{1}{2}r(r-1)}$ another curve, which is of high order, worth mentioning. It is the transform of the (r^2-1)-ic curve j^{r^2-1} (§ 4) in which the plane σ_2 meets $J_{r-1}^{r^2-1}$ and its degree is $r(r+1)(r-2)/2$, for $j_1^{r^2-1}$ has $r(r+1)/2$ r-fold points coincident with $j_0^{\frac{1}{2}r(r+1)}$. It meets every one of the lines $g_1^{\frac{1}{2}r(r+1)}$ r times. Since any r-ic curve γ_1^r through the points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2 meets $j_1^{r^2-1}$ in $r(r+1)(r-2)/2$ points besides the $r(r+1)/2$ r-fold points, this curve is met by every $\Gamma_1^{\frac{1}{2}r(r-1)}$, proper or improper, in $r(r+1)(r-2)/2$ points.

19. Attention is here called to the fact that in § 10 another manner of setting up the correspondence between plane r-ic curves and $(r-1)$-space $\frac{1}{2}r(r-1)$-ic curves, for in that paragraph we saw that an $(r-1)$-ic curve λ_1^{r-1} in σ_2 transforms into an $\frac{1}{2}(r-1)(r-2)$-ic curve $L_1^{\frac{1}{2}(r-1)(r-2)}$ in an $(r-2)$-space. If r is substituted for $r-1$, we have an r-ic curve λ_1^r in σ_2 transforming into an $\frac{1}{2}r(r-1)$-ic curve $L_1^{\frac{1}{2}r(r-1)}$ in an $(r-1)$-space. But in this correspondence the curve $L_1^{\frac{1}{2}r(r-1)}$ is not the intersection of an $(r-1)$-space and a $\Sigma_2^{\frac{1}{2}r(r-1)}$ although in another correspondence it is, but that of an $(r-1)$-space and a $\Sigma_2^{\frac{1}{2}r(r+1)}$. Every $\Sigma_2^{\frac{1}{2}r(r+1)}$ has only $(r+1)(r+2)/2$ curves of order $r(r-1)/2$ each lying in an $(r-1)$-space. (§ 15.)

20. There is another correspondence between plane r-ic curves and $(r-1)$-space $\frac{1}{2}r(r-1)$-ic curves in this connection. Pass an $(r+1)$-ic curve through the points $j_0^{\frac{1}{2}r(r+1)}$ in σ_2. This curve transforms into $r(r+1)/2$ lines and a curve of order $r(r+1)/2$ on $\Sigma_2^{\frac{1}{2}r(r-1)}$. This curve lies in S_r and not in S_{r-1}. But if r is substituted for $r+1$, we have the desired correspondence; and this correspondence can be easily shown to be identical with that mentioned in the opening paragraph.

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ON CLASS NUMBER RELATIONS IMPLIED BY REPRESENTATIONS AS SUMS OF AN ODD NUMBER OF SQUARES.

By E. T. Bell.

Part I.

Representations as sums of 5, 7, 9, 11, 13, 15, 17, 19, 21 squares involving class numbers.

1. Except for the classical examples obtained from representations as sums of 3 squares, the binary quadratic class number relations implicit in the formulas for the number $N_r(n)$ of representations of n as a sum of r integer squares with roots $\gtrless 0$, have not been investigated in the case of r *odd*, as may be verified by consulting Dickson's *History of the Theory of Numbers*, vol. 3, chap. 6. Such relations are of greatest interest, because finite in number, when, in addition to the class number functions, are involved only functions of the real divisors alone of integers. These exist for 3, 5, 7, 9, 11, 13 squares. Although 15 squares have some interesting properties in common with the preceding, they furnish no similar class number relations, and both series of results cease with the case of representations as sums of 15 squares. In this paper we shall give all the developments necessary for writing out the complete set of such class number relations and will exhibit several.

In addition to $N_r(n)$ above defined we shall need $N_r(n, s)$ the number of representations of n as a sum of r integer squares precisely s of which are odd with roots $\gtrless 0$, and $r-s$ even with roots $\gtrless 0$. Evidently, $N_r(0, 0) = 1$, $N_r(4n, 0) = N_r(4n, 0) = N_r(n)$.

2. From all the representations enumerated by $N_r(n, s)$ select those in which the s odd squares occupy the first s places, and let the number of such representations be $N_r'(n, s)$. Then

$$N_r(n, s) = \frac{(r+s)!}{r!\ s!}\ N_r'(n, s).$$

Henceforth the letters n, a denote arbitrary integers > 0. Whenever a occurs under the summation sign $\sum$ without indicated limits, the sum is with respect to all $a = 1, 2, 3, \cdots$ *that render the arguments of the summands* > 0. As customary, $0! = 1$.

In the usual notation, $G(n)$ is the whole number of classes, and $F(n)$, $F_1(n)$ the number of odd, of even classes respectively of binary quadratic

forms of determinant $-n$, with all the conventions usually imposed, as in H. J. S. Smith's *Report on the Theory of Numbers*, so that

$$F(0) = 0, \quad F_1(0) = -\frac{1}{12}, \quad G(0) = \frac{1}{12} = E(0),$$

where $E(n-1) = F(n-1) - F_1(n-1)$, and a class equivalent to $a(x^2+y^2)$ counts for $\frac{1}{2}$ in F or F_1, and a class equivalent to $2a(x^2+xy+y^2)$ contributes $\frac{1}{3}$ to F_1. For reductions of formulas we have $F(4n) = 2F(n)$,

$$\begin{aligned}
F_1(8n-5) &= \tfrac{1}{3}F(8n-5), & E(8n-5) &= \tfrac{2}{3}F(8n-5);\\
F_1(8n-1) &= F(8n-1), & E(8n-1) &= 0;\\
F_1(4n-3) &= 0, & E(4n-3) &= F(4n-3);\\
F_1(4n-2) &= 0, & E(4n-2) &= F(4n-2);\\
F_1(4n) &= G(n), & E(4n) &= E(n),
\end{aligned}$$

and there are the well known theorems,

$$\begin{aligned}
N_3(8n-5) &= N_3(8n-5, 3) = 8F(8n-5);\\
N_3(4n-3) &= N_3(4n-3, 1) = 12F(4n-3);\\
N_3(4n-2) &= N_3(4n-2, 2) = 12F(4n-2);\\
N_3(n-1) &= 12E(n-1),
\end{aligned}$$

which will be used immediately.

From the definitions we see at once that

$$N_r'(n-1, \varrho) = \sum N_s'(a-1, \sigma)\, N_{r-s}'(n-a, \varrho-\sigma),$$

and therefore

$$N_r(n-1, \varrho) = \frac{r!\,\sigma!\,(s-\sigma)!\,(\varrho-\sigma)!\,(r-s-\varrho+\sigma)!}{s!\,\varrho!\,(r-\varrho)!\,(r-s)!} \sum N_s(a-1,\sigma) N_{r-s}(n-a,\varrho-\sigma).$$

In this put $s = 3$, $\sigma = 1, 2, 3$, and reduce:

$$(1)\quad N_r(n-1, \varrho) = \frac{4r(r-1)(r-2)}{\varrho(r-\varrho)(r-\varrho-1)} \sum F(4a-3) N_{r-3}(n+2-4a, \varrho-1),$$

$$(2)\quad N_r(n-1, \varrho) = \frac{4r(r-1)(r-2)}{\varrho(\varrho-1)(r-\varrho)} \sum F(4a-2) N_{r-3}(n+1-4a, \varrho-2),$$

$$(3)\quad N_r(n-1, \varrho) = \frac{8r(r-1)(r-2)}{\varrho(\varrho-1)(\varrho-2)} \sum F(8a-5) N_{r-3}(n+4-8a, \varrho-3);$$

and from the obvious identities

$$N_r(n-1, r-s) = \frac{r!}{s!\,(r-s)!} \sum N_s(a-1)\, N_{r-s}(n+3-4a, r-s),$$

$$N_r(n) = \sum N_s(a-1)\, N_{r-s}(n+1-a),$$

find in a similar way

$$(4)\quad N_r(n-1,\, r-3) = 2r(r-1)(r-2)\sum E(a-1)N_{r-s}(n+3-4a,\, r-3),$$

$$(5)\quad N_r(n) = 12E(n)+12\sum E(a-1)N_{r-s}(n+1-a).$$

It is convenient to have separately the reduced forms of (1), (2), (3) when in them respectively $\varrho = 1, 2, 3$:

$$(1.0)\quad N_r(4n-3,1) = 4r\left[F(4n-3)+\sum_{s=1}^{n-1} F(4s-3)N_{r-s}(n-s)\right],$$

$$(2.0)\quad N_r(4n-2,2) = 2r(r-1)\left[F(4n-2)+\sum_{s=1}^{n-1} F(4s-2)\,N_{r-s}(n-s)\right],$$

$$(3.0)\quad 3N_r(8n-5,3) = 4r(r-1)(r-2)\left[F(8n-5)+\sum_{s=1}^{n-1} F(8s-5)\,N_{r-s}(2n-2s)\right],$$

$$3N_r(8n-1,3) = 4r(r-1)(r-2)\sum F(8a-5)\,N_{r-s}(2n-1-2a).$$

3. When $r-3 = 2, 4, 6, 8, 10, 12, 14, 16, 18$ complete systems of results for $N_{r-s}(n-1)$, $N_{r-s}(n-1, \varrho)$ are available in the work of Glaisher*. Hence (5) gives $N_r(n-1)$ for $r = 5, 7, 9, 11, 13, 15, 17, 19, 21$. Moreover all of the numerical functions occuring in these determinations can be rapidly computed by several linear recurrences. This is well known for the class numbers; the rest of the recurrences are due for the most part to Glaisher. The numerical computation of $N_r(n-1)$ for $r = 5, 7, \cdots, 21$ is thus reduced to a practicable form.

For certain values of ϱ when $r = 5, 7, 9, 11, 13, 15$, the formulas (1), (2), (3) express $N_r(n-1, \varrho)$ *in terms of class numbers and functions of real divisors alone (finitely)* and likewise for (4) with $r = 5, 7, 9, 11$. The complete system of such results is obtained in §§ 5—11. This system is of particular interest because the like does not exist for any odd number squares >15.

Having found this system we may eliminate from its members in all possible independent ways the functions N_r to get *a complete system of class number relations of a new type,* whose nature is sufficiently evident from the specimens in Part II and the general description in § 12.

By comparing the values of the N_r found in this paper with those given for 5, 7, 9, 11, 13 squares on a former occasion† we are enabled in several instance to effect a material reduction of the class number relations.

* Quart. Journ., vol. 38 (1907), pp. 1-63, 178-237, 289-352; hereafter cited as Glaisher.

† Amer. Journ., vol. 42 (1920), pp. 168-188; hereafter cited as (A).

Since in both the present paper and (A) the enumeration of odd square theorems of the type discussed is exhaustive, it follows that we have here the sufficient means for writing out all the reduced class number relations of this kind.

In what follows frequent use is made of Glaisher's work for an even number of squares. All of his theorems used are rewritten in (A) in the proper notation for our present purpose. To save space we shall use (A) directly in writing down results.

The numbering of the formulas has been devised to obviate the necessity of reproducing any of the calculations by which they were obtained. The first decimal in the number of a formula indicates which one of the fundamental relations (1), (2), (3), (4), (5) has been used in the derivation; figures before the decimal give the value of ϱ used in (1), (2), (3), or of r in (4), (5). When several results follow from a given ϱ or r, they are distinguished by additional digits after the decimal point. In this way we can dispense with the writing of all detailed calculations. For example, (7.3) comes from (3) by putting $\varrho = 7$; (9.41), (9.42) are two theorems which follow from (4) by putting $r = 9$.

4. From work by Mordell* and Bulyguin (reference later), it follows that all finite functions of real divisors occurring in the expressions for $N_{2s}(n)$, $N_{2s}(n, \sigma)$ $(s = 1, 2, 3, \cdots)$ are polynomials, for appropriate values of r, in one or more of the following four:

$\zeta'_r(n)$ = the sum of the rth powers of all the odd divisors of n;

$\zeta''_r(n)$ = the like for the even divisors;

$\theta'_r(n)$ = the sum of the rth powers of all those divisors of n that are of the form $4k+1$;

$\theta''_r(n)$ = the like for the divisors of the form $4k+3$.

Let $n = d\delta = t\tau$ be resolutions of n into pairs of conjugate positive integer divisors d, δ and t, τ, of which τ is always odd. Then, for our present purpose, we construct from the four fundamental functions the following, which suffice, so far as real divisors are concerned, for representations as sums of $2s$ squares (when the appropriate values of r are assigned):

$$\begin{aligned}
\zeta_r(n) &= \zeta'_r(n) + \zeta''_r(n) \equiv \sum d^r, \\
\varrho_r(n) &= \zeta'_r(n) - \zeta''_r(n) \equiv \sum (-1)^{d-1} d^r, \\
\varrho'_r(n) &= \zeta_r(n) - 2^{1-r}\zeta''_r(n) \equiv \sum (-1)^{\delta-1} d^r, \\
\alpha_r(n) &= \zeta_r(n) - 2^{-r}\zeta''_r(n) \equiv \sum t^r, \\
\beta_r(n) &= [1 + 2(-1)^n]\,\zeta'_r(n) + (1 - 2^{1-r})\,\zeta''_r(n) \equiv -\sum (-1)^{d+\delta} d^r,
\end{aligned}$$

* Quart. Journ., vol. 48 (1920), pp. 93–104.

$$\xi_r(n) = \theta'_r(n) - \theta''_r(n) \equiv \sum(-1)^{(\tau-1)/2}\tau^r,$$
$$\xi'_r(n) = n^r[\theta'_{-r}(n) - \theta''_{-r}(n)] \equiv \sum(-1)^{(\tau-1)/2}\tau^r,$$
$$\lambda_r(n) = [1+2(-1)^n]\,\zeta'_r(n),$$
$$\mu_r(n) = 4\xi'_r(n) - \xi_r(n),$$

in which all of the implied reductions can be easily verified. Of these $\alpha_r(n)$ is the sum of the rth powers of all those divisors of n whose conjugates are odd; $\xi_r(n)$ is the excess of the sum of the rth powers of all those divisors of n that are of the form $4k+1$ over the like sum for the divisors of the form $4k+3$; $\xi'_r(n)$ is the similar function in which the rth powers of those divisors of n whose conjugates are of the form $4k+1$ are taken positively. The rest may be similarly defined.

For numerical reductions (used in simplifying results written down later) we note that if m is odd and $n = 2^s m$, then

$$\zeta'_r(n) = \zeta_r(m),\ \zeta''_r(2n) = 2^r\zeta_r(n);$$
$$\xi'_r(n) = 2^{rs}\,\xi'_r(m) = (-1)^{(m-1)/2}2^{rs}\,\xi_r(m),\ \xi_r(n) = \xi_r(m);$$
$$\alpha_r(2n) = 2^r\,\alpha_r(n) = 2^{r(s+1)}\,\alpha_r(m) = 2^{r(s+1)}\,\zeta_r(m);$$
$$\beta_r(2n) = (2^r+1)\,\zeta'_r(n) + (2^r-2)\,\zeta''_r(n),\ \beta_r(m) = 3\zeta_r(m);$$
$$\beta_r(n) = \zeta'_r(n) - \zeta''_r(n) + 2\alpha_r(n) = \varrho_r(n) + 2\alpha_r(n);$$
$$\varrho'(n) = 2\alpha_r(n) - \zeta_r(n),\ \varrho'_r(m) = \zeta_r(m);$$
$$\lambda_1(2n) = 3\zeta'_1(n),\ \lambda_1(m) = -\zeta_1(m).$$

The functions χ, Ω, W, Θ, G occurring in § 5 are as defined by Glaisher; cf. his index to the paper cited. It may be mentioned in passing that the necessary developments for a systematic study of these functions, and the entire class to which they belong, by straightforward, simple and elementary methods, without recourse to the modular function theory of Klein-Fricke, are given in a little known memoir by the late V. Bulyguin.*

For zero or negative values of the argument the value of each of the numerical functions defined in this section is zero.

All formulas from now on have been verified numerically, each for at least two values of n.

5. Refer to (A) pp. 175, 176, 178, 180 and Glaisher pp. 210, 292 for the even square theorems required in writing down the following from (5):

* Bull. de l'Acad. impér. des Sci. de St. Petersburg, 1914, pp. 389–404. This work is based on the rare book of Nazimoff (in Russian).

$$(5.51)\quad N_5(n) = 12E(n)+48\sum E(a-1)\xi_0(n+1-a),$$

$$(7.52)\quad N_7(n) = 12E(n)-96(-1)^n\sum(-1)^a E(a-1)\lambda_1(n+1-a),$$

$$(9.53)\quad N_9(n) = 12E(n)+48\sum E(a-1)\mu_2(n+1-a),$$

$$(11.54)\quad N_{11}(n) = 12E(n)+192(-1)^n\sum(-1)^a E(a-1)\varrho_3(n+1-a),$$

$$(13.55)\quad 5N_{13}(n) = 60E(n)+48\sum E(a-1)[16\xi_4'(n+1-a)+\xi_4(n+1-a)+8\chi_4(n+1-a)],$$

$$(15.56)\quad N_{15}(n) = 12E(n)-96(-1)^n\sum(-1)^a E(a-1)[\beta_5(n+1-a)-2\Omega(n+1-a)],$$

$$(17.57)\quad 61N_{17}(n) = 732E(n)+48\sum E(a-1)[64\xi_6'(n+1-a)-\xi_6(n+1-a)+364W(n+1-a)]$$

$$(21.59)\quad 18005N_{21}(n) = 216110E(n)+48\sum E(a-1)[13\xi_8(n+1-a)+3328\xi_8'(n+1-a)+97504\chi_8(n+1-a)-61200G(n+1-a)-6120G(2n+2-2a)].$$

6. From (A) p. 175 and (1)–(4) we get for 5 squares:

$$(2.1)\quad N_5(4n-2) = 80\sum F(4a-3)\xi_0(4n+1-4a),$$

$$(2.2)\quad N_5(4n-2) = 40F(4n-2)+160\sum F(4a-2)\xi_0(n-a),$$

$$(5.4)\quad N_5(4n-2) = 480\sum E(a-1)\xi_0(2n+1-2a),$$

$$(3.2)\quad N_5(4n-1) = 80\sum F(4a-2)\xi_0(4n+1-4a),$$

$$(4.3)\quad N_5(4n,4) = 80\sum F(8a-5)\xi_0(4n+5-8a);$$

$$(1.11)\quad N_5(8n-7) = 20F(8n-7)+80\sum F(4a-3)\xi_0(2n-1-a),$$

$$(3.11)\quad N_5(8n-5) = 160\sum F(8a-7)\xi_0(4n+1-4a),$$

$$(3.31)\quad N_5(8n-5) = 80F(8n-5)+320\sum F(8a-5)\xi_0(n-a),$$

$$(4.22)\quad N_5(8n-4,4) = 80\sum F(8a-6)\xi_0(4n+1-4a),$$

$$(1.12)\quad N_5(8n-3,1) = 20F(8n-3)+80\sum F(4a-3)\xi_0(2n-a),$$

$$(5.3)\quad N_5(8n-3,5) = 32\sum F(8a-5)\xi_0(4n+1-4a),$$

$$(3.12)\quad N_5(8n-1) = 160\sum F(8a-3)\xi_0(4n+1-4a),$$

$$(3.22)\quad N_5(8n-1) = 320\sum F(8a-5)\xi_0(2n+1-2a),$$

$$(4.21)\quad N_5(8n,4) = 80\sum F(8a-2)\xi_0(4n+1-4a).$$

From identities similar to

$$N_5(4n) = N_5(4n,0)+N_5(4n,4), \qquad N_5(4n,0) = N_5(n),$$

which give the first of the following, we get from the above in conjunction with (5.51) the additional cases

$$(6)\quad N_5(4n) = 12E(n) + 16\sum[3E(a-1)\xi_0(n+1-a)+5F(8a-5)\xi_0(4n+5-8a)];$$

$$(7)\quad N_5(8n-4) = 12E(2n-1) + 16\sum[3E(a-1)\xi_0(2n-a)+5F(8a-6)\xi_0(4n+1-4a)],$$

$$(8)\quad N_5(8n-3) = 20F(8n-3) + 16\sum[5F(4a-3)\xi_0(2n-a)+2F(8a-5)\xi_0(4n+1-4a)],$$

$$(9)\quad N_5(8n) = 12E(2n) + 16\sum[3E(a-1)\xi_0(2n+1-a)+5F(8a-2)\xi_0(4n+1-4a)].$$

7. In a similar way the system for seven squares is obtained:

$$(1.1)\quad N_7(4n-3,1) = 28F(4n-3)+224(-1)^n\sum(-1)^a F(4a-3)\lambda_1(n-a),$$

$$(5.2)\quad N_7(4n-3,5) = 168\sum F(4a-2)\zeta_1(4n-1-4a),$$

$$(5.3)\quad N_7(4n-3,5) = 672\sum F(8a-5)\zeta_1(2n+1-4a),$$

$$(2.1)\quad N_7(4n-2,2) = 168\sum F(4a-3)\zeta_1(4n+1-4a),$$

$$(2.2)\quad N_7(4n-2,2) = 84F(4n-2)+672(-1)^n\sum(-1)^a F(4a-2)\lambda_1(n-a),$$

$$(6.3)\quad N_7(4n-2,6) = 112\sum F(8a-5)\zeta_1(4n+3-8a),$$

$$(4.1)\quad N_7(4n,4) = 280\sum F(4a-3)\zeta_1(4n+3-4a),$$

$$(4.2)\quad N_7(4n,4) = 560\sum F(4a-2)\zeta_1(2n+1-2a),$$

$$(4.3)\quad N_7(4n,4) = 560\sum F(8a-5)\zeta_1(4n+5-8a);$$

$$(5.11)\quad N_7(8n-7,5) = 1344\sum F(8a-3)\zeta_1(2n-1-2a),$$

$$(6.21)\quad N_7(8n-6,6) = 448\sum F(8a-2)\zeta_1(2n-1-2a),$$

$$(3.11)\quad N_7(8n-5) = 560\sum F(4a-3)\zeta_1(4n-1-2a),$$

$$(3.21)\quad N_7(8n-5) = 280\sum F(4a-2)\zeta_1(8n-3-4a),$$

$$(3.31)\quad N_7(8n-5) = 280F(8n-5)+6720\sum F(8a-5)\zeta_1'(n-a),$$

$$(7.41)\quad N_7(8n-4,4) = 6720\sum E(2a-2)\zeta_1(2n+1-2a),$$

$$(5.12)\quad N_7(8n-3,5) = 1344\sum F(8a-7)\zeta_1(2n+1-2a),$$

$$(6.22)\quad N_7(8n-2,6) = 448\sum F(8a-6)\zeta_1(2n+1-2a),$$

$$(3.12)\quad N_7(8n-1,3) = 560\sum F(4a-3)\zeta_1(4n+1-2a),$$

$$(3.22)\quad N_7(8n-1,3) = 280\sum F(4a-2)\zeta_1(8n+1-4a),$$

$$(3.32)\quad N_7(8n-1,3) = 2240\sum F(8a-5)\zeta_1(2n+1-2a),$$

$$(7.3)\quad N_7(8n-1,7) = 128\sum F(8a-5)\zeta_1(2n+1-2a),$$

$$(7.42)\quad N_7(8n,4) = 6720\sum E(2a-1)\zeta_1(2n+1-2a).$$

From these, by referring to (A) p. 178 for the "census" of seven squares, we can write down several more, e. g., $N_7(4n-3)$ is the sum of $N_7(4n-3,1)$, $N_7(4n-3,5)$. These formulas, rather more complicated than the preceding, may be omitted.

8. For nine squares, the system is

$$(1.1)\quad N_9(4n-3,1) = 36F(4n-3)+144\sum F(4a-3)\mu_2(n-a),$$

$$(5.1)\quad N_9(4n-3,5) = 8064\sum F(4a-3)\xi_2'(n-a),$$

$$(5.2)\quad N_9(4n-3,5) = -504\sum F(4a-2)\xi_2(4n-1-4a),$$

$$(5.3)\quad N_9(4n-4,5) = 4032\sum F(8a-5)\xi_2'(2n+1-4a),$$

$$(5.21)\quad N_9(4n-2,2) = 144F(4n-2)+576\sum F(4a-2)\mu_2(n-a),$$

$$(6.2)\quad N_9(4n-2,6) = 5376\sum F(4a-2)\xi_2'(n-a),$$

$$(6.3)\quad N_9(4n-2,6) = -672\sum F(8a-5)\xi_2(4n+3-8a),$$

$$(3.1)\quad N_9(4n-1,3) = 1344\sum F(4a-3)\xi_2'(2n+1-2a),$$

$$(7.3)\quad N_9(4n-1,7) = 4608\sum F(8a-5)\xi_2'(n+1-2a),$$

$$(4.1)\quad N_9(4n,4) = -504\sum F(4a-3)\xi_2(4n+3-4a),$$

$$(4.2)\quad N_9(4n,4) = 2016\sum F(4a-2)\xi_2'(2n+1-2a);$$

$$(9.3)\quad N_9(8n-7,9) = -64\sum F(8a-5)\xi_2(4n-1-4a),$$

$$(9.42)\quad N_9(8n-6,6) = -8064\sum E(2a-1)\xi_2(4n-1-4a),$$

$$(7.11)\quad N_9(8n-5,7) = -1152\sum F(8a-3)\xi_2(4n-1-4a),$$

$$(3.31)\quad N_9(8n-5,3) = 672F(8n-5)+2688\sum F(8a-5)\mu_2(2n-2a),$$

$$(9.41)\quad N_9(8n-2,6) = -8064\sum E(2a-2)\xi_2(4n+3-4a),$$

$$(7.12)\quad N_9(8n-1,7) = -1152\sum F(8a-7)\xi_2(4n+3-4a),$$

$$(3.32)\quad N_9(8n-1,3) = 2688\sum F(8a-5)\mu_2(2n+1-2a),$$

$$(8.2)\quad N_9(8n,8) = -288\sum F(8a-6)\xi_2(4n+3-4a).$$

As before several further formulas can be obtained from these.

9. For eleven squares,

$$(1.1)\quad N_{11}(4n-3,1) = 44F(4n-3)-704(-1)^n\sum(-1)^a F(4a-3)\varrho_3(n-a),$$

$$(5.1)\quad N_{11}(4n-3,5) = 29568\sum F(4a-3)\alpha_3(n-a),$$

$$(2.2)\quad N_{11}(4n-2,2) = 220F(4n-2)-3520(-1)^n\sum(-1)^a F(4a-2)\varrho_3(n-a),$$

$$(6.2)\quad N_{11}(4n-2,6) = 29568\sum F(4a-2)\alpha_3(n-a),$$

$$(7.3)\quad N_{11}(4n-1,7) = 42240\sum F(8a-5)\alpha_3(n+1-2a);$$

$$(9.11)\quad N_{11}(8n-7,9) = 56320\sum F(8a-7)\alpha_3(n-a),$$

$$(10.21)\quad N_{11}(8n-6,10) = 11264\sum F(8a-6)\alpha_3(n-a),$$

$$(3.31)\quad N_{11}(8n-5,3) = 1320F(8n-5)+147840\sum F(8a-5)\zeta_3(n-a),$$

$$(3.3)\quad N_{11}(8n-5,11) = 2048\sum F(8a-5)\alpha_3(n-a),$$

$$(11.42)\quad N_{11}(8n-4,8) = 506880\sum E(2a-1)\alpha_3(n-a),$$

$$(9.12)\quad N_{11}(8n-3,9) = 56320\sum F(8a-3)\alpha_3(n-a),$$

$$(10.22)\quad N_{11}(8n-2,10) = 11264\sum F(8a-2)\alpha_3(n-a),$$

$$(3.32)\quad N_{11}(8n-1,3) = 21100\sum F(8a-5)\zeta_3(2n+1-2a),$$

$$(11.41)\quad N_{11}(8n,8) = 506880\sum E(2a-2)\alpha_3(n+1-a).$$

10. For thirteen squares there are but two results of the type considered,

$$(5.3)\quad 5N_{13}(8n-7,5) = -41184\sum F(8a-5)\xi_4(4n-1-4a),$$

$$(9.3)\quad N_{13}(8n-7,9) = -4576\sum F(8a-5)\xi_4(4n-1-4a).$$

11. For fifteen squares,

$$(3.3)\quad N_{15}(8n-5,3) = 3640F(8n-5)+29120\sum F(8a-5)\beta_5(2n-2a),$$

$$(7.3)\quad N_{15}(8n-5,7) = 13178880\sum F(8a-5)\alpha_5(n-a),$$

$$(11.3)\quad N_{15}(8n-5,11) = 2975520\sum F(8a-5)\alpha_5(n-a).$$

We note the interesting result, $7N_{15}(8n-5,7) = 33N_{15}(8n-5,11)$.

For 17, 19, $\cdots$ squares there are no similar theorems.

PART II.
Class number relations.

12. The relations obtainable from the preceding theorems are all of the types

$$\sum_i c_i K(a_i) D(n-a_i) = 0,$$

where n is fixed, and the sum referring to i continues so long as $n-a_i>0$, the a_i, c_i, being integers, and K a class number function, D *a function of the real divisors alone of the argument*; or a similar sum equated to a simple sum of divisor functions whose arguments form an arithmetic progression of the *second* order.

We note first two general relations. From (1), (2), (3), on changing ϱ into $\varrho+3$ and r into $r+3$, we get, for $r\geqq\varrho+2$,

$$(10)\quad \begin{aligned}&(\varrho+1)(\varrho+2)\sum F(4a-3)N_r(n+2-4a,\ \varrho+2)\\ &\quad=(\varrho+1)(r-\varrho-1)\sum F(4a-2)N_r(n+1-4a,\varrho+1)\\ &\quad=2(r-\varrho)(r-\varrho-1)\sum F(8a-5)N_r(n+4-8a,\varrho);\end{aligned}$$

and similarly from (1), (2), (3), (4),

$$(11)\quad \begin{aligned}&3r(r-1)(r-2)\sum E(a-1)N_r(n+2-4a,\ r)\\ &\quad=(r-1)(r-2)\sum F(4a-3)N_r(n+2-4a,\ r-1)\\ &\quad=6(r-2)\sum F(4a-2)N_r(n+1-4a,\ r-2)\\ &\quad=12\sum F(8a-5)N_r(n+4-8a,\ r-3).\end{aligned}$$

13. Examining the preceding lists we see that frequently the same $N_r(n)$ or $N_r(n,s)$ occurs more than once, or that by suitable choice of n this can be brought about. By equating the several expressions for a given function we obtain the required class number relations. In certain cases a further reduction is possible, whereby the class number sum is equal to a sum of divisor functions only. This will be so if the N_r function whose values are equated is among those given in (A), but not otherwise. The appropriate divisor sums for 5, 7, $\cdots$, 13 squares are taken without further reference directly from (A), where they may easily be located by referring to the section which concerns the particular value of r (= number of squares) under discussion. Thus, referring to §§ 5, 6 and (A) for the values of $N_5(4n-2)$ we find on equating them the sixfold class number

$$(\mathrm{I})\quad \begin{aligned}&3E(4n-2)+12\sum E(a-1)\xi_0(4n-1-a)\\ &\quad=20\sum F(4a-3)\xi_0(4n+1-4a)\\ &\quad=10\ F(4n-2)+30\sum F(4a-2)\xi_0(n-a)\\ &\quad=120\sum E(a-1)\xi_0(2n+1-2a)\\ &\quad=10\sum\zeta_1(4n-1-4a^2+4a)\\ &\quad=10\zeta_1(2n-1)+20\sum\zeta_1(2n-1-2a^2).\end{aligned}$$

Sums involving ξ_0 can be reduced by means of $\xi_0(2n) = \xi_0(n)$, $\xi_0(4n-1) = 0$. Similarly from sums containing E may be omitted all $E(n)$ where $n \equiv 1 \bmod 8$. In the same way from $N_5(8n-5)$ we get the fivefold relation,

$$\begin{aligned}(\text{II}) \quad & 3E(8n-5)+12\sum E(a-1)\xi_0(8n-4-a)\\ &= 40\sum F(8a-7)\xi_0(4n+1-4a)\\ &= 20F(8n-5)+80\sum F(8a-5)\xi_0(n-a)\\ &= 20\sum \zeta_1(4n-3-2a^2+2a)\\ &= 5\zeta_1(8n-5)+10\sum \zeta_1(8n-5-4a^2).\end{aligned}$$

From $N_5(8n-1)$ we find

$$\begin{aligned}(\text{III}) \quad & 12\sum E(a-1)\xi_0(8n-a)\\ &= 20\sum F(4a-2)\xi_0(8n+1-4a)\\ &= 40\sum F(8a-3)\xi_0(4n+1-4a)\\ &= 80\sum F(8a-5)\xi_0(2n+1-2a)\\ &= 5\zeta_1(8n-1)+10\sum \zeta_1(8n-1-4a^2).\end{aligned}$$

These are sufficient to show the kind obtainable from 5 squares. The few remaining can be written out in the same way.

14. Of the numerous relations implied by 7 and 9 squares we shall write only two.

From $N_7(8n-5)$ we get

$$\begin{aligned}(\text{IV}) \quad & 12E(8n-5)+96\sum (-1)^a E(a-1)\lambda_1(8n-4-a)\\ &= 560\sum F(4a-3)\zeta_1(4n-1-2a)\\ &= 280\sum F(4a-2)\zeta_1(8n-3-4a)\\ &= 280F(8n-5)+6720\sum F(8a-5)\zeta_1'(n-a)\\ &= 280\sum \xi_2(4n-3-2a^2+2a)\\ &= -35\xi_2(8n-5)+2\xi_2(8n-5-4a^2);\end{aligned}$$

and from $N_9(4n-3, 5)$,

$$\begin{aligned}(\text{V}) \quad & 16\sum F(4a-3)\xi_2'(n-a)\\ &= -\sum F(4a-2)\xi_2(4n-1-4a)\\ &= 8\sum F(8a-5)\xi_2'(2n+1-4a)\\ &= 8\sum \alpha_3(n-1-a^2+a).\end{aligned}$$

15. As the complete system implied by 11 squares is short we give it in full. Referring to §§ 5, 9 and (A) p. 183 we find

$$\text{(VI)}\quad 8\sum(-1)^a E(a-1)\varrho_3(8n-a)$$
$$= -2640\sum F(8a-5)\zeta_3(2n+1-2a)$$
$$= 33\sum \xi_4(2n+a-a^2).$$

To find the second line of this we have used $N_{11}(8n-1) = N_{11}(8n-1, 3) + N_{11}(8n-1, 7)$, and the third line has been reduced by means of $\xi_4(2n) = \xi_4(n)$. There is but one further relation from 11 squares,

$$\text{(VII)}\quad 240\sum E(2a-2)\alpha_3(16n-8-a)$$
$$= \xi_4'(8n-1) + 2\sum \xi_4'(32n-4-a^2).$$

16. From 13 squares there is only the single relation

$$\text{(VIII)}\quad \sum F(8a-5)\xi_4(4n-1-4a) = -80\sum \alpha_5(n-1-\frac{1}{2}a^2+\frac{1}{2}a).$$

For the reasons stated in (A) § 22 the type of relations which we are considering ends here.

TABLE OF QUADRATIC RESIDUES.

By A. A. Bennett.

The use of the law of quadratic reciprocity renders the determination of the quadratic character of a given integer with respect to a given prime a simple matter particularly where the prime is not unduly large. Despite the simplicity of the procedure the actual computation at best requires time, and when the same type of question must be answered repeatedly in connection with a single problem the use of a table is a convenience not to be despised. For questions relating to the distribution of quadratic residues and of quadratic nonresidues for a given prime current methods are entirely inadequate although the conclusions may be at once obvious in an explicit table. For such reasons as these I have prepared the following table of quadratic characters covering the set of primes less than 331. The entries in the table are the values of the exponent x in the relation, $\left(\frac{n}{p}\right) = (-1)^x$, where the left hand member is the Legendre symbol. Thus $x = 0$ if n is a quadratic residue with respect to p, and $x = 1$, if n is a quadratic nonresidue with respect to p. This notation has the advantage of being capable of extension to tables for cubic residues with respect to primes of the form, $3k+1$, by use of the indices, 0, 1, 2.

	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	101	103	107	109	113	127	131	137
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	.	1	1	0	1	0	0	1	0	1	0	1	0	1	0	1	1	1	1	0	0	0	0	0	0	1	0	1	0	1	0	1	0
3	.	.	1	1	0	1	1	1	0	1	0	0	1	1	0	1	0	0	1	0	1	1	1	1	0	0	1	0	0	0	1	0	1
4	.	.	0	1	0	0	1	0	0	0	1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	1	1	1	0	1	1	0	1
5	.	.	.	1	0	0	1	1	1	0	0	0	1	0	1	0	1	0	1	0	1	1	0	1	1	0	1	1	1	0	0	1	1
6	.	.	.	.	1	1	1	0	0	1	0	1	1	0	0	0	1	1	1	1	0	1	0	0	1	1	0	0	1	0	1	0	0
7	.	.	.	.	1	1	1	0	1	0	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	1	0	0	0	1	0	0	0
8	.	.	.	.	1	0	0	1	1	0	0	0	0	1	0	1	0	1	0	0	1	1	0	0	0	1	1	0	1	1	0	1	0
9	.	.	.	.	0	1	1	1	0	1	1	0	0	0	1	0	1	1	0	1	1	0	0	0	1	1	1	0	0	1	1	0	1
10	.	.	.	.	.	0	1	0	1	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	1	0	0	0	1	1
11	.	.	.	.	.	.	1	1	0	1	1	1	1	1	0	1	0	0	1	1	0	1	1	1	0	0	0	1	1	0	1	0	1
12	.	.	.	.	.	.	0	1	1	0	1	1	0	0	1	0	1	0	1	1	1	1	0	1	1	1	0	0	0	1	0	1	1
13	.	.	.	.	.	.	.	1	1	1	1	0	1	0	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
14	.	.	.	.	.	.	.	0	1	1	0	1	1	0	1	1	0	0	0	0	0	1	1	1	1	0	0	0	1	1	0	0	0
15	.	.	.	.	.	.	.	.	1	1	0	1	1	0	1	0	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
16	.	.	.	.	.	.	.	.	0	1	1	0	1	0	1	0	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
17	.	.	.	.	.	.	.	.	.	1	1	0	1	0	1	0	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
18	.	.	.	.	.	.	.	.	.	.	1	0	1	0	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
19	.	.	.	.	.	.	.	.	.	.	.	0	1	0	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
20	.	.	.	.	.	.	.	.	.	.	.	1	1	0	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
21	.	.	.	.	.	.	.	.	.	.	.	1	1	0	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
22	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
23	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
24	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
25	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
26	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
27	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	1	1	1	1	0	0	0	1	1	0	0	0
29	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	1	0	0	0	1	1	0	0	0
30	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	0	0	0	1	1	0	0	0
31	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	0	0	0	1	1	0	0	0
32	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	0	0	1	1	0	0	0
33	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	0	1	1	0	0	0
34	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0	0	1	1	0	0	0
35	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	0	1	1	0	0	0
36	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	1	0	0	0
37	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	0	0	0
38	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	0
39	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0	0
40	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	0
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	139	149	151	157	163	167	173	179	181	191	193	197	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281	283	293	307	311	313	317
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	1	1	0	1	1	1	0	0	1	0	1	0	1	1	0	0	1	1	0	0	1	0	1	0	1	1	0	0	0	1
3	1	1	0	0	1	0	1	0	0	0	0	1	0	1	1	0	0	1	0	0	1	0	0	0	1	0	1	1	0	0	0	0	1
4	0	0	0	1	0	0	1	0	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
5	0	0	1	1	0	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
6	0	0	1	1	0	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
7	0	1	0	1	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
8	1	0	0	1	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
9	0	1	1	1	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
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12	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
13	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
14	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
15	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
16	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
17	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
18	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
19	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
20	1	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
21	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
22	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
23	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
24	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
25	1	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
26	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
27	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
28	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
29	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
30	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
31	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
32	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
33	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
34	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
35	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
36	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
37	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
38	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
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41	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
42	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
43	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
44	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
45	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
46	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
47	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
48	1	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
49	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
50	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
51	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
52	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
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54	0	0	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
55	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
56	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0
57	0	1	1	0	1	0	1	1	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0

																		59	61	67	71	73	79	83	89	97	101	103	107	109	113	127	131	137
58	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	0	1	1	1	1	1	0	0	1	1	1	1	0	1
59	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	1	1	1	1	1	1	0	0	1	1	0	0	0
60	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	0	1	0	1	1	1	0	0	1	0	0	0	0	0
61	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	1	0	1	1	1	0	0	0	0	0	0	1
62	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	0	1	0	1	1	1	0	0	0	0	0	0
63	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	0	0	1	0	0	1	1	0	0	0	0	0
64	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	0	0	1	0	0	0	1	1	0	1	0	0	0
65	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	1	1	0	0	1	1	1	1	1	1	1	1
66	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1
67	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	1	0	0	1	1	1	1	0	0	1	1	0
68	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	0	0	1	1	1	1	0	0	1	1	0
69	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	0	1	0	1	1	1	1	0	0	1	1
70	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	0	1	0	1	1	1	1	0	0	1	1
71	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0	1	1	0	1	1	1	1	0	0	1	1
72	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	1	1	0	0	1	1	1	0	0	1	0
73	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	0	1	1	1	1	0	0	1	0
74	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	1	1	1	1	1	1	0	0	1	0
75	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	1	0	0	1	1	0	0	1	0
76	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	0	0	1	1	1	0	0	1	0
77	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	1	1	1	1	1	1	0	0	1	0
78	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	1	0	0	1	1	0	0	1	0
79	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	1	1	1	1	1	0	0	1	0
80	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0	1	1	1	1	1	0	0	1	0
81	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0	1	1	1	1	1	0	0	1	0
82	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	1	1	0	0	1	0
83	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	1	0	0	1	0
84	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	1	0	0	1	0
85	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	1	1	0	0	1	0
86	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	1	1	1	0	0	1	0
87	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	1	1	1	1	0	0	1	0
88	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	1	0	0	1	0
89	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	0	0	1	0
90	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	1	0	0	1	0
91	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	0	0	1	0
92	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	1	0	0	1	0
93	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	0	0	1	0
94	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	1	0	0	1	0
95	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	0	0	1	0
96	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	1	0	0	1	0
97	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	0	0	1	0
98	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	0	0	1	0
99	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	0	0	1	0
100	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	1	0	0	1	0
101	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	0	0	1	0
102	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	0	0	1	0
103	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	0	0	1	0
104	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	0	0	1	0
105	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	0	0	1	0
106	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	0	1	0
107	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1	0
108	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0	1	0
109	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	0
110	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0
111	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0
112	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0
113	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
114	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

	139	149	151	157	163	167	173	179	181	191	193	197	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281	283	293	307	311	313	317
58	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	1	0	1	1	0	1	0	0	1	0	0
59	1	1	0	1	1	1	1	0	0	0	0	0	1	0	1	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	1	1	0
60	1	1	1	1	0	1	1	0	0	1	1	0	1	1	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
61	1	0	1	1	0	0	1	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
62	1	1	0	1	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
63	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
64	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
65	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
66	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
67	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
68	1	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
69	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
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71	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
72	1	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
73	1	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
74	1	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
75	1	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
76	1	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
77	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
78	0	1	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
79	0	1	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
80	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
81	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
82	1	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
83	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
84	1	1	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
85	1	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0
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127	.	1	1	0	0	0	0	1	0	1	1	1	1	1	0	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	0	1	0	0	1	1
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141	.	.	.	.	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1	0	1	0	1	0	1	1	0	0	1	0	0	0	1	0	0	0
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149	.	.	.	.	.	1	1	1	1	0	0	1	0	1	1	1	1	1	1	0	1	1	1	0	1	1	0	0	1	1	1	0	0	0	0	0
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151	.	.	.	.	.	.	1	0	1	0	0	1	1	1	0	0	1	0	1	1	1	0	0	1	1	0	1	1	1	0	0	1	1	0	0	1
152	.	.	.	.	.	.	1	0	0	0	1	0	1	1	1	1	1	0	1	1	0	1	1	0	1	1	1	0	1	0	1	0	1	1	0	1
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157	.	.	.	.	.	.	.	1	0	0	1	1	1	0	0	0	1	1	1	1	0	1	0	1	1	0	1	1	0	1	0	1	0	0	1	1
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163	.	.	.	.	.	.	.	.	1	0	1	1	1	0	1	0	1	0	1	1	1	1	0	1	0	1	1	1	0	0	1	1	0	1	1	1
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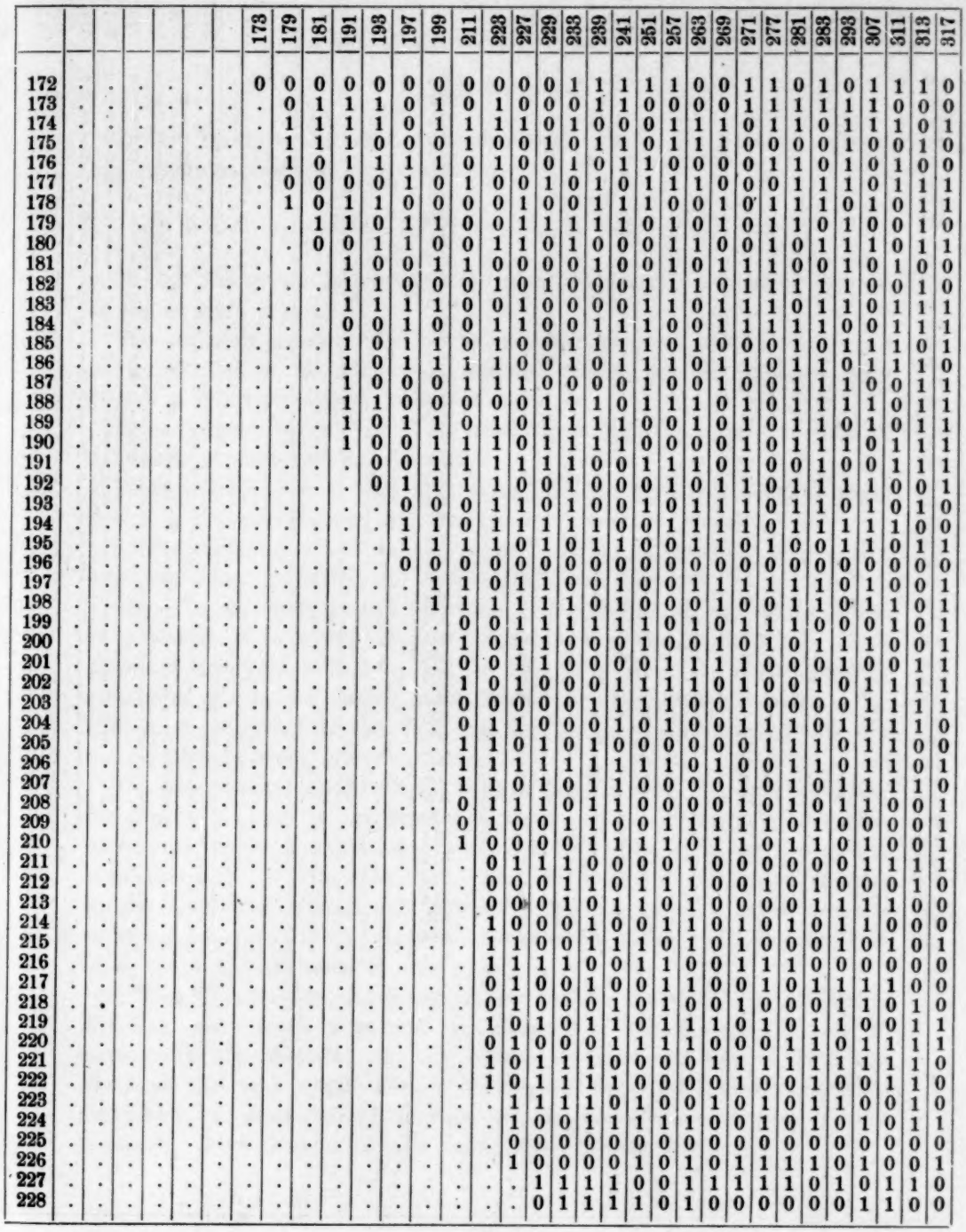
170	.	.	.	.	.	.	.	.	.	1	1	0	0	0	1	1	0	0	1	0	0	0	1	1	1	0	1	1	0	1	1	1	0	1	0	1
171	.	.	.	.	.	.	.	.	.	1	0	1	1	1	1	0	1	0	0	1	0	0	1	1	1	1	1	1	1	0	1	1	0	1	0	1



	233	239	241	251	257	263	269	271	277	281	283	293	307	311	313	317
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268	.	.	.	.	.	.	0	0	0	1	1	0	1	0	1	0
269	.	.	.	.	.	.	.	1	1	1	0	0	1	0	1	1
270	.	.	.	.	.	.	.	1	0	1	1	1	0	1	1	1
271	.	.	.	.	.	.	.	.	1	0	1	0	1	1	1	1
272	.	.	.	.	.	.	.	.	1	0	1	0	0	1	1	1

										277	281	283	293	301	311	313	317
273	.	.	.	.	.	.	.	.	.	0	0	1	1	0	0	1	0
274	.	.	.	.	.	.	.	.	.	0	0	1	1	0	0	0	0
275	.	.	.	.	.	.	.	.	.	1	1	0	1	0	1	0	0
276	.	.	.	.	.	.	.	.	.	0	0	1	0	0	1	1	1
277	.	.	.	.	.	.	.	.	.	.	0	1	0	0	0	0	0
278	.	.	.	.	.	.	.	.	.	.	1	0	0	0	0	0	0
279	.	.	.	.	.	.	.	.	.	.	0	1	0	1	1	1	0
280	.	.	.	.	.	.	.	.	.	.	0	0	1	0	0	0	0
281	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	0	0
282	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	1	1
283	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1	0
284	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	0	1
285	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	1	1
286	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	0	0
287	.	.	.	.	.	.	.	.	.	.	.	.	0	0	1	0	1
288	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1
289	.	.	.	.	.	.	.	.	.	.	.	.	0	0	0	0	0
290	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1
291	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0	0
292	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0	1	0
293	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	0
294	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0	0	0
295	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0	1
296	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1
297	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	1
298	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1
299	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	1	1
300	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0	1
301	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0	0
302	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0	0
303	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1
304	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0	1
305	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1	0	1
306	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	0
307	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0
308	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	0
309	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1
310	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	0
311	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	0
312	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0	1
313	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0
314	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
315	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
316	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	0

ANHARMONIC GROUPS.

By A. S. Hathaway.

1. By "reciprocals" and "complements" $Rx = 1/x$, $Cx = -1-x$ one generates a group of six elements:

$$(x) = (x,\ RCx,\ CRx,\ Rx,\ Cx,\ RCRx = CRCx)$$

such that the group is repeated by substituting for x any other element of the group. Hence:

Two different groups have no number in common. All elements of a group are real, or all imaginary, and all rational or all irrational, with respect to a predetermined rational domain Ω.

By a "square" or a "cube" we shall mean only the square or cube of a number of such rational domain.

When $a+b+c = 0$, the six ratios between any pair form a group "viz." in the above order, a/b, b/c, c/a, and their reciprocals.

2. The elements of a real group are the six anharmonic ratios, of a four-point range, $(ABCD) = AB\cdot CD/AD\cdot BC$, *etc.*, in which $+1$ is harmonic. There is an alternate theory in which -1 is harmonic, and the negatives of elements of a group in one theory is a group in the other. Such alternate theory is defined by $Rx = 1/x$, $Cx = 1-x$, so that in general, a function Fx in one theory corresponds to $-F(-x)$ in the other.

3. A *major* group is one that contains no *equal* elements. There are three and only three *minor* groups:

The two rational groups (0), (1), each of three unequal elements (including the form $R0 = \infty$ in the first), and the group (ω) of the imaginary cube roots of 1, *which is rational or irrational, as* -3 *is square or non-square,* since $(2\omega+1)^2 = -3$. Defining $\varepsilon = 2$ or 1, as -3 is square or non-square (ω rational or irrational) the number of *finite, rational, minor elements* is $2\varepsilon+3$ (0, -1, 1, -2, $-1/2$ and sometimes ω and ω^2). These definitions of ω and ε are unaltered in this paper.

4. The F parameters of a group (x) are the six functions $F(x)$, $F(RCx)$, $F(CRx)$, etc., (each argument an element of the group). If two F parameters are identical, the six reduce to three; to two, if three are identical, and to a single value or *invariant of the group* if all are identical. Obviously, *a symmetrical function of the non-identical values of any parameter is an invariant.* We define three parameters:

$$Ux = 1+x+x^2,\quad Vx = x+x^2 = Ux-1,\quad Wx = (x-1)(x+2)(2x+1),$$

and then identically, $Wx^2 = 4Ux^3 - 27Vx^3$ ($Wx^2 = Wx \cdot Wx$, not $W(x^2)$, etc.).

U and V are three-valued, since identical for arguments x, Cx, viz.

$$Ux,\ URCx = Ux/(x+1)^2, \quad UCRx = Ux/x^2,$$
$$Vx,\ VRCx = Vx/(x+1)^3, \quad VCRx = Vx/x^3,$$

and identically, $2x+1 = Ux/URCx - Ux/UCRx$.

5. The product of the three U parameters above is an *invariant*, whose reciprocal is

$$Ix = Vx^2/Ux^3 = \text{"}index\text{"}\ of\ (x),$$

this determines two other invariants, $Jx = Ix/I(1) = 27Ix/4$ and $Hx = 2 - 27Ix = 2(Wx^2 - 27Vx^2)/(Wx^2 + 27Vx^2) = \text{"}harmonic\text{"}\ of\ (x)$.

(*a*) **Even and odd groups.** Note that the index and three U parameters of a *rational* group are *all* squares or *all* non squares, and rational groups will be named *"even"* when these are squares, *"odd"*, when non squares. Similarly, a rational group is *cubic* or *non cubic* as the index and three V parameters are *all* cubes or *all* non cubes. Every rational group is either even or odd, cubic or noncubic.

6. *A given group corresponds to one and only one index* (*and similarly for a J or H invariant*). For, $Ix = k$ is an equation of sixth degree for the six elements of the group of index k. It has equal roots for the minor indices only, $I0 = 0$, $I(1) = 4/27$, $I(\omega) = \infty$, (reducing for *finite, unequal* roots to $Vx = 0$, $Wx = 0$, $Ux = 0$). The defined invariants are infinite for the group (ω), and invariant relations do not apply to this group.

7. **Roots and periods.** The *"roots"* of a group (x) are the two reciprocal roots of the quadratic, $r + r^{-1} = Hx$. The cube roots of its roots are the *subroots* of (x), $r^{\frac{1}{3}} = s$, ωs, $\omega^2 s$, etc., and the *periods* of (x) are the three periods $\omega^\alpha s + 1/\omega^\alpha s$, $\alpha = 0, 1, 2$, the same three for all subroots s.

(*a*) *The roots of a group are rational functions of any element x and ω, and each subroot s is a rational function of a corresponding element x and ω, linear as to x and s.*

For solving $r + r^{-1} = Hx = 2(Wx^2 - 27Vx^2)/(Wx^2 + 27Vx^2)$ we find

$$r = \frac{[Wx + 3(2\omega + 1)Vx]^2}{Wx^2 + 27Vx^2},$$

a rational function of x and ω, whence,

$$r = (x - \omega^2)^6/Ux^3 = (x - \omega^2)^3/(x - \omega)^3.$$

Hence, *corresponding to x, we have a subroot* $s = (x-\omega^2)/(x-\omega)$ a rational function of x and ω linear in x and s, which gives $x = \omega\ (s-\omega)/(s-1)$.

Note that *this x is the algebraic solution of the sextic* $Ix = k$ *by* $s^3+s^{-3}=2-27k$. It is easily verified that ωs and RCx, $\omega^2 s$ and CRx, $1/s$ and Cx, etc., correspond by the same linear equation as s and x.

8. *Each period of a group (x) is a rational function of x and conversely x is a rational function of the periods. Hence, a group is rational when and only when its periods are all rational.*

For, from corresponding values $s = (x-\omega^2)/(s-\omega)$, etc., we find the periods,

$$s+s^{-1} = 2-3/Ux,\quad \omega s+1/\omega s = 2-3/URCx,\quad \omega^2 s+1/\omega^2 s = 2-3/UCRx$$

which are rational functions of x. Conversely, from the values of Ux, $URCx$, $UCRx$, and the identity of §4 (noting that the sum of the periods is zero), $x = (p_2+1)/(p_1-2)$, p_1, p_2, being the first two periods above.

(*a*) *When ω is rational* ($\varepsilon = 2$), *a group (x) is rational when and only when its subroots are all rational.*

(*b*) *When ω is irrational* ($\varepsilon = 1$), *the subroots of a rational major group are all irrational.* For, if a subroot s be rational, the period $\omega s + 1/\omega s = \omega\,(s-s^{-1}) - 1/s$ is irrational (and the group also) except $s = \pm 1$.

Note. The periods of (x) are the roots of the cubic $t^3-3t = Hx$, which are therefore all real when and only when x is real. Thus, when (x) is real, $-2 \leqq Hx \leqq 2$, $0 \leqq Ix \leqq \frac{4}{27}$, $0 \leqq Jx \leqq 1$.

9. **Resultant groups.** A product of the roots of given groups (one root from each group) determines the root of *a resultant group.*

Since subroots are cube roots of roots, a similar product of subroots determines a subroot of a resultant group. We have seen that an element x corresponds to a subroot $(x-\omega^2)/(x-\omega)$, and y to $(y-\omega^2)/(y-\omega)$ whose product is $(z-\omega^2)/(z-\omega)$ where $z = (xy-1)/(x+y+1)$. Thus, *resultant groups are rational when the component groups are rational.* Since any group may be repeated to make two or more components, we have:

(*a*) *A group is rational when its root is an integral power of a root of a rational group, or a product of integral powers of roots of rational groups.*

Since the two roots of a group are reciprocal numbers:

(*b*) *A rational group is the resultant of rational groups, all but one of which may be arbitary.* For if r be the given root, s, t, the roots of the arbitrary rational groups, then $u = rst$ is the root of a rational group, and $r = us^{-1}t^{-1}$ is the root of a resultant of the rational groups of roots s, t, u.

10. **Modular groups and exponents.** A root of unity determines a *modular group* and the exponent to which such root pertains is *the exponent* of the group. Only modular groups have exponents, and of minor groups, (0), (1) are modular groups (exponents 1, 2) and (ω) is not modular. We let M denote all modular groups, and M_n, all *rational* modular groups, supposed $n+2$ in number, so that n is defined as "*the number of major M-groups in the rational domain* Ω".

(*a*) *A resultant of M groups is an M group, of* M_n *groups, an* M_n.

11. *If given M groups of exponents h, k, be rational, then all M groups are rational whose exponents are divisors of the least common multiple of h and k.*

In the first place, all groups of exponents h, k, are rational, since their roots are integral powers of the roots of the given groups. We can find two roots ϱ, σ, of exponents h, k such that the exponent of $\varrho\sigma$ is m, the least common multiple h and k, whose integral powers therefore determine all roots whose exponents are divisors of m, and these determine rational groups, since ϱ, σ, do so (§ 9*a*).

12. *The domain* M_n *consists of those groups, and those only, whose exponents* are divisors *of* $2(n+1)$.

Let $2m$ be the least common multiple of all exponents of the M_n groups. Thus (§ 11), these exponents consist of all divisors of $2m$, and those only, or the roots of the groups are the roots of $x^{2m}=1$. The number of M_n groups is therefore $m+1$, the number of harmonics $r+r^{-1}$ of the roots of $x^{2m}=1$. Since this number is $n+2$ by hypothesis, therefore $2m=2(n+1)$.

Incidentally, $\sum \operatorname{tot} q_r = 2n$, where q_r is a divisor, greater than 2, of $2(n+1)$. For, $\operatorname{tot} q_r$ is the number of roots of exponent q_r, which is twice the number of groups of that exponent, exponents 1 and 2 excepted.

13. Hence the harmonics of the n major groups of M_n are the roots of $H_n z$, which is defined by

(*a*) $H_n(r+r^{-1}) = (r^{2(n+1)}-1)/r^n(r^2-1)$, a function of n'th degree.

(*b*) $H_n z = \sum(-1)^s(n-s)!\, z^{n-2s}/s!\,(n-2s)!$, $s=0$ to $n/2$ or $(n-1)/2$.

For, by (*a*), $H_{m+2}z = zH_{m+1}z - H_m z$, $m=1, 2, \cdots$, so that $H_n z$ is a symmetric function of n'th degree in the roots of $t^2 = zt-1$, and hence of the form $\sum(ArBs)z^r(-1)^s$, where A, B, are undetermined constants of the gamma coefficient $(Ar\,Bs) = (Ar+Bs)\,\Gamma(r+s)/\Gamma(r+1)\,\Gamma(s+1)$. From $H_1 z = z$, $H_2 z = z^2-1$, we find $A=B=1$, and thence* (*b*).

14. It appears that every domain Ω is a domain $R(\varepsilon, n)$, where $\varepsilon = 2$ or 1 as it does or does not contain ω, and n is the number of its contained major M groups. Since the powers of a primitive root ϱ of $x^{6(n+1)}=1$,

* Cf. Bull. Amer. Math. Soc., vol. 22, pp. 9, 43.

are subroots of M_n groups, and $\varrho+\varrho^{-1}$, etc., are binary periods of the same, the minimum domain $R(\varepsilon, n)$ is $\Omega\ (1, \varrho)$ or $\Omega\ (1, \varrho+\varrho^{-1})$ as $\varepsilon = 2$ or 1. Also, if σ be a root of exponent k prime to $6(n+1)$, $\Omega(1, \varrho, \sigma+\sigma^{-1})$, $\Omega(1, \varrho+\varrho^{-1}, \sigma)$ are respectively domains $R(2, n)$, $R(1, n)$, but $\Omega\ (1, \varrho, \sigma)$, $\Omega(1, \varrho+\varrho^{-1}, \sigma+\sigma^{-1})$ are domains R(2, n'), R(1, n'), $n'+1 = k(n+1)$.

(*a*) *The number of major M groups whose harmonics are rational is* $3n+2$ *for the minimum domain* $R(\varepsilon, n)$ (*noted above*), *and is* $3n+2+k-1$, when as above $\sigma+\sigma^{-1}$ or σ *is adjoined* as $\varepsilon = 2$ or 1. For, the rational binary periods of roots of unity in the minimum domain $R(\varepsilon, n)$ are those of $x^{6(n+1)} = 1$, of which $3n+2$ determine major groups, and in the $R(\varepsilon, n)$ domain of adjoined periods or roots of $x^k = 1$, the added rational binary periods are those of $x^{2k} = 1$ which determine major groups (since k is odd, $\pm(\varrho+\varrho^{-1})$, rational).

15. A *group transformation* Gx is a function that transforms a group into a group. Hence it must satisfy certain identities $GTx = T'Gx$, where T, T' are generated by R, C. The identities $GRx = RGx$, $GCx = CGx$, determine a normal group transformation Gx.

The linear group transformations will be determined simply by the fact that *minor group must transform into minor group*, and are the six ratios of $x-1$, $x+2$, $-2x-1$; and since these are numbers whose sum is zero, their ratios form a group, *i. e. all linear transformations are equivalent as to groups, but different as to elements.*

16. The linear group transformation,

$$Kx = -(x+2)/(2x+1)$$

is *normal* ($KR = RK$, $KC = CK$), and also $KKx = x$, so that we may define x, Kx, as *mutually conjugate numbers* and (x), (Kx) as *mutually conjugate groups.* The following are conjugate relations:

$$UKx = 3Ux/(2x+1)^2, \quad VKx = -Wx/(2x+1)^3, \quad WKx = 27Vx/(2x+1)^3,$$

whence

$$IKx = Wx^2/27Ux^3 = 4/27 - Ix, \quad JKx = 1-Jx, \quad HKx = -Hx.$$

"The sum of corresponding invariants of conjugate groups is constant."
"The sum of corresponding roots of conjugate groups is zero."

(*a*) *Groups occur in conjugate pairs, minor to minor,* (0) *to* (1), (ω) *to itself, and hence, major to major; also real to real, rational to rational. and modular to modular.*

17. There is one and only one major selfconjugate group, since its index is 2/27. Since its root satisfies $r^2+1 = 0$, it is a modular group of exponent 4, and therefore *rational or irrational as* $n+1$ *is even or odd.*

It is readily determined (by $Rx = Kx$ or $x = -2+\sqrt{3}$) that it is the group $(-2+\sqrt{3}) = (-2-\sqrt{3})$, which is rational or irrational as 3 is square or non-square, so that,

(*a*) *3 is square or nonsquare as* $n+1$ *is even or odd.*

(*b*) *Rational conjugate groups are both even or both odd when* $n+1$ *is even, and one even one odd or both odd, when* $n+1$ *is odd.*

From $UKx = 3Ux/(2x+1)^2$, and 5*a*, 17*a*.

18. **Products.** We define a "product" group transformation,

$$Gx = -x(x+2)/(2x+1) = xKx,$$

and a group product, $(x)(Kx) = (xKx)$, so that elements of the product are products of conjugate elements of the factors. The transformation G is normal, (G commutative with R, C). If the root of (x) is r, the root of (xKx) is r^2, so that (xKx) is one resultant of (x) with (x) of which the other is (0). Since identically, $x+Kx = 2CGx$, *the elements of* (Gx) *are the arithmetical means of the conjugate elements of its factors.*

19. *A given group* (a) *resolves in one and only one way into conjugate factors* $(x)(Kx) = (a)$*; and such factors are rational when and only when* (a) *is an even group.*

For we must have conjugate numbers, $xKx = a$. This is a quadratic whose roots are conjugate elements, $x = Ca \pm \sqrt{Ua}$, and rational when and only when a is rational and Ua is a square, or (a), an even group (5*a*).

Such conjugate factors of (a) are its "*primary*" factors. The primary factors of each of these determine four "*secondary*" factors, etc., so that a group (a), has 2^k unique kth order factors.

20. *The product of major conjugate groups is a major group, the major self conjugate excepted, whose self product is the minor group* (1).

For, the primary factors of the minor groups are,

$$(0) = (0)(1) \qquad (\omega) = (\omega)(\omega), \qquad (1) = (-2+\sqrt{3})(-2-\sqrt{3}).$$

We have the product relations

$$UGx = Ux^2/(2x+1)^2, \qquad VGx = VxWx/(2x+1)^3,$$

$$WGx = -\mathrm{H}xUx^3/(2x+1)^3,$$

$$IGx = Vx^2Wx^2/Ux^6 = 27Ix \cdot IKx = Ix(4-27Ix),$$

$$JGx = 4Jx(1-Jx), \qquad HGx = Hx^2-2.$$

From the last identity, we have in terms of the roots, $r+r^{-1} = Hx$,

$$(a) \qquad HG^m x = r^{2^m}+r^{-2^m}, \qquad m = 1, 2, 3, \cdots.$$

21. *If* (a) *be a group of exponent* k, *then as* k *is even or odd, the exponent of* (a) (Ka) *is* $k/2$ *or* k, *and the exponents of the primary factors of* (a) *are both* $2k$ *or one* k, *the other* $2k$.

For if r be a root of (a) the root of the product is r^2, and of the factors, $\pm\sqrt{r}$ (20a).

(a) *The domain* M *contains all products and all factors of its groups, the domain* M_n, *all products of its groups (making the even* M_n *groups) and all primary factors of its even groups, but no factors of its odd groups.*

22. **Subscripts of modular groups.** We define the *subscript* of a group of exponent k as the number $2(n+1)/k$. Thus the rational condition is that the subscript is an integer (§ 12).

(a) *A rational modular group is even or odd as its subscript is even or odd.* For, it is even or odd as its primary factors are rational or irrational. If k be its exponent, the exponent of one primary factor is $2k$ (21), which is rational only when $2(n+1)/2k$ is an integer, or $2(n+1)/k$ an *even* integer.

(b) *The exponents of all even* M_n *groups are the divisors of* $n+1$ *and the equation for their harmonics* $z = r + r^{-1}$ is

$$H_k z = 0 \quad \text{or} \quad H_k z + H_{k-1} z = 0, \quad \text{as} \quad k = (n-1)/2 \quad \text{or} \quad n/2.$$

For, by definition of $H_n z$ (§ 13), we have identically,

$$H_k z = (r^{2(k+1)} - 1)/r^k (r^2 - 1), \qquad H_k z + H_{k-1} z = (r^{2k+1} - 1)/r^k (r - 1),$$

and which function determines the harmonics in question depends upon whether $n+1 = 2(k+1)$ or $= 2k+1$.

(c) *When* $n+1$ *is odd, conjugate groups of* M_n *are one even, one odd.* Let (a) be a group of M_n and let k be the exponent of (a) (Ka). Since this is an even group, its subscript, $2(n+1)/k$ is even, whence k is odd (since $n+1$ is odd by hypothesis). Therefore by § 21, the exponents of (a), (Ka) are one k, one $2k$, (one even one odd).

23. *The number of even major groups of* M_n *is the number of even integers in the series* $1, 2, 3, \cdots, n$, i. e., $(n-1)/2$ *or* $n/2$ *as* $n+1$ *is even or odd.* This follows since k (§ 22 b) is the number of *major harmonics of the even groups.* Also as follows, the even major groups are the products of all pairs of major conjugate factors the self conjugate excepted, which exists in M_n when $n+1$ is even. (17).

(a) *The major self conjugate group is rational and even or odd as* $(n+1)/2$ *is an even or odd integer* (since this is its subscript). Also as follows: such group being rational, $n+1$ is even, and M_n consists of a set of $(n-1)/2$ even groups and another of $(n+1)/2$ odd groups, and the self-conjugate

must be in the set with an odd number of groups since each set will be in conjugate pairs (§ 17*b*).

(*b*) *When* $n+1$ *is even*, $2, -2+\sqrt{3}, -2-\sqrt{3}$ *are square or nonsquare as* $(n+1)/2$ *is even or odd.* For these numbers appear to square factors in the index and U parameters of the major selfconjugate.

24. **Numbers of rational modular groups.** The following table of numbers will represent the $n+2$ groups in M_n so that even numbers represent even groups, odd numbers odd groups (§ 23), and vertical pairs (of sum $n+1$) represent conjugate groups:

$$\begin{array}{cccccccc} 1 & 2 & 3 & \cdots & n/2 & \text{or} & (n+1)/2, & 0 \\ n & n-1 & n-2 & \cdots & 1+n/2 & \text{or} & (n+1)/2, & n+1 \end{array}$$

The number of the even group (0) is 0, the numbers of its conjugate (1) and of the self conjugate $(-2+\sqrt{3})$ when it exists in M_n, are their subscripts $n+1$, $(n+1)/2$ (22*a*). We next assign an *even* number to an *even* group, and to its conjugate, the difference of that number from $n+1$, until all even numbers are assigned, which exhaust as required all numbers and groups when $n+1$ is *odd* (§ 22*c*), but when $n+1$ is *ever* only the even numbers and even groups (§ 17*b*), and the odd numbers and odd groups are exhausted similarly. Note that when the sum of two numbers is $n+1$, they are both even or both odd when $n+1$ is even, and one even one odd when $n+1$ is odd.

25. **Evennes.** The *evennes* of a rational modular group is the evennes of its subscript (an m-even integer being one with 2^m as relatively prime factor, so that such number is even or odd as $m>0$ or $m=0$).

(*a*) *A group of evenness* m *and even exponent factors rationally into* 2^h *factors or order* $h \leqq m$, *each of evenness* $m-h$. For, since its exponent k is even, the exponents of its two primary factors are $2k$, of its four secondary factors $4k$, of its 2^h factors of order h, $2^h k$ (§ 21), and since 2^m is relatively prime factor of $2(n+1)/k$, 2^{m-h} is relatively prime factor of their common subscript $2(n+1)/2^h k$, (an integral subscript determines rational factors).

26. **Cycles.** A cycle of order m is a set of m different groups:

$$(x)\ (Gx)\ \cdots\ (G^{m-1}x)+$$

where $+$ signifies that the next term is $(G^m x) = (x)$.

It is necessary and sufficient for such cycle that any component group (x) is *a modular group, of odd exponent* k *that is a primitive divisor of the form* $2^m \pm 1$ *for the given order* m. For, $(G^m x) = (x)$, gives $HG^m x = Hx$, i. e., $r^{2^m \pm 1} = 1$, where r is a root of the group (x) (§ 20, *a*).

The common exponent k of every group of a cycle will be called *the exponent of the cycle.* The exponent determines the order, say $m = \text{ord. } k$. Conversely the solutions of ord. $k = 1, 2, 3, 4,$ are $k = (1, 3), 5, (7, 9), (15, 17), \cdots$.

The number of all cyclic groups of order k is the number of harmonics $r+r^{-1}$ of exponent k or $(\text{tot. } k)/2$. Also, the number of all groups whose orders are divisors of m is 2^m, the degree in Hx of $HG^m x = Hx$.

27. **Multiplication table of rational modular groups.** This is a table which shows the products of all conjugate pairs, and the order of evenness of groups by their numbers, in further extension of § 24.

A group pertains to a number of the same evenness, except a cyclic group, which possesses the maximum evenness (that of $2(n+1)$) and corresponds to a number of the same or higher evenness.

When $n+1$ is odd, we illustrate by $n+1 = 15$, the exponents of rational cycles are 15, 5, 3, 1, of orders 4, 2, 1, 1 and the table is,

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 14 & 13 & 12 & 11 \end{matrix} + \begin{matrix} 5 & 6 \\ 10 & 9 \end{matrix} + \begin{matrix} 7 \\ 8 \end{matrix} + \begin{matrix} 0 \\ 15. \end{matrix}$$

The highest evenness, possessed only by cyclic groups, is 1, whose numbers are even, and their conjugates, odd numbers. Assigning 14 to any group of the cycle of fourth order, and 1 to its conjugate, then

$$1\times 14 = 2, \quad 2\times 13 = 12, \quad 3\times 12 = 4, \quad 4\times 11 = 14, \text{ etc.}$$

Incidentally, we have $q-1 = \sum \text{tot. } q_r$, where q is an odd integer, and $q_r > 1$ is any divisor of q. For, each member is the number of major groups in the domain M_{q-1}.

28. When $n+1 = 2^k(2g+1)$, the divisors of $2g+1$ are *the exponents of the rational cyclic groups.* These and their conjugates are therefore the same groups that make the primitive table $n+1 = 2g+1$, as above, but their numbers are 2^k times their primitve numbers since evenness has been increased by k (21). Of the unassigned number pairs between two successive pairs thus assigned by the primitive table, those on the left of the conjugate-cyclic group $2^k(2s+1)$ are its rational factors, and the middle pair $[2^{k-1}(4s+1)$ or $2^{k-1}(4s+3)$, and its difference from $n+1]$ have the proper evenness ($k-1$, § 25*a*) for its primary factors (assigned in any predetermined order). Thence, for an assigned group $2^r(2s+1)$ $r<k$, the middle unassigned pair on its left or right as it is in the upper or lower line of the table $[2^{r-1}(4s+1)$ and its difference from $n+1]$ have the proper evenness for its primary factors, $(r-1)$, assigned again in any predetermined order.

For illustration, we give part of the table for $n+1 = 60$ for convenience, marking, the conjugate cyclic group (of evenness 2) with the multiplication sign $\times$ on its left, to indicate that the preceding intermediate groups are its factors.

$$\begin{matrix} 17 & 18 & 19\times 20 & 21 & 22 & 23 & 24 \\ 43 & 42 & 41 & 40 & 39 & 38 & 37\times 36 \end{matrix} + \begin{matrix} 25 & 26 & 27\times 28 \\ 35 & 34 & 33 & 32 \end{matrix} + \begin{matrix} 29 & 30 & 0 \\ 31 & 30\times 60. \end{matrix}$$

For example, $20 = 18\times 42$, $18 = 17\times 43$, $42 = 19\times 41$, etc.

29. **Domain of a prime modulus** $p>3$. The preceding theory applies to this domain by interpreting "equal" as "congruent, mod p". The rational domain consists of the p residues $0, \pm 1, \pm 2, \ldots \pm(p-1)/2$, (and their congruents). It includes the rational domain $\Omega(1)$ by defining the fraction a/b as the solution of the congruence $bx = a$, mod p, which, if a is prime to p, and b is not, becomes the form ∞ of this theory. It also includes square and cube roots, etc., as symbolic solutions of $x^2 = a$, $x^3 = b$, etc., which have rational solutions when a, b, are respectively "quadratic" and "cubic" residues mod p, which are equivalent, therefore, to the preceding terms "square" and "cube", as applied to this domain.

Thus, -3 is square or non-square (quadratic or non quadratic residue mod p) as ω is rational or irrational, or as $\varepsilon = 2$ or 1. Hence, the number of rational and finite minor residues is $2\varepsilon+3\,(p>3)$ and including with these the $6n$ major residues, we have $p = 6n+2\varepsilon+3$.

30. Beginning with a rational group (a), the series of groups (a), (Ga). ..., must finally repeat, since every group is rational, and there are only p finite rational elements. Thus, *in the domain mod p, every rational group,* (ω) *excepted, is a modular group whose exponent is a divisor of* $2(n+1)$. The preceding table for $n+1 = 15$ would apply for $p = 89$ $(\varepsilon = 1)$ and may be assigned in groups (x) as follows:

$$\begin{matrix} (2) & (27) & (23) & (7) \\ (4) & (6) & (8) & (28) \end{matrix} + \begin{matrix} (5) & (13) \\ (24) & (16) \end{matrix} + \begin{matrix} (3) \\ (12) \end{matrix} + \begin{matrix} (0) \\ (1). \end{matrix}$$

Similarly for $n+1 = 60$, $p = 361$, $\varepsilon = 2$.

31. *There are (mod p) p finite rational harmonics* $r+r^{-1} = 0, \pm 1$, etc. Of these (excluding ± 2), $p-2$ are major harmonics; $3n+2$ are binary periods of $x^{6(n+1)} = 1$, whose roots are therefore rational or irrational mod p as $\varepsilon = 2$ or 1, and by $(14a)$, the remaining, $k-1 = p-2-(3n+2)$ $= 3n+2\varepsilon-1$ major harmonics, are the major binary periods of $x^{6n+4\varepsilon} = 1$, whose roots are therefore rational or irrational, mod p, as $\varepsilon = 1$ or 2. Hence, whether $\varepsilon = 2$ or 1, the roots of $x^{p-1} = 1$ and

binary periods of $x^{p+1} = 1$ are rational mod p, and roots of the latter irrational, *i. e., the rational domain, mod p, contains the domain* $\Omega(1, \varrho, \sigma+\sigma^{-1})$, where ϱ, σ, are primitive roots of $x^{p-1} = 1$, $x^{p+1} = 1$. (Cf. §§ 8, 14.)

32. *We have the identity,*

$$(z^2-4)\, H_k z \,.\, H_{k-1} z \equiv z^p - z, \mod p, \qquad 2k = p-1,$$

for the roots of each member mod p, are $0, \pm 1, \pm 2, \ldots, \pm k$.

The analytical identity is in fact, for any odd integer p,

$$(z^2-4)\, H_k z\, H_{k-1} z = z^p - z + p \sum (-1)^s \frac{(p-1-s)!}{s!\,(p-2s)!}\, z^{p-2s}$$
$$s = 1, 2, \ldots, k.$$

For, the first member, for $z = r+r^{-1}$ is $r^p + r^{-p} - (r+r^{-1})^p$ $(13a)$, and defining $F_p(z) = r^p + r^{-p}$, it will have the same difference equation as $H_p z$, and only differs from it in the constants A, B, which are $A = 1$, $B = 2$.

33. The integral symmetric functions of the roots of the cubic $t^3 = tUx + Vx$, are of the form $\sum (A\alpha B\beta)\, Ux^\alpha Vx^\beta$, $2\alpha + 3\beta =$ weight $=$ degree $= m$. For the symmetric s_m, $A = 2$, $B = 3$, and for the homogeneous products π_m, $A = B = 1$. Also the roots of the cubic are $t = x+1$, $-x$, -1. Hence,

$$s_m = (x+1)^m + (-x)^m + (-1)^m = m \sum \frac{(\alpha+\beta-1)!}{\alpha!\,\beta!}\, Ux^\alpha Vx^\beta,$$
$$(2\alpha + 3\beta = m)$$

$$(a)\quad \pi_m = [(x-1)(x+1)^{m+2} + (x+2)(-x)^{m+2} - (2x+1)(-1)^{m+2}]\, Wx$$
$$= \sum \frac{(\alpha+\beta)!}{\alpha!\,\beta!}\, Ux^\alpha Vx^\beta, \qquad (2\alpha + 3\beta = m)$$

For $m = p = 6n + 2\varepsilon + 3$, the values of α, β satisfying $2\alpha + 3\beta = p$ are

$$\alpha = 3(n-r) + \varepsilon, \qquad \beta = 2r+1, \qquad r = 0, 1 \ldots, n.$$

These will be retained as the values of α, β, with summations $r = 0$ to $r = n$ and p is a prime unless otherwise stated. Other degrees besides $2\alpha + 3\beta = p$ may be so expressed, as $2\alpha + 3(\beta-1) = p-3$, $2(\alpha-1) + 3(\beta+1) = p+1$, but in the latter case, there is a term outside of Σ, corresponding to the non negative values $\alpha - 1, \beta + 1$ for $r = -1$.

34. Let $A_r = \dfrac{(\alpha+\beta-1)!}{\alpha!\,\beta!}$. This is an integer, since $2\alpha + 3\beta = p = a$ *prime.* Then,

(a) $$s_p = (x+1)^p - x^p - 1 = pUx^\varepsilon Vx \sum A_r Ux^{3(n-r)} Vx^{2r}$$
$$= pUx^{3n+\varepsilon} Vx \sum A_r Ix^r, \quad (Ix = Vx^3/Ux^3)$$

(b) $$Vx\,Wx \cdot \pi_{p-3} = x(x-1)s_p - 2Ux(x^p - x) = Vx\,Wx\,Ux^{3n+\varepsilon} \cdot \sum \beta A_r Ix^r$$

(c) $$s_{p+1} = (x+1)^{p+1} + x^{p+1} + 1 = Ux^{3n+\varepsilon+2}\left[2 + (p+1)\sum \frac{\alpha}{\beta+1} A_r Ix^{r+1}\right]$$

35. *The function of n'th degree, $\sum \beta A_r y^r$ resolves, mod p, into the n linear factors $y - Ix$, corresponding to every major group (x).*

For, the second member of (34*b*), and therefore the third, will be divisible by p for every residue x, but for a major residue x, $Ux\,Vx\,Wx$ is prime to p, and hence $\sum \beta A_r Ix^r$ is divisible by p for every major residue x.

Since $2\alpha + 3\beta = p$, $\sum \alpha A_r y^r$ and $\sum (\alpha+\beta) A_r y^r$ have the same resolutions mod p.

(a) *Identically, mod p,* $H_n(2 - 27y) \equiv H_n 2 \cdot \sum \beta A_r y^r$, since they have the same roots, $y \equiv Ix$, mod p. (x a major residue.)

36. *The functions of degree $n+1$,*

$$\sum \frac{\beta}{r+1} A_r y^{r+1} \quad \text{and} \quad \sum \frac{\beta}{r+1} A^r y^{r+1} - 16/3$$

completely resolve into finite linear factors $y - Ix$, mod p, but in the first, Ix is a quadratic residue (square) mod p, and in the second, a non quadratic residue (non square). Also, $y - Ix$ or $(y - Ix)^2$ is highest factor, as x is minor or major residue.

The second member of (34*c*) is $2Ux$, mod p, (Ux *not* $= 0$, since Ix is finite. Also, $2\beta + 1 = 2(r+1)$, $2\alpha = -3\beta$, and $Ux^{(p-1)/2} = 1$ or -1, as Ux is square or nonsquare, mod p, and therefore Ix the same (5*a*). Thus, the functions named are divisible by p for $y = Ix$, respectively as Ix is square or nonsquare, mod. p. Also, a root is repeated when their common derivative, $\sum \beta A_r y^r = 0$ for the same root, or when x is a major residue, and each such root repeated once gives $n+1$ factors of first degree, viz. $2 + 2(n-1)/2$ and $2(n+1)/2$ if $n+1$ be even, and $1 + 2n/2$ if $n+1$ be odd.

Note that unrepeated major square indices, mod p, are the roots of $H_k(2 - 27y) = 0$ or $H_k(2 - 27y) + H_{k-1}(2 - 27y) = 0$, as $k = (n-1)/2$ or $= n/2$ (22*b*).

37. To transform these congruences into those whose roots are $z = Jx$, we substitute $y = 4z/27$. *Defining* $Fz = \sum \frac{2r+1}{r+1} B_r z^r + 1$, *where,*

$$B_0 = 1, \quad B_{r+1} = B_r f(6r), \quad fr = \frac{(r+3)(r+5)(r+7)}{(r+6)(r+9)^2}$$

we have $F(z) \equiv 0$, or $F(z)-36 \equiv 0$, *with roots* $z = Jx$ *as* (x) *is even or odd with repeated roots for major groups; and* $F'z = 0$ *for roots* $z = Jx$, *of major groups.* In fact, we have $A_r = \left(\frac{27}{4}\right)^r B_r \bmod p$, since

$$(a)\quad A_0 = 1,\quad A_{r+1} = A_r \frac{27}{4} \frac{(6r+3-p)(6r+5-p)(6r+7-p)}{(6r+6)(6r+9)(6r+9-3p)} \equiv \frac{27}{4} A_r f(6r), \quad \bmod p.$$

Note that making the following infinite series

$$(b)\qquad 1+\frac{5\cdot 7}{6\cdot 9}z+\frac{5\cdot 7\cdot 11\cdot 13}{6\cdot 9\cdot 12\cdot 15}z^2+\frac{5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19}{6\cdot 9\cdot 12\cdot 15\cdot 18\cdot 21}z^3+\cdots$$

and selecting *any prime* $p = 6n+2s+3$, the corresponding function $F'z = \sum \beta B_r z^r$ is the first $n+1$ terms of this series.

38. In the preceding notation, we have the following identities to the modulus p:

(a) $Fz(Fz-36) \equiv 36z(z-1)(F'z)^2$ (by identical linear factors mod p),
(b) $F'(1-z) \equiv (-1)^n F'z$, $(JKx = 1-Jx)$,
(c) $F(1-z) \equiv Fz$ or $= 36-Fz$, as $n+1$ is even or odd (17b, 22c),
(d) $F(4z-4z^2) \equiv 4z(1-z)(F'z)^2$, $[JGx = 4Jx(1-Jx)]$.

39. *If* x *be a major residue mod* p, $s_p = (1+x)^p - x^p - 1$ *is not divisible by* p^2 *when the discriminant of* $\sum_0^n A_r y^r$ *is prime to* p. For if s_p is divisible by p^2 for one value of x, it is so for all congruent values, of which one can be found so that π_{p-3} is divisible by p^2 (34a, b), so that $\sum A_r y^r$, $\sum \beta A_r y^r$ have a common factor mod p, whence, from $\beta = 2r+1$, $\sum r A_r y^{r-1}$ and $\sum (n-r) A_r y^r$ have a common factor mod p, *i. e.*, the discriminant of $\sum A^r y^r$ is divisible by p. Similary, s_p *is not divisible by* p^3 *for a major residue* x, *except such discriminant be divisible by* p^2.

40. Assuming that the above discriminant is prime to p, we have some important results. Thus, *the sum of the* p'*th powers of three integers is not divisible by* p^2, *except one of the integers or the difference of the cubes of two be divisible by* p. Also, $x^p + y^p + z^p = 0$ *is not solvable in integers prime to* p, for if so, it is known that $2^p - 2$, $3^p - 3$ are divisible by p^2, and their difference s_p for the major residue $x = 2$, is divisible by p^2, which is not possible when the given discriminant is prime to p. When this discriminant is divisible by p, since the factors of $\sum A_r y^r$ are *square* factors mod p, there cannot be more than $n/2$ *different* linear factors mod p, whence $x^p + y^p + z^p = 0$ is in no case solvable in integers prime to p when either 2 or 3 is a *primitive* residue mod p.

41. The author appears to be in measurable distance of proving that *the discriminant above is always prime to* p, and offers a brief summary of leading features so far obtained.

42. Take positive integers a, b, $p_m = am+b$, and polynomials Px, Qx, with integral coefficients. Then beginning with the (constant) polynomials $L_0x = M_0x = 1$, an infinite series of polynomials L_mx, M_mx with integral coefficients is determined by the identities,

$$(a)\quad \begin{aligned}(x+p_{m+1})L_{m+1}x &= a_mP(x+am)L_mx - b_mQxM_m(x+a)\\ -xM_{m+1}x &= c_mP(x+am)L_mx - d_mQxM_m(x+a)\end{aligned}\quad (m=0,1,2,\cdots).$$

In fact, from the linear factors of the first members a_m and b_m, also c_m and d_m, are definite relatively prime integers defined by:

$$(b)\quad \begin{aligned}\alpha_m a_m &= (-1)^mQ(-p_{m+1})M_m(-p_m), & \beta_m c_m &= Q0\,M_m a\\ \alpha_m b_m &= (-1)^mP(-p_1)L_m(-p_{m+1}), & \beta_m d_m &= P(am)L_m0\end{aligned}$$

where α_m, β_m, are the *positive* greatest common divisors of their integral second members.

43. We may write (42a) with $\Delta_m = a_m d_m - b_m c_m$,

$$(a)\quad \begin{aligned}d_m(x+p_{m+1})L_{m+1}x + b_mxM_{m+1}x &= \Delta_mP(x+am)L_mx,\\ c_m(x+p_{m+1})L_{m+1}x + a_mxM_{m+1}x &= \Delta_mQxM_m(x+a),\end{aligned}$$

whence

$$(b)\quad \begin{aligned}L_{m+1}0 &= \frac{\beta_m\Delta_m}{p_{m+1}} = \Pi_0^m\frac{P(ar)\Delta_r}{p_{r+1}d_r}, \qquad r=0,\ 1,\ 2,\ \cdots,\ m.\\ (-1)^{m+1}M_{m+1}(-p_{m+1}) &= \frac{\alpha_m\Delta_m}{p_{m+1}} = \Pi_0^m\frac{Q(-p_{r+1})\Delta_r}{p_{r+1}a_r}\end{aligned}$$

In particular define Px, Qx, as *quadratics* with *real negative roots*,

$$Px=(x+a+\alpha')(x+a+\alpha),\quad Qx=(x+\beta)(x+\beta'),\quad \alpha'\geqq\alpha>0,\quad \beta\geqq\beta'>0,$$

so that α, α', β, β' are positive integers, or only their symmetrical functions are such, as their respective quadratics are reducible or irreducible. This also holds when defined for a prime modulus p. Further, let the inequalities exist, $b>\alpha'$, $(a+\alpha)>\beta>\alpha$, so that $a>a+\alpha-\beta>0$, and

$$(a)\quad \begin{aligned}&\beta<a+\alpha<a+\beta<2a+\alpha<2a+\beta<\cdots\\ &\cdots<ma+\beta<(m+1)a+\alpha<(m+1)a+\alpha'<p_{m+1}.\end{aligned}$$

The *k'th range* of this series is $ak+\beta<x<a(k+1)+\alpha$ or $x=ak+\beta+\lambda$, where $0<\lambda<a+\alpha-\beta$. We also let λ, μ, with integral subscripts denote numbers of the same range as λ, exclusive of the end values 0, $a+\alpha-\beta$, except $\mu_{m0}=0$, $\lambda_{mm+1}=a+\alpha-\beta$, for every value of m.

44. *The functions* $L_m x$, $M_m x$ *are of the form*

$$L_m x = L_m \Pi_0^{m-1}(x+ar+\beta+\lambda_{mr+1}), \qquad M_m x = \Pi_1^m (x+ar-a+\mu_{mr}),$$
$$L_m > 0, \quad M_m > 0.$$

For, this is true for $m = 1$, and assuming that it is true for a given number m, it is true for the next number $m+1$, as follows (42a),

$$(x+p_{m+1})\, L_{m+1}x = a_m L_m (x+ma+a+\alpha')\ \Pi_0^m (x+ar+\beta+\lambda_{mr+1})$$
$$+ b_m M_m (-x-\beta')\ \Pi_0^m (x+ar+\beta+\mu_{mr})$$

and for $-x M_{m+1}x$, write c_m, d_m, for a_m, b_m.
Note that the other factors of the P, Q, quadratics are included by $r = m$, $r = 0$, respectively in the first and second terms. Also that a_m, b_m, c_m, d_m, are positive by 42b and that all linear factors are positive for $x = -ak-\beta$ or $x = -a(k+1)-\alpha$, except the k or $k+1$ of them given by $r = 0$ to $k-1$ or $r = 0$ to k, which are negative, except a zero factor in the first term for $r = k = m$, and one in the second, for $r = k = 0$. Thus we have, $(-1)^k L_{m+1}(-ak-\beta) > 0$, $(-1)^{k+1} L_{m+1}[-a(k+1)-\alpha] > 0$, and similarly for $M_{m+1}x$ (the necessary and sufficient conditions for the required forms).

45. *The numbers* $(mk) = \mu_{mk}$, μ_{m+1k+1}, λ_{m+1k+1}, λ_{mk+1} ($k = 0$ to m), *are either in ascending or descending order of magnitude,* as may be shown by substituting the k'th roots of the functions in 41a, which determines the *ratios* of their differences (taken one from a following one) as *always positive.* The order is ascending for $k = 0$, $k = m$, ($\mu_{m0} = 0$, $\lambda_{mm+1} = a+\alpha-\beta$), and the author is working on the idea that the series,

$$\mu_{11},\ \mu_{21},\ \mu_{22}\cdots\ \mu_{m1}\ \mu_{m2}\cdots\ \mu_{mm}\cdots\ \lambda_{mm}\,\lambda_{mm-1}\cdots\ \lambda_{22}\,\lambda_{21}\,\lambda_{11}$$

is *ascending,* with a common limit λ between λ and μ terms. This implies that the series (mk) above is *ascending for all values of* k, and will be established by showing that λ_{mk}, μ_{mk} respectively *decrease* and *increase* when k increases by 1 ($k < m$).

46. Without using the above possibility, we can show from the ratio of $L_{m+1}(-\beta')$ to $L_{m+1}0$ that,

$$(a)\ \Pi_0^m (a_r d_r/\Delta_r) < \Pi_6^m (1+s_r),\ s_r = \beta'(b-\alpha')/[a(r+1)+b]\,[a(r+1)+\alpha'-\beta']$$

and hence the limit of the first member for limit $m = \infty$ is a finite number $K > 1$, where $\log K < \beta'(b-\alpha')/a^2 +$ first g terms of $\sum s_r$ and where g is the *smallest number* such that the denominator of s_g is greater than $2a^2$.

Note that K is an infinite product of factors all greater than 1 which must therefore decrease to the limit 1, as m increases.

47. Let $p = an+b = $ *prime number*, and $\sum_0^n A_r y^r$ an integral polynomial satisfying $A_{r+1} \equiv \frac{ar+b}{r+1} \cdot \frac{Q(ar)}{P(ar)} A_r$, mod p. All coefficients are prime to p, if A_0 be so since Px, Qx, are irreducible mod p or resolve into integral factors prime to p as, $Q(ar) = (ar+\beta)(ar+\beta')$, etc.

(*a*) *The discriminant of the preceding function is divisible by* p *only when* p *is a common divisor of* α_{n-1}, β_{n-1}. *The degree of the polynomial whose square is a factor mod* p *of* $\sum_0^n A_r y^r$, *equals the number of successive pairs* α_r, β_r, $r = n-1, n-2, \cdots$, *each with a common factor* p.

For, the discriminant is the resultant of $\sum (n-r) A_r y^r$, $\sum r A_r y^{r-1}$, which may be replaced mod p by $\sum (ar+b) A_r y^r$, $\sum (ar+b) Q(ar) A_r y^r / P(ar)$. The limits for $\sum$ are nominally $r = 0$, $r = n$, with terms $\equiv 0$ mod p for some values of r. Thus in the latter $r = 1$, $r = n$, or substituting $r+1$ for r and the value of A_{r+1} mod p in terms of A_r, the limits become $r = 0$, $r = n-1$, and the term $r = n$ is divisible by p.

Denoting these polynomials by $F_{n-1} y$, $f_{n-1} y$, respectively, we have (42 a, b, $x = ar$).

$$F_{n-k-1} y = a_{k-1} F_{n-k} y - b_{k-1} f_{n-k} y = \sum (ar+b)_{k+1} \frac{L_k(ar) A_r y^r}{P(ar)_k},$$

(*b*)
$$f_{n-k-1} y = \frac{1}{y} [d_{k-1} f_{n-k} y - c_{k-1} F_{n-k} y] = \sum (ar+b)_{k+1} \frac{Q(ar) M_k(ar+a) A_r y^r}{P(ar)_{k+1}}$$

where

$$(ar+b)_{k+1} = (ar+b)(ar+b+a) \cdots (ar+b+ka),$$
$$P(ar)_{k+1} = P(ar) P(ar+a) \cdots P(ar+ka).$$

These functions have the same common factor, mod p, as for $k = 0$, and if such a factor exists, we have by $k = n-1$, mod p, $L_{n-1} 0 \equiv 0$, $M_{n-1} a \equiv 0$, and hence, $L_{n-1}(-p) \equiv 0$, $M_{n-1}(-p+a) \equiv 0$ or by 42*b* with $m = n-1$, α_{n-1}, β_{n-1} have the common factor p. So, by $k = n-2$, a quadratic factor requires $\alpha_{n-2} \equiv 0$, $\beta_{n-2} \equiv 0$, *mod* p, *etc.*

48. $\sum A_r y^r$ becomes the function of §§ 34, 39, by $\alpha = \alpha' = \beta' = 3$, $a = 6$, $b = 2\varepsilon + 3$, $\beta = 9 - 2\varepsilon$, and K (§ 46) is less than $\sqrt{2}$. Also inequalities indicate a value probably less than $p_{m+1} K^2$ for the greatest common divisor of α_m, β_m, and if this be true, p_{m+1} would be greater than one-half such g. c. d. (in the present case an even number for $m = 1, 2, \cdots$) and could not be a divisor of it whether a prime or not since it is odd.

SOME GENERAL DETERMINANT THEOREMS IN TENSOR NOTATION.*

By Clyde M. Cramlet.

In the usual treatment of the theory of determinants certain elementary transformations are shown to leave a determinant unchanged. The problem of determining whether these include all possible transformations which do not affect the value of a determinant will be solved by finding the most general transformations which merely multiply a determinant by a constant. The restrictions on the transformation for which this constant is unity, will be apparent. Incidentally an illuminating proof of the multiplication rule is given.

After proving that all of the fundamental properties of determinants follow from the expansions by cofactors, these expansions are used as a basis for some further theorems. The particular interest attaching to these formulas lies in the fact that they are expressed in the notation of tensor analysis. Should one wish to use for example Laplace's development in general tensor analysis, he should find a *rule* giving the expansion, with emphasis laid upon the form that would revive the determinant. The vanishing expansions are, it will appear, even more important, and a usable set of equations is desirable that will give all expansions.

It will be shown that a determinant a is but one component of a tensor having components of a's, $-a$'s and o's. This gives a clue to the importance of the vanishing expansions.

In the last article an application is made to alternating and to relative tensors.

Eddington† has used the symbol $\varepsilon_{r_1 \cdots r_k}$ in connection with determinants, ascribing to it an alternating character to take care of the sign of a term. A distinct improvement on this is the symbol used by Murnaghan.‡ He shows that

$$\delta^{r_1 \cdots r_k}_{s_1 \cdots s_k} = \sum \delta^{r_1}_{s_1} \cdots \delta^{r_k}_{s_k}$$

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* Presented to the American Mathematical Society June 19, 1925.

† Mathematical Theory of Relativity, § 48. Eddington's proof that this is not a tensor is not sound, as will appear in § 5 of this paper.

‡ See Murnaghan, Amer. Math. Monthly, vol. 32, p. 233.

where the $\sum$ implies that a sum be formed of all of the products of the $\delta^{r_1}_{s_1}$, $\delta^{r_2}_{s_2}$, $\cdots$, $\delta^{r_k}_{s_k}$ that can be formed by permutation of the $r_1 \cdots r_k$ when each product has a $\pm$ sign prefixed according as the permutation is even or odd. It follows immediately that $\delta^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ is a tensor and that its components are the same in every system because it is expressed as a sum of tensors of like character all of which possess the property of having identical components in all systems.* This property is not enjoyed by Eddington's symbol.

The explicit representation of a determinant will be taken in the form $a = \delta^{r_1 \cdots r_n}_{1 \cdots n} \cdot a_{r_1 1} \cdots a_{r_n n}$ where the r's are indices of summation. Any rearrangement of the r's is permissible since they are the umbral indices, and any permutation of one pair of indices on an element, $a_{r_p p}$ with another pair merely rearranges the product and changes the sign of δ an even number of times. This leads to the equation

$$a = \delta^{\varrho_1 \cdots \varrho_n}_{s_1 \cdots s_n} \cdot a_{\varrho_1 s_1} \cdots a_{\varrho_n s_n}, \qquad (s_1 \cdots s_n \text{ not summed}). \tag{1.1}$$

If each term in the expanded form of this equation has all pairs of indices on an element permuted so that all of the ϱ's are brought to some order, say $r_1 \cdots r_n$, then all of the permutations will be among the s's. Corresponding changes on δ will effect an even number of changes of signs. If in addition, the upper and lower indices be interchanged the equation may be written

$$a = \delta^{\sigma_1 \cdots \sigma_n}_{r_1 \cdots r_n} \cdot a_{r_1 \sigma_1} \cdots a_{r_n \sigma_n} \qquad (r\text{'s not summed}). \tag{1.11}$$

Any interchange of the subscripts of δ merely changes a sign, or if two subscripts are identical causes δ to vanish. Hence if a_{rs} are components of a tensor we may write the tensor identity

$$a_{r_1 \cdots r_n \, s_1 \cdots s_n} \equiv \delta^{\varrho_1 \cdots \varrho_n}_{s_1 \cdots s_n} a_{r_1 \varrho_1} \cdots a_{r_n \varrho_n}. \tag{1.12}$$

An abbreviated form for writing this equation may be illustrated by writing (1.12) as

$$a_{(r_n)(s_n)} \equiv \delta^{(\varrho_n)}_{(s_n)} (a_{r_n \varrho_n}). \tag{1.13}$$

As in deriving (1.11) from (1.1) we may show that

$$a_{(r_n)(s_n)} \equiv \delta^{(\varrho_n)}_{(r_n)} (a_{s_n \varrho_n}) \tag{1.14}$$

and thus that

$$a_{(r_n)(s_n)} \equiv a_{(s_n)(r_n)}. \tag{1.15}$$

* Compare the proof by Murnaghan, Bull. Amer. Math. Soc., vol. 31, p. 323, where the true simplicity is obscured. In fact, this property will obtain if, instead of the $\pm$ signs $n!$ arbitrary scalars are prefixed to the products. More will be said about this elsewhere.

The components of this tensor are $\pm a$ when its sets (r_n) and (s_n) are equal, and composed of distinct indices according as these sets can be reduced to the same order by an even or odd number of transpositions. The components vanish when the sets are unequal or when there are any repeated indices. A determinant then from our view point is but a component of this tensor.

From (1.1) immediately .

$$a_{\varrho_1 t_1}\frac{\partial a}{\partial a_{\varrho_1 s_1}} = \delta^{\varrho_1 \cdots \varrho_n}_{s_1 \cdots s_n} a_{\varrho_1 t_1} a_{\varrho_2 s_2} \cdots a_{\varrho_n s_n} \quad (s\text{'s not summed}),$$

$$= \delta^{s_1}_{t_1} \cdot a \tag{1.2}$$

since by (1.1), a is recovered when $t_1 = s_1$, but when $t_1 = s_k \neq s_1$, l_1 and l_k can be interchanged without changing the product of the elements, but with a change of the sign of δ thus permitting all terms to cancel in pairs.

The quantities a^{rs} are introduced by the equation

$$\frac{\partial a}{\partial a_{rs}} = a \cdot a^{rs} \tag{1.3}$$

and (1.2) may be written

$$a_{rt} \cdot a^{rs} = \delta^s_t, \tag{1.4}$$

justifying the contravariant notation. The theory to this point is sufficient for the solution of linear equations, for by the usual method of multiplying the linear set $a_{ir}\, x_r = k_i$ by a^{is} there is obtained $\delta^s_r\, x_r \equiv x_s = a^{is}\, k_i$. These new coefficients can be computed by (1.1) and (1.3). As in the derivation of (1.4) from (1.1) the equations

$$a_{tr}\, a^{sr} = \delta^s_t \tag{1.5}$$

may be obtained from (1.11). These equations enable us to solve the equation $x_s = a^{is}\, k_i$ for the k_i's in the illustration above.

By multiplying (1.3) by da_{is},

$$\frac{da}{a} = a^{is}\, da_{is}.$$

By differentiating (1.5)

$$a_{tr}\, da^{sr} + a^{sr}\, da_{tr} = 0, \tag{1.6}$$

Hence

$$\frac{da}{a} = a^{ij}\, da_{ij} = -a_{ij}\, da^{ij}. \tag{1.7}$$

Introducing $\bar{a} = \frac{1}{a}$, (1.7) gives

$$\frac{d\bar{a}}{\bar{a}} = a_{ij}\, da^{ij}.$$

Hence

$$\frac{\partial \bar{a}}{\partial a^{ij}} = \bar{a} \cdot a_{ij};$$

and

$$a^{rk} \frac{\partial \bar{a}}{\partial a^{ik}} = \bar{a}\, \delta_i^r.$$

If it can be shown that (1.2) incorporates the fundamental properties of a determinant $a \equiv |a_{ik}|$ it will follow that (1.9) defines $\bar{a} = |a^{ik}|$ and that any theorem dealing with a can be interpreted as a theorem of $\bar{a}$ by a mere change of indices.

2. Analogous to (1.2),

$$a_{ir} \frac{\partial \varphi}{\partial a_{ij}} = \varphi \cdot \delta_r^j \tag{2.1}$$

may be considered as a set of n^2 differential equations with n^2 independent variables a_{ir}. By setting $j = r$ but suspending the summation convention, φ is by Euler's theorem, homogeneous of degree one in the elements of the rth column. By multiplying by $a_{rj}\, a^{sr}$ and setting $s = r$, φ is seen to be homogeneous in the elements of the rth row. If the summation is not suspended, φ is seen to be homogeneous of the nth degree in its elements. But the homogeneity valid in Euler's formula is of a more general character than that ascribed to determinants. It might, therefore, be supposed that these equations would define a function more general than determinants. It is the remaining $n^2 - n$ equations, however, which prevent this possibility. Multiplying (2.1) by a^{sr} the equivalent set

$$\frac{\partial \varphi}{\partial a_{sj}} = \varphi \cdot a^{sj}$$

is found and from this may be obtained the single total equation

$$\frac{\partial \varphi}{\partial a_{ir}}\, da_{ir} = \varphi \cdot a^{ir}\, da_{ir}. \tag{2.2}$$

Hence by 1.3

$$\frac{d\varphi}{\varphi} = \frac{da}{a}. \tag{2.3}$$

Conversely (2.1) follows from (2.3), so $\varphi = A \cdot a$ is the general solution of (2.1) where A is an integration constant. Thus the equation (2.1) defines a determinant to within a constant factor.

3. The n^2 equations (2.1) are equivalent to the single equation (2.2). If (2.2) is transformed by the general transformations

$$c_{\mu\nu} = c_{\mu\nu}(a_{11} \cdots a_{nn}) \tag{3.1}$$

the equation (2.2) becomes

$$\left[\frac{\partial \varphi}{\partial c_{\alpha\beta}} - a^{ik} \cdot \varphi \cdot \frac{\partial a_{ik}}{\partial c_{\alpha\beta}}\right] dc_{\alpha\beta} = 0.$$

The condition that this be the same form as (2.2) is that

$$a^{ik} \frac{\partial a_{ik}}{\partial c_{\alpha\beta}} = c^{\alpha\beta}. \tag{3.2}$$

These n^2 equations are equivalent to the one equation

$$a^{ik} \frac{\partial a_{ik}}{\partial c_{\alpha\beta}} dc_{\alpha\beta} = c^{\alpha\beta} dc_{\alpha\beta}$$

for the $dc_{\alpha\beta}$ are arbitrary differentials. This may be written

$$a^{ik} da_{ik} = c^{ik} dc_{ik}. \tag{3.21}$$

To find a solution of (3.2) assume

$$c_{ik} = k_{is} a_{ks}, \tag{3.31}$$

the k_{is} being constants and not tensors. Then $c^{ik} = k^{is} a^{ks}$; and $c^{si} dc_{sk} = a^{is} da_{ks}$ is the set of differential equations obtained by the elimination of constants. The sum of the n equations obtained from these by setting $i = k$ gives (3.21).

In a similar manner

$$c_{ik} = k_{is} a_{sk} \tag{3.32}$$

$$c_{ik} = k_{si} a_{ks} \tag{3.33}$$

$$c_{ik} = k_{si} a_{sk} \tag{3.34}$$

are found to satisfy (3.21). Since (3.21) is equivalent to the set of n^2 equations (3.2) it follows that the general solution contains n^2 arbitrary constants. These equations (3.3) then replace (3.21) as the necessary and

sufficient condition that the set of differential equations (2.1) be invariant, that is, equivalent to

$$c_{ir}\frac{\partial \varphi}{\partial c_{is}} = \varphi \cdot \delta^{j}_{r}. \tag{3.4}$$

Since $\varphi = c$ is a solution of (3.4) it is a solution of (2.1). Since c and a are both solutions of (2.1)

$$c = A \cdot a \tag{3.5}$$

where A is a constant by (2.4).

To determine A set $a_{ks} = \delta^{k}_{s}$ then by (1.1)

$$a = \delta^{(\varrho_k)}_{(s_k)}\left(\delta^{s_k}_{\varrho_k}\right) = \delta^{(s_k)}_{(s_k)} = 1, \qquad (s_k) \text{ not summed}$$

and $c_{ik} = k_{ik}$ by any of the four equations (3.3). Hence $A = k$ where k is the determinant of the k_{ik}.

This gives the multiplication rule which is ordinarily stated,

$$|k_{is}\,a_{ks}| = |k_{is}\,a_{sk}| = |k_{si}\cdot a_{ks}| = |k_{si}\,a_{sk}| = |k_{rs}|\cdot|a_{ij}|.$$

The equation (3.5) has now become

$$c = k \cdot a. \tag{3.51}$$

Since this equation is a consequence of the invariancy of (2.1), the equations (3.3) and (3.51) are equivalent.

4. By a repetition of the method by which (1.2) was derived

$$a_{\alpha_1 s_1}\cdots a_{\alpha_k s_k}\cdot\frac{\partial^k a}{\partial a_{\alpha_1 r_1}\cdots\partial a_{\alpha_k r_k}} = a\cdot\delta^{r_1\cdots r_k}_{s_1\cdots s_k} \tag{4.1}$$

which may be written in the abbreviated form,

$$(a_{\alpha_k s_k})\frac{\partial^k a}{(\partial a_{\alpha_k r_k})} = a\,\delta^{(r_k)}_{(s_k)}. \tag{4.2}$$

This may be recognized as a Laplacian development, giving not only the expansion but the sign of the determinant obtained and all the vanishing cases.

By multiplying by $(a^{\beta_k s_k})$, (4.2) becomes

$$\frac{\partial^k a}{(\partial a_{\beta_k r_k})} = a\cdot\delta^{(r_k)}_{(s_k)}\cdot(a^{\beta_k s_k}).$$

Accordingly

$$\frac{\partial^{k+l} a}{(\partial a_{\alpha_k r_k})(\partial a_{\beta_l s_l})} = a \cdot \delta^{(r_k)(s_l)}_{(\varrho_k)(\sigma_l)} (a^{\alpha_k \varrho_k}) \cdot (a^{\beta_l \sigma_l}).$$

And since

$$k!\, l!\, \delta^{(r_k),\,(s_l)}_{(\varrho_k)\,(\sigma_l)} = \delta^{(m_k)}_{(\varrho_k)} \cdot \delta^{(n_l)}_{(\sigma_l)} \cdot \delta^{(r_k),\ (s_l)}_{(m_k),\ (n_l)},$$

$$\text{(4.3)} \qquad \frac{\partial^{k+l} a}{(\partial a_{\alpha_k r_k})(\partial a_{\beta_l s_l})} = \frac{1}{a \cdot k!\, l!} \cdot \frac{\partial^k a}{(\partial a_{\alpha_k m_k})} \cdot \frac{\partial^l a}{(\partial a_{\beta_l n_l})} \cdot \delta^{(r_k)\ (s_l)}_{(m_k),\,(n_l)}.$$

Or instead, if both members had been multiplied by $(a_{\beta_l \gamma_l})$

$$\text{(4.4)} \qquad (a_{\beta_l \gamma_l}) \frac{\partial^{k+l} a}{(\partial a_{\alpha_k r_k})(\partial a_{\beta_l s_l})} = \frac{1}{k!} \frac{\partial^k a}{(\partial a_{\alpha_k m_k})} \cdot \delta^{(r_k),\ (s_l)}_{(m_k),\,(\gamma_l)}.$$

5. Let $\varepsilon_{(s_n)}$ be an alternating covariant tensor of rank n. Then

$$\text{(5.1)} \qquad \varepsilon_{(\sigma_n)} = \frac{1}{n!} \varepsilon_{(\varrho_n)} \cdot \delta^{(\varrho_n)}_{(\sigma_n)}.$$

When this equation is substituted in the transformation equation

$$\text{(5.2)} \qquad \begin{aligned} \varepsilon'_{(s_n)} &= \varepsilon_{(\sigma_n)} \left(\frac{\partial x_{\sigma_n}}{\partial x'_{s_n}} \right), \\ \varepsilon'_{(s_n)} &= \frac{1}{n!} \varepsilon_{(\varrho_n)} \left[\delta^{(\varrho_n)}_{(\sigma_n)} \left(\frac{\partial x_{\sigma_n}}{\partial x'_{s_n}} \right) \right]. \end{aligned}$$

When $(\varrho_n) \equiv (s_n)$ in the quantity within the brackets, this quantity is by (1.1) equal to J the Jacobian. As the (ϱ_n) take all permutations in the summation an even number of changes of sign occur because of the alternating character of both tensors, so

$$\text{(5.3)} \qquad \varepsilon'_{(s_n)} = J \cdot \varepsilon_{(s_n)}.$$

This equation demonstrates that the alternating property is preserved in the new system. From these equations we see that

$$\text{(5.4)} \qquad \varepsilon_{(s_n)} J = \varepsilon_{(\sigma_n)} \left(\frac{\partial x_{\sigma_n}}{\partial x'_{s_n}} \right).$$

In the right member of (5.4) there is a sum of products with signs attached by the tensor while on the left these signs have been "absorbed" in the J.

The argument is reversible and leads to the observation that the "Relative Tensor" of Veblen and Thomas* is expressible as a tensor. The relative tensor of weight one has the following transformation equation:

$$(5.5)\qquad T'^{\,p_1\cdots p_m}_{\;q_1\cdots q_r} = J\cdot T^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_r}\frac{\partial x'_{p_1}}{\partial x_{\alpha_1}}\cdots\frac{\partial x'_{p_m}}{\partial x_{\alpha_m}}\cdot\frac{\partial x_{\beta_1}}{\partial x'_{q_1}}\cdots\frac{\partial x_{\beta_r}}{\partial x'_{q^r}}.$$

The alternating suffixes (s_n) may be annexed to the component on the left of this equation if they are annexed to each component in the sum on the right, thus

$$T'^{\,p_1\cdots p_m}_{\;q_1\cdots q_r(s_n)} = J\cdot T^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_r(s_n)}\frac{\partial x'_{p_1}}{\partial x_{\alpha_1}}\cdots\frac{\partial x'_{p_m}}{\partial x_{\alpha_m}}\cdot\frac{\partial x_{\beta_1}}{\partial x'_{q_1}}\cdots\frac{\partial x_{\beta_r}}{\partial x'_{q_r}};$$

for they have the effect of merely multiplying the equation by 1, -1 or 0. As in (5.4) this may be written as

$$(5.6)\quad T'^{\,p_1\cdots p_m}_{\;q_1\cdots q_r(s_n)} = T^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_r(\sigma_n)}\left(\frac{\partial x_{\sigma_n}}{\partial x'_{s_n}}\right)\cdot\frac{\partial x'_{p_1}}{\partial x_{\alpha_1}}\cdots\frac{\partial x'_{p_m}}{\partial x_{\alpha_m}}\cdot\frac{\partial x_{\beta_1}}{\partial x'_{q_1}}\cdots\frac{\partial x_{\beta_r}}{\partial x'_{q_r}}.$$

By a mere repetition of this process the method for writing a relative tensor of weight k (k an integer) will be apparent. The ordinary formulas for differentiation will now apply.

The tensor $a_{(r_n)(s_n)}$ defined in equation (1.13) illustrates this idea. It is alternating in the suffixes (r_n) and (s_n) so the tensor transformation equation

$$a'_{(r_n)(s_n)} = a_{(\varrho_n)(\sigma_n)}\left(\frac{\partial x_{\varrho_n}}{\partial x'_{r_n}}\right)\left(\frac{\partial x_{\sigma_n}}{\partial x'_{s_n}}\right)$$

may be written as $a'_{(r_n)(s_n)} = J^2 a_{(r_n)(s_n)}$.

*Trans. Amer. Math. Soc., vol. 26, 375.

SOME CONDITIONS UNDER WHICH A CONTINUUM IS A CONTINUOUS CURVE.*

By Harry Merrill Gehman.†

Altho there are a number of definitions of a continuous curve,‡ it is at times difficult to apply any one of them to a given point set. In certain problems that we have been considering, a continuum M has been obtained as the sum of a collection of continuous curves, and the problem arises of finding the conditions under which M is a continuous curve. The following two theorems are a contribution to the solution of this problem.

Theorem 1. *A bounded continuum M is a continuous curve, if it can be expressed as the sum of a collection (C) of continuous curves, such that:* (1) *at most a finite number are of diameter greater than any given positive number, and* (2) *if (D) is any infinite collection of the elements of (C), then some element of (C) has points in common with infinitely many of the elements of (D).*

Proof. By the theorem of R. L. Moore and R. L. Wilder,§ if M is not a continuous curve, there exist two concentric circles, K_1 and K_2, and a countable infinity of subcontinua of M: $\bar{M}$, M_1, M_2, M_3, $\cdots$, such that: (1) each of these continua contains at least one point on K_1 and at least one point on K_2, and is a subset of the point set H, which is composed of K_1 and K_2 and all points of the plane between them, (2) no two of these continua have points in common, and each (save possibly $\bar{M}$) is a maximal connected subset of the points common to M and H, (3) the set $\bar{M}$ is the sequential limiting set of the sequence of sets: M_1, M_2, M_3, $\cdots$, and (4) there exists a connected subset of M which contains $M_1+M_2+M_3+\cdots$, but which contains no point of the maximal connected subset containing $\bar{M}$ of the points common to M and H.

Let K_3 be a circle concentric with K_1 and K_2, whose radius is half the sum of the radii of K_1 and K_2. Let P_1, P_2, P_3, $\cdots$ be a sequence of points on K_3, such that P_i is in M_i. If any continuous curve J of (C) contained

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† National Research Fellow in Mathematics.

‡ For these definitions, see: R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 289-302. We shall refer to this paper hereafter as "Report".

§ *Report*, p. 296, and R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 371.

an infinite number of points of the sequence $P_1, P_2, P_3, \cdots$, it would also contain a point of $\bar{M}$ on the circle K_3, and would have in common with $\bar{M}$ and with an infinite number of the continua $M_1, M_2, M_3, \cdots$, a continuum of diameter at least equal to r, the difference in the radii of K_3 and K_1 (or K_2). In that case, J would fail to be connected im kleinen at some points which it has in common with $\bar{M}$, which is of course impossible. Therefore any element of (C) can contain at most a finite number of points of $P_1, P_2, P_3, \cdots$.

Since by hypothesis only a finite number of elements of (C) are of diameter greater than r, only a finite number of them can contain more than one point of $P_1, P_2, P_3, \cdots$, and by the preceding paragraph, each of these contains only a finite number of these points. We can therefore select an infinite collection (D) of the elements of (C), such that each one contains one and only one point of $P_1, P_2, P_3, \cdots$, and is of diameter less than $r/2$.

By hypothesis, the collection (C) contains an element J' which has points in common with infinitely many of the elements of (D). By the method by which (D) was selected, J' has a continuum of diameter greater than $r/2$ in common with infinitely many of the sets $M_1, M_2, M_3, \cdots$, and has therefore a continuum of diameter at least $r/2$ in common with $\bar{M}$. In this case, J' fails to be connected im kleinen at some points which it has in common with $\bar{M}$, and is therefore not a continuous curve, which is contrary to hypothesis. The continuum M is therefore a continuous curve.

THEOREM 2. *Every subcontinuum of a bounded continuum M is a continuous curve, if M can be expressed as the sum of a collection (C) of continuous curves, such that*: (1) *at most a finite number are of diameter greater than any given positive number*; (2) *if (D) is any infinite collection of the elements of (C), then some element of (C) has points in common with infinitely many of the elements of (D)*; (3) *every subcontinuum of each element of (C) is also a continuous curve*; (4) *the collection (C) contains a subset (F), such that the point set G consisting of all points lying in elements of (F), can be expressed as the sum of a finite number of arcs,† and contains all points common to two elements of (C).*

Proof. If some subcontinuum of M is not a continuous curve, then if we add to it the set G, the maximal connected subset of the sum that contains the given subcontinuum of M will not be a continuous curve.‡ We shall call this set M^*. The state of affairs described in the proof of Theorem 1 exists in M^*. If the limiting set $\bar{M}$ contains a point P which

† Including also a point as a special type of "arc" with its end points identical.

‡ H. M. Gehman, *Concerning the subsets of a plane continuous curve.* Annals Math., vol. 27 (1925), p. 30, Corollary to Theorem 1.

is not a point of the closed set G, a circle K can be constructed with center at P such that no point of G is contained or enclosed by K. If $\bar{M}$ is a subset of G, a point P of $\bar{M}$ can be found which is not a limit point of $G-\bar{M}$, and a circle K can be constructed with center at P such that no point of $G-\bar{M}$ is contained or enclosed by K.

In either case, there exists within the circle K a new pair of circles K_1 and K_2, and a new sequence of subcontinua of M^*: $\bar{N}$, N_1, N_2, N_3, $\cdots$, having the properties described in the Moore-Wilder Theorem. The set $\bar{N}$ is a subset of $\bar{M}$ and may therefore contain points of G, but each of the other continua N_i is a subset of $M-G$. Therefore, by (4), each continuum N_i can be expressed as the sum of a collection of mutually exclusive closed sets, namely the point sets which the various elements of (C) have in common with N_i. If there is more than one of these closed sets, there are an uncountable infinity of them.† In this latter case, since there are only a finite number of elements of (C) which have points in common with one of the continua N_i and with either $\bar{N}$ or the exterior of K, there is therefore an uncountably infinite collection (D) of elements of (C) which have no points in common with G, and for which no element of (C) having the property described in (2) exists, which is contrary to hypothesis.

Each of the continua N_i is therefore contained in one and only one of the elements of (C). Since there are an infinite number of the continua N_i, each of diameter at least equal to r (the difference in radii of K_1 and K_2), and since there are only a finite number of elements of (C) of diameter equal to or greater than r, it follows that an element J of (C) must contain an infinite number of continua of the sequence: N_1, N_2, N_3, $\cdots$. This set J then contains a subcontinuum which is not a continuous curve,‡ which is contrary to hypothesis. Therefore every subcontinuum of M is a continuous curve.

We shall give examples to show that the conclusions of these two theorems do not necessarily follow in case the collection (C) fails to satisfy any numbered part of the hypothesis.

Example 1. Let M be the sum of the countable collection (C) composed of C_1 = the straight-line interval from (0,0) to (1,0); C_2 = the interval from (0,0) to (0,1); C_n = the interval from $(1/n, 0)$ to $(1/n, 1)$, for $n = 3, 4, 5, \cdots$, and let (F) be C_1. This collection satisfies all the hypotheses of Theorems 1 and 2, except (1), and in each case the conclusion is false, as M itself is not a continuous curve.

Example 2. The set M of Example 1 can also be expressed as the sum of the uncountable collection (C) of elements, each of which is a single

† W. Sierpiński, *Un théorème sur les continus*, Tôhoku Math. Journ., vol. 13 (1918), p. 300.

‡ H. M. Gehman, loc. cit., p. 39, Theorem 5.

point of M. Let (F) be the point (0,0). This collection satisfies all the hypotheses of Theorems 1 and 2, except (2), and again the conclusion is false.

Example 3. Let M be the sum of the two continuous curves, C_1 and C_2, where C_1 is the interval from (0, 1/2) to (1, 1/2), and C_2 consists of the intervals from (0, 0) to (1, 0), from $(0, 1/2^n)$ to $(1, 1/2^n)$, for $n = 2, 3, 4, \cdots$, and from $(k/2^n, 0)$ to $(k/2^n, 1/2^n)$, for $n = 1, 2, 3, \cdots$ and $k = 1, 3, 5, \cdots, 2^n-1$. Let (F) be C_1. This collection satisfies all the hypotheses of Theorem 2, except (3), since the continuum consisting of the intervals in C_2 of lines with slope equal to zero, and the interval from (1/2, 0) to (1/2, 1/2), is not a continuous curve, and hence the conclusion is false.

Example 4. The set M of Example 3 can also be expressed as the sum of the countably infinite collection (C), where C_1 consists of the intervals of lines in M with slope equal to infinity, and the interval from (0, 0) to (1, 0); C_2 is the interval from (0, 1/2) to (1, 1/2); C_3 is the interval from (0, 1/4) to (1/2, 1/4); C_4 is the interval from (1/2, 1/4) to (1, 1/4); C_5 is the interval from (0, 1/8) to (1/4, 1/8); etc. Since G is a closed set containing all points common to two elements of (C), and since every point of M on the x-axis is a limit point of such points, it follows that G must contain the interval from (0, 0) to (1, 0), and therefore all of C_1. But in that case, G cannot be expressed as the sum of a finite number of arcs. Therefore this collection satisfies all the hypotheses of Theorem 2, except (4), and yet the conclusion is false, as was pointed out under Example 3.

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SUCCESSIVE DERIVATIVES OF A FUNCTION OF SEVERAL FUNCTIONS.*

By L. S. Dederick.

1. **Introduction.** If $x = f(t)$ and $\varphi(x) = \varphi[f(t)] = F(t)$, Bruno's Formula† enables one to write down the various terms of the nth derivative of $F(t)$ in terms of the derivatives of $\varphi(x)$ and $f(t)$ without finding the derivatives of $F(t)$ of lower orders. Certain cases of a generalization of this to a function of several functions have been treated by E. G. Gallop.‡ A general formula in terms of total differentials has been given by Leo Königsberger.§ The present paper derives a general formula explicitly in terms of the derivatives, and develops certain algebraic properties of this formula. These have some interest in themselves, and also have applications which are important in the study of the higher singularities of transformations of two or more variables. The method of obtaining the general formula is similar to that given by Franz Meyer‖ for proving Bruno's Formula and attributed by him to R. Dedekind. His requirement, however, that the functions be analytic is not made here.

2. **The general formula.** If we apply the multinomial theorem to expand a power of the form $(a_q x^q + a_{q+1} x^{q+1} + \cdots + a_n x^n)^m$ we may write the total coefficient of x^r in the expansion as

$$\sum \frac{m! \prod a_i^j}{\prod j!}, \tag{1}$$

where the summation includes every possible product for which i and j are integers satisfying the conditions $q \leqq i \leqq n$, $j \geqq 0$, $\sum j = m$, and $\sum ij = r$. The case where every $j = 0$, and hence $m = r = 0$, is apparently trivial, but actually useful. It is convenient to have a fixed method of expanding a sum of this kind. One such method is to arrange the factors in the numerator of each term according to descending values of i, $i_1 > i_2 > \cdots$, and to arrange the terms according to descending values first of i_1, then

* Presented to the American Mathematical Society, Dec. 29, 1925.

† Francesco Faà de Bruno, Annales de Tortolini (1855) and Théorie des Formes Binaires, p. 304.

‡ Trans. Cambr. Phil. Soc., vol. 17, p. 231.

§ Math. Ann., vol. 27, p. 473.

‖ Ibid., vol. 36, p. 464.

in succession of $j_1, i_2, j_2, \cdots$. Thus for $q = 2$, $n = 7$, $m = 4$, $r = 14$, we should have as the expansion of (1), $12a_7 a_3 a_2^2 + 12a_6 a_4 a_2^2 + 12a_6 a_3^2 a_2 + 6a_5^2 a_2^2 + 24a_5 a_4 a_3 a_2 + 4a_5 a_3^3 + 4a_4^3 a_2 + 6a_4^2 a_3^2$.

By applying this formula to combinations of Taylor's expansions in finite form, we can obtain the derivative of the nth order of a function of several functions. The discussion will be carried out for a function of two functions. The procedure will not be quite the simplest possible for this case, being chosen so as to extend without change to a function of three or more functions.

THEOREM 1. *If the function $\varphi(x, y)$ has all partial derivatives of order n or lower continuous at and near (x_0, y_0), and x and y are functions of t having the values x_0 and y_0 respectively when $t = t_0$, and having their derivatives of order n continuous at and near this point; if also we denote by $F(t)$ the function $\varphi(x, y)$ when regarded as a function of t, and use the notation*

$$\varphi_{m,m'} = \frac{\partial^{m+m'} \varphi}{\partial x^m \, \partial y^{m'}}, \quad \alpha_i = \frac{d^i x}{dt^i}, \quad \beta_i = \frac{d^i y}{dt^i};$$

then

$$F^{(n)}(t_0) = \sum \frac{n! \, \varphi_{m,m'} \prod \alpha_i^j \beta_{i'}^{j'}}{\prod j! \, j'! \, (i!)^j \, (i'!)^{j'}}, \tag{2}$$

where the arguments on the right side are x_0, y_0, and t_0, the products have in general several factors of each of the types indicated, and the summation is to extend to all possible terms with indices integers satisfying the conditions,

$$\begin{gathered} m, \, m' \geqq 0, \quad 1 \leqq m+m' \leqq n, \quad 1 \leqq i \leqq n, \quad 1 \leqq i' \leqq n, \quad j, \, j' \geqq 0, \\ \sum j = m, \quad \sum j' = m', \quad \sum ij + \sum i'j' = n. \end{gathered} \tag{3}$$

To prove this, let $h = t - t_0$, $k = x - x_0$, and $k_1 = y - y_0$. Then by Taylor's theorem we may write

$$k = \sum_{i=1}^{n} \frac{\alpha_i h^i}{i!}, \qquad k_1 = \sum_{i=1}^{n} \frac{\beta_i h^i}{i!},$$

where the arguments of all derivatives of order less than n are t_0 and those of the derivatives of order n are $t_0 + \theta h$, where $0 < \theta < 1$. Also

$$\Delta\varphi = \varphi(x, y) - \varphi(x_0, y_0) = \sum \frac{\varphi_{m,m'} \, k^m k_1^{m'}}{m! \, m'!},$$

where the summation extends to all values of m and m' which are zero or positive integers such that $1 \leqq m + m' \leqq n$, and where the partial

derivatives of order less than n have the arguments (x_0, y_0), while those of order n have the arguments $(x_0+\theta_1 k, y_0+\theta_1 k_1)$ where $0<\theta_1<1$.

If in $\Delta\varphi$ we substitute the previous expansions for k and k_1 we get an expansion of $\Delta\varphi$ of the form

$$\Delta\varphi = a_1 h + a_2 h^2 + \cdots + a_{n-1} h^{n-1} + a_n(h)\, h^n \tag{4}$$

with coefficients $a_1, a_2, \cdots$ involving the derivatives of φ with respect to x and y, and of the latter with respect to t. It is easy to see that the coefficients $a_1, \cdots, a_{n-1}$ are constants, whereas a_n is a function of h as indicated. If we denote by $A(r, m)$ the coefficient of h^r in the expansion of k^m, we have from (1) that

$$A(r,m) = \sum \frac{m! \prod \alpha_i^j}{\prod j! \,(i!)^j}$$

where $1 \leqq i \leqq n$, $j \geqq 0$, $\sum j = m$, $\sum ij = r$, with a similar expression for $B(r', m')$ the coefficient of $h^{r'}$ in the expansion of $k_1^{m'}$. Now the coefficient of h^n in the expansion of $\Delta\varphi$ is clearly

$$a_n = \sum \frac{\varphi_{m,m'}\, A(r,m)\, B(r',m')}{m!\, m'!}$$

where $r + r' = n$, and this may now be written

$$a_n = \sum \frac{\varphi_{m,m'} \prod \alpha_i^j \beta_{i'}^{j'}}{\prod j!\, j'!\, (i!)^j\, (i'!)^{j'}}$$

with the conditions, $1 \leqq m + m' \leqq n$, $\sum j = m$, $\sum j' = m'$, and $\sum ij + \sum i'j' = n$.

On the other hand, since $\varphi(x, y) = F(t)$ we may write

$$\Delta\varphi = \Delta F = h F'(t_0) + \frac{1}{2!} h^2 F''(t_0) + \cdots \tag{5}$$
$$\cdots + \frac{h^n}{n!} F^{(n)}(t_0 + \theta_2 h),\ 0 < \theta_2 < 1.$$

The two expansions (4) and (5) for $\Delta\varphi$ both hold through some interval for h including $h = 0$. If we equate them, we know that the coefficients of h^n on the two sides remain finite as h approaches zero. This is sufficient to enable us to equate the corresponding constant coefficients on the two sides. If we subtract the terms thus shown equal and divide by h^n, we see that the coefficients of h^n on the two sides are equal for all values

of h in the interval except zero, and hence by their continuity also when $h = 0$. Setting $F^{(n)}(t_0) = n!\, a_n(0)$ completes the proof.

3. **Algebraic property.** If we regard $F^{(n)}$ expressed in the form (2) as a polynomial in the ordinary derivatives α_i and β_i of order higher than the first, we should speak of the terms involving only the first order ordinary derivatives as the "constant term", and the factors of any term other than the ordinary derivatives of order above the first, as the "coefficient" of that term. In this sense we have the theorem that the total "coefficient" of any term in $F^{(n)}$ is the product of a numerical factor and the partial derivative of some order of the "constant term" in some derivative of F of lower order than n. (A more explicit statement of the theorem follows the proof.)

Let the degrees of the specified term in the α's and β's, excluding α_1 and β_1 be respectively σ and σ', and the respective weights τ and τ'. Then we shall have from (3) $m - \sigma = \sum ij - \tau$ and $m' - \sigma' = \sum i'j' - \tau'$. From these $m + m' = n - (\tau + \tau' - \sigma - \sigma')$. If we call this number n' and the required coefficient $C_n(\sigma, \tau, \sigma', \tau')$, we may write

$$C_n(\sigma, \tau, \sigma', \tau') = \frac{n!}{\prod j!\, j'!\, (i!)^j\, (i'!)^{j'}} \sum_{m, m'} \frac{\varphi_{m, m'}\, \alpha_1^{m-\sigma}\, \beta_1^{m'-\sigma'}}{(m-\sigma)!\, (m'-\sigma')!}$$

where $\sum j = \sigma$, $\sum ij = \tau$, $\sum j' = \sigma'$, $\sum i'j' = \tau'$, and $m + m' = n'$. But the sum here indicated is obviously the partial derivative of

$$\sum \frac{\varphi_{m, m'}\, \alpha_1^m\, \beta_1^{m'}}{m!\, m'!}$$

of order σ and σ' with respect to α_1 and β_1 respectively; and this, except for the factor $n'!$ is the constant term in $F^{(n')}$. Hence we have

$$C_n(\sigma, \tau, \sigma', \tau') = \frac{n!}{n'! \prod j!\, j'!\, (i!)^j\, (i'!)^{j'}} \frac{\partial^{\sigma+\sigma'}}{\partial \alpha_1^{\sigma}\, \partial \beta_1^{\sigma'}} C_{n'}\ (0, 0, 0, 0),$$

and we may now state explicitly

THEOREM 2. *The total coefficient in $F^{(n)}$ of a product of the ordinary derivatives of order above the first of degrees σ and σ' and weights τ and τ' in the α's and β's respectively, is a numerical factor multiplied by the partial derivative of order σ with respect to α_1 and σ' with respect to β_1 of the terms in $F^{(n')}$ containing only first order ordinary derivatives, where $n' = n - (\tau + \tau' - \sigma - \sigma')$.*

It may be noted that the value of n' as well as the number of times the "constant term" of $F^{(n')}$ is to be differentiated with respect to each

argument, is determined by the degrees and weights in the α's as a whole and the β's as a whole. The numerical factor, on the other hand, is not thus determined, but depends on the orders and powers of the individual α's and β's. Thus the "constant term" in F''' is $\varphi_{30}\alpha_1^3+3\varphi_{21}\alpha_1^2\beta_1+3\varphi_{12}\alpha_1\beta_1^2+\varphi_{03}\beta_1^3$. Its second partial derivative with respect to α_1 is $6\varphi_{30}\alpha_1+6\varphi_{21}\beta_1$. Therefore the coefficient of $\alpha_4\alpha_2$ in F^{VII} is this expression multiplied by $7!/(3!4!2!)$ or $105\ (\varphi_{30}\alpha_1+\varphi_{21}\beta_1)$, whereas the coefficient of α_3^2 which has the same degree and weight, has the numerical factor $7!/[(3!)^3 2!]$ which makes the coefficient $70\,(\varphi_{30}\alpha_1+\varphi_{21}\beta_1)$. The values of σ and τ in this illustration are, however, the smallest for which the distinction occurs.

THEOREM 3. *The total coefficient in $F^{(n)}$ of a product of ordinary derivatives all of order greater than r, the product being of degrees σ and σ' and weights τ and τ' in the α's and β's respectively, is a numerical factor multiplied by the partial derivative of order σ with respect to α_r and σ' with respect to β_r of the terms in $F^{(n')}$ containing no ordinary derivatives of order greater than r, where $n'=n-\tau-\tau'+r\sigma+r\sigma'$.*

This theorem reduces to Theorem 2 when $r=1$. The proof is from this by induction. Let $D_n(\sigma,\tau,\sigma',\tau')$ be the total coefficient referred to in the theorem where the derivatives are of order greater than r, and let $C_n(\sigma,\tau,\sigma',\tau')$ be the coefficient of a similar product including derivatives of order greater than $r-1$. Also let $\prod(\sigma,\tau,\sigma',\tau')$ denote the product $\prod j!\,j'!\,(i!)^j\,(i'!)^{j'}$ where $\sum j=\sigma$, $\sum j'=\sigma'$, $\sum ij=\tau$, $\sum i'j'=\tau'$; and let $C_n=C_n(0,0,0,0)$, $D_n=D_n(0,0,0,0)$, and

$$H_r(a,b,c)=\frac{\partial^{a+b}}{\partial\alpha_r^a\,\partial\beta_r^b}C_{ra+rb+c}.$$

Then if we assume that

$$(6)\qquad C_n(\sigma,\tau,\sigma',\tau')=\frac{n!\,[(r-1)!]^{\sigma+\sigma'}}{p!\prod(\sigma,\tau,\sigma',\tau')}H_{r-1}(\sigma,\sigma',n-\tau-\tau')$$

where $p=n-\tau-\tau'+(r-1)(\sigma+\sigma')$, our assumption is verified when $r=2$ by reference to the preceding theorem.

Now we can readily see that

$$D_n(\sigma,\tau,\sigma',\tau')=\sum_{k,k'}C_n(\sigma+k,\ \tau+rk,\ \sigma'+k',\ \tau'+rk')\,\alpha_r^k\,\beta_r^{k'}$$

where k and k' take on such values as the exponents of α_r and β_r have in $D_n(\sigma,\tau,\sigma',\tau')$. It should be understood that $D_n(\sigma,\tau,\sigma',\tau')$ denotes the coefficient of a particular product, which may not be completely determined by the arguments, and that the other functions of these arguments,

particularly $\prod(\sigma, \tau, \sigma', \tau')$ are suitably restricted to agree with this (Compare the remark following Theorem 2).

From the assumption and the definitions we may now write

$$D_n(\sigma, \tau, \sigma', \tau')$$
$$= \sum_{k,k'} \frac{n!\,[(r-1)!]^{\sigma+\sigma'+k+k'}\, H_{r-1}(\sigma+k, \sigma'+k', n-\tau-\tau'-rk-rk')}{p'!\,\prod(\sigma+k, \tau+rk, \sigma'+k', \tau'+rk')}\, \alpha_r^k\, \beta_r^{k'}$$
$$= \frac{n!\,[(r-1)!]^{\sigma+\sigma'}}{\prod(\sigma, \tau, \sigma', \tau')} \sum_{k,k'} \frac{H_{r-1}(\sigma+k, \sigma'+k', n-\tau-\tau'-rk-rk')}{p'!\,k!\,k'!\,r^{k+k'}}\, \alpha_r^k\, \beta_r^{k'}$$

where $p' = n-\tau-\tau'+(r-1)(\sigma+\sigma')-k-k'$.

Similarly

$$D_{n'} = n'! \sum_{l,l'} \frac{H_{r-1}(l, l', n'-rl-rl')}{(n'-l-l')!\,l!\,l'!\,r^{l+l'}}\, \alpha_r^l\, \beta_r^{l'}$$

and from this,

$$\frac{\partial^{\sigma+\sigma'}}{\partial \alpha_n^{\sigma}\, \partial \beta_n^{\sigma'}} D_{n'} = n'! \sum_{l,l'} \frac{H_{r-1}(l, l', n'-rl-rl')}{(n'-l-l')!\,(l-\sigma)!\,(l'-\sigma')!\,r^{l+l'}}\, \alpha_r^{l-\sigma}\, \beta_r^{l'-\sigma'}$$

Now if we let $n' = n-\tau-\tau'+r\sigma+r\sigma'$, $l = \sigma+k$, $l' = \sigma'+k'$, we have $n'-l-l' = p'$ and

$$\frac{\partial^{\sigma+\sigma'}}{\partial \alpha_r^{\sigma}\, \partial \beta_r^{\sigma'}} D_{n'} = \frac{n'!}{r^{\sigma+\sigma'}} \sum_{k,k'} \frac{H_{r-1}(\sigma+k, \sigma'+k', n-\tau-\tau'-rk-rk')}{p'!\,k!\,k'!\,r^{k+k'}}\, \alpha_r^k\, \beta_r^{k'}$$

Hence

$$D_n(\sigma, \tau, \sigma', \tau') = \frac{n!\,[(r-1)!]^{\sigma+\sigma'}}{\prod(\sigma, \tau, \sigma', \tau')} \cdot \frac{r^{\sigma+\sigma'}}{n'!} \cdot \frac{\partial^{\sigma+\sigma'}}{\partial \alpha_r^{\sigma}\, \partial \beta_r^{\sigma'}} D_{n'}$$
$$= \frac{n!\,(r!)^{\sigma+\sigma'}}{n'!\,\prod(\sigma, \tau, \sigma', \tau')} \cdot \frac{\partial^{\sigma+\sigma'}}{\partial \alpha_r^{\sigma}\, \partial \beta_r^{\sigma'}} D_{n'}.$$

Since this may be obtained from (6) by replacing C by D and $r-1$ by r, the induction is completed and the theorem thereby established.

This theorem, as well as Theorems 1 and 2 may be extended without change in statement or proof to a function of any number of functions.

4. **Case where one function equals its argument.** The principal application of the preceding material to higher singularities of transformations of two variables occurs when $x-x_0 = t-t_0$, or without any essential loss of generality when $x = t$. This gives $\alpha_1 = 1$, $\alpha_i = 0$ if $i > 1$. Formula (2) then becomes

$$F^{(n)} = \sum \frac{n!\,\varphi_{p-m,m} \prod \beta_i^j}{(p-m)! \prod j! (i!)^j}$$

where $1 \leqq p \leqq n$, $1 \leqq i \leqq n$, $0 \leqq j \leqq m \leqq p$, $\sum j = m$. $\sum ij = n - p + m$. From these conditions it follows that $m = 0$ only if $p = n$. If at (x_0, y_0) the partial derivatives of φ of order less than ν all vanish, this gives

$$F^{(\nu)} = \sum_{m=0}^{\nu} \frac{\nu!\,\varphi_{\nu-m,m}\,\beta_1^m}{(\nu-m)!\,m!} \tag{7}$$

a polynomial in β_1 of degree not greater than ν containing no higher ordinary derivative. It is essentially the same polynomial as $d^\nu \varphi(x, y)$, merely being the non-homogeneous polynomial in $\frac{dy}{dx}$ corresponding to the homogeneous polynomial in dx and dy which is the differential.

The vanishing of the partial derivatives of φ of order less than ν also puts restrictions on the degree and weight of the products of the higher derivatives $\beta_2, \beta_3, \cdots$ actually appearing in $F^{(\nu+1)}$, $F^{(\nu+2)}$, etc. This may be expressed as follows.

THEOREM 4. *If in Theorem* 1, $\alpha_1 = 1$, *and* $\alpha_i = 0$ *when* $i > 1$, *and if all the partial derivatives of* φ *of order less than* ν *are zero, but not all those of order* ν, *then* $F^{(\nu)}$ *contains no ordinary derivatives above the first order. Moreover, if* σ *and* τ *denote the degree and weight of any product of ordinary derivatives above the first order occurring in* $F^{(n)}$ *where* $n > \nu$, *then* $\sigma \leqq n - \nu$, *and* $\tau \leqq \sigma + n - \nu \leqq 2(n-\nu)$.

To prove this we may apply the conditions of this theorem to Theorem 2 and get $n - \tau + \sigma \geqq \nu$. We have also $\tau \geqq 2\sigma$ since β_2 is the lowest order derivative in the product. The required inequalities follow immediately from these.

If ν has the meaning of this theorem and we denote by C_n the aggregate of terms in $F^{(n)}$ containing no ordinary derivatives above the first order, and by $C_n^{(i)}$ the partial derivative of C_n with respect to β_1, of order i, then Theorems 2 and 3 enable us to write

$$F^{(\nu)} = C_\nu,$$

$$F^{(\nu+1)} = \frac{\nu+1}{2} C_\nu' \beta_2 + C_{\nu+1},$$

$$F^{(\nu+2)} = \frac{(\nu+2)(\nu+1)}{6} C_\nu' \beta_3 + \frac{(\nu+2)(\nu+1)}{8} C_\nu'' \beta_2^2 + \frac{\nu+2}{2} C_{\nu+1}' \beta_2 + C_{\nu+2},$$

and so on.

If any value of β_1 for which (7) vanishes is a simple root of that polynomial, then for this value of β_1, $F^{(\nu+1)}$ is a linear function of β_2, of which we know by Theorem 2 that the coefficient of β_2 is not zero. If, however, β_1 is a multiple root of (7), $F^{(\nu+1)}$ is for this value a mere constant independent of β_2. This constant may be zero. To determine the degree and character of the lowest derivative of F which does not vanish identically in $\beta_2 \cdots$ for a value of β_1 for which $F^{(\nu)}$ vanishes, we may state the following

THEOREM 5. *If in Theorem* 1 *we make the following hypotheses*:

a) $\alpha_1 = 1$, $\alpha_i = 0$ *when* $i > 1$;

b) C_n *is to denote the aggregate of terms in* $F^{(n)}$ *containing no ordinary derivative above the first order, and* $C_n^{(i)}$ *the partial derivative of* C_n *of order* i *with respect to* β_1;

c) *all the partial derivatives of* φ *of order less than* ν *are zero at* (x_0, y_0), *but not all those of order* ν;

d) *for a particular value of* β_1 $C_{\nu+i}^{(j-i)} = 0$ *when* $0 \leqq i \leqq j < \lambda$;

e) *for this value of* β_1 $C_{\nu+i}^{(\lambda-i)} = 0$ *when* $i < \lambda - \varrho$, *but* $C_{\nu+\lambda-\varrho}^{(\varrho)} \neq 0$;

then $F^{(\nu)} = F^{(\nu+1)} = \cdots = F^{(\nu+\lambda-1)} = 0$, *and* $F^{(\nu+\lambda)}$ *is a polynomial of degree* ϱ *in* β_2, *and does not involve* β_3 *or any higher ordinary derivative.*

To prove this consider first the "constant terms" in $F^{(\nu)}, \cdots, F^{(\nu+\lambda-1)}$, namely $C_\nu, \cdots, C_{\nu+\lambda-1}$. These are all found to be zero by setting $i = j$ in d). Next consider any product of higher derivatives of degree σ and weight τ, when $n < \nu + \lambda$. By Theorem 2 the coefficient of this product is $C_{n-\tau+\sigma}^{(\sigma)}$ multiplied by a numerical factor. Comparing this with d) we have $\sigma = j - i$, $n - \tau + \sigma = \nu + i$, whence $2\sigma - \tau = \nu - n + j$. But since the lowest order β is β_2 we have $\tau \geqq 2\sigma$. Hence $n \geqq \nu + j$. On the assumption $n < \nu + \lambda$, then, we have $j < \lambda$, and therefore d) implies that the coefficient of the product in question is zero provided $j - i$ is not negative. This condition is obviously fulfilled. The apparent requirement that i be not negative need not be made, as $C_n = 0$ identically in β_1 in that case and hence also its derivatives of all orders vanish. We have then the first part of the theorem, that $F^{(n)} = 0$ if $n < \nu + \lambda$.

If in the preceding we replace $n < \nu + \lambda$ by $n = \nu + \lambda$ we get $j \leqq \lambda$. For the cases where $j < \lambda$ the coefficients are zero as before. This leaves only $j = \lambda$ to consider. For this case $\lambda - i = \sigma$, $\nu + i = n - \tau + \sigma$. Adding these to $n = \nu + \lambda$, we get $\tau = 2\sigma$, which shows that there are no terms with β_3 or higher derivatives. If $\sigma > \varrho$, we have $\lambda - i > \varrho$ or $i < \lambda - \varrho$. This by e) makes the coefficient of the term zero. If, however, $\sigma = \varrho$, then $i = \lambda - \varrho$ and the coefficient does not vanish. This completes the proof that $F^{(\nu+\lambda)}$ is a polynomial in β_2 of degree ϱ.

This theorem can also be stated in a simpler but less accurate and less complete form as follows:

If at (x_0, y_0) $d^\nu \varphi(x, y)$ is the lowest order differential of φ which does not vanish identically and if $\beta_1 = \frac{dy}{dx}$ is a root of this of order λ and a root of $d^{\nu+1}\varphi$ of order $\lambda - 1$, of $d^{\nu+2}\varphi$ of order $\lambda - 2$, and so on, then for this value of β_1, $F^{(\nu+\lambda)}$ will be the lowest derivative of F which does not vanish identically in $\beta_2 \cdots$; and if the order of multiplicity of β_1 as a root of any of these differentials is less than that indicated, the index of the first non-vanishing derivative of F is correspondingly reduced.

We have seen here that if a certain value of β_1 causes $F^{(\nu)}$ to vanish, then the lowest derivative of F which for this value of β_1 does not vanish identically in β_2 and the higher β's, involves β_2 and not β_3 or any higher β. This may vanish for certain particular values of β_2. The following theorems show that in this case the lowest derivative of F which for these values of β_1 and β_2 does not vanish identically in $\beta_3 \ldots$, actually involves β_3 and not β_4 or any higher β, and similarly for succeeding cases.

THEOREM 6. *If β_i actually appears in $F^{(n)}$ when certain particular values are assigned to $\beta_1, \cdots, \beta_{i-2}$, then for these same values β_{i-1} appears in $F^{(n-1)}$.*

To prove this assume that β_i appears in some product of β's in $F^{(n)}$ with the exponent j and let σ and τ be the degree and weight of this product respectively, when β_1 is excluded. Then its coefficient in β_1 by Theorem 2 is $C^{(\sigma)}_{n-\tau+\sigma}$ multiplied by a numerical factor. If now we replace β_i^j by $\beta_i^{j-1}\beta_{i-1}$ we can see that σ is unchanged but that τ is diminished by one. If, however, we are considering the coefficient of this new product as it appears in $F^{(n-1)}$, where n is also diminished by one we see that $C^{(\sigma)}_{n-\tau+\sigma}$ is unchanged. Hence this product in $F^{(n-1)}$ has a coefficient which differs only by a numerical factor from that of the product containing β_i^j in $F^{(n)}$, and consequently will be different from zero if that coefficient is;

THEOREM 7. *If the successive derivatives of F are formed, and in each those values of $\beta_1, \beta_2 \cdots$ substituted which make all the previous derivatives zero, then the β's will appear in these derivatives of F one at a time and in regular order without omissions.*

This follows immediately from Theorem 6.

THEOREM 8. *In Theorem 5 λ can not exceed ν.*

In condition d) of that theorem if we set $i = 0$, we have $C_\nu^{(j)} = 0$ for all values of j less than λ. But we know from (7) that C_ν is of degree not greater than ν and hence $C_\nu^{(j)} \neq 0$ for some value of j not greater than ν. Therefore $\lambda \leqq \nu$.

5. **Enumeration of cases.** If β_1 is a root of order λ of $F^{(\nu)}$, then we learn from the preceding theorems that the first non-vanishing derivative

of F for this value of β_1 may be any of following: if $\lambda > 1$, $F^{(\nu+1)}$ equal to a constant; if $\lambda > 2$, $F^{(\nu+2)}$ linear in β_2, or a constant; if $\lambda > 3$, $F^{(\nu+3)}$ quadratic or lower in β_2; and so on up to $F^{(\nu+\lambda-1)}$ of degree $\lambda - 2$ or lower in β_2. Finally if all these vanish, $F^{(\nu+\lambda)}$ will be actually of degree λ in β_2. It is a little remarkable that the degree of the first non-vanishing derivative may be equal to λ or less than $\lambda - 1$, but not equal to $\lambda - 1$.

If β_2 has a value which is a simple root of $F^{(\nu+\lambda)}$, we can infer from Theorem 3 that $F^{(\nu+\lambda+1)}$ is a linear function of β_3. If, however, β_2 is a double root of $F^{(\nu+\lambda)}$, $F^{(\nu+\lambda+1)}$ is merely a constant, possibly zero. In this latter case, $F^{(\nu+\lambda+2)}$ is quadratic in β_3. If β_2 is a root of $F^{(\nu+\lambda)}$ of higher order, the various possibilities for the character of the first non-vanishing derivative of F as a function of β_3 are precisely similar to the preceding case for β_2. We can infer also from Theorem 3 that for the similar special values of $\beta_3, \beta_4 \cdots$, the behavior of the successive derivatives is exactly similar.

The results just enumerated are the ones having particular application to singularities of transformations.

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PLANE CUBIC CURVES IN THE GALOIS FIELDS OF ORDER 2^n.

By Alan D. Campbell.

We define a plane cubic curve in a Galois Field of order 2^n as the locus of all points whose coördinates (x, y, z) satisfy an equation of the form:

$$(1) \qquad \begin{aligned} K(x, y, z) \equiv\ & ax^3+by^3+cz^3+dx^2y+exy^2+fx^2z+gxz^2 \\ & +hy^2z+jyz^2+kxyz = 0 \end{aligned}$$

whose coefficients and variables represent numbers in this field, which we shall denote by $GF[2^n]$. We note that in $GF[2^n]$ we have

$$(\alpha x+\beta y+\gamma z)^2 = \alpha^2x^2+\beta^2y^2+\gamma^2z^2,$$

also every number is a perfect square and has just one square root.* If $k = 0$ in (1), it is easy to prove that every cubic projectively equivalent to (1) will lack the term in xyz.

In this paper we first develop some general theory concerning the first polars of points with respect to (1), and concerning nodes, cusps, and points of inflection on (1),† and then we derive typical cubics of the different classes of projectively equivalent plane cubic curves in $GF[2^n]$. There are some arbitrary coefficients that appear in the equations of some of the typical cubics, but we have no data as to the equivalence of these cubics for the different values of these coefficients.

For use in this discussion we note that the general conic in $GF[2^n]$:

$$(2) \qquad ax^2+by^2+cz^2+fyz+gzx+hxy = 0$$

has the discriminant:‡

$$(3) \qquad \Delta \equiv af^2+bg^2+ch^2+fgh.$$

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* See Dickson, "Linear Groups", pp. 13, 36, 44, 47.

† We define nodes, cusps, and points of inflection as in Hilton, "Plane Algebraic Curves", p. 23.

‡ We note that, if we send (2) into an equivalent conic (2′) with discriminant Δ' by a transformation with determinant D, we have $\Delta' = D^2\Delta$.

We note also the following classes of conics in $GF[2^n]$*

(4) Class *1* of nondegenerate conics, $x^2+xy=0$.

(5) Class *2* of real line pairs, $xy=0$.

(6) Class *3* of conjugate imaginary line pairs, $x^2+y^2+\alpha xy=0$, where $\alpha \neq 0$ makes the equation:

$$x^2+\alpha x+1=0 \tag{7}$$

irreducible† in $GF[2^n]$‡.

(8) Class *4* of double lines, $x^2=0$.

Proceeding as in Hilton loc. cit. pp. 89, 90 we find that the first polar (polar conic) of a point $P(X, Y, Z)$ with respect to (1) is:

$$x^2(aX+dY+fZ)+y^2(eX+bY+hZ)+z^2(gX+jY+cZ) +kXyz+kYzx+kZxy=0, \tag{9}$$

with discriminant:

$$\Delta \equiv k^2K(X, Y, Z). \tag{10}$$

Therefore the polar conic of $P(X, Y, Z)$ is degenerate, first if P lies on the cubic, second if $k=0$ and P is any point in the plane. In the latter case alone is (9) double line.§

It is easy to prove that from any point P (not an inflection, node, or cusp) on (1) there are two tangents to the cubic‖ if $k \neq 0$, and one tangent if $k=0$; and that these tangents are the polar conic of P.¶ If P is an inflection the polar conic consists of the tangent at P and the tangent from P. If P is a node the polar conic is the pair of tangents at P. If P is a cusp (this can happen only when $k=0$) the polar conic is the tangent at P. Also if P is not on (1) the polar conic cuts (1) in the points of contact of tangents to (1) from P.

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* Dickson, loc. cit. p. 197.

† It follows immediately from the discussion given by L. E. Dickson, *Linear Groups*, p. 199, that (7) is irreducible if and only if $1/\alpha^2$ is a solution of

$$\delta+\delta^2+\delta^4+\cdots+\delta^{2^{n-1}}=1.$$

‡ Dickson. loc. cit. p. 11.

§ We note that the tangent at a point (X, Y, Z) is gotten by interchanging in (9) x and X, y and Y, z and Z, respectively.

‖ Not counting the tangent at P.

¶ Polar conic of course with respect to (1).

If the cubic (1) has a node or cusp $P'(x'y'z')$, then the polar conic (9) of every point (X, Y, Z) passes through P'. This gives the equations of condition for the existence of P' on (1):

$$(11)\quad ax'^2+ey'^2+gz'^2+ky'z' = 0,\quad dx'^2+by'^2+jz'^2+kx'z' = 0,$$
$$fx'^2+hy'^2+cz'^2+kx'y' = 0.$$

If $k = 0$, (11) gives the simpler condition:

$$(12)\qquad \begin{vmatrix} a, & e, & g \\ d, & b, & j \\ f, & h, & c \end{vmatrix} = 0.$$

We shall test each typical cubic separately for the existence on it of real inflections. We do this indirectly by searching for tangents at inflections, i. e., by finding for what values of its coefficients the general equation of a straight line, when solved simultaneously with this typical cubic, gives us an equation that is a perfect cube in x and y, or x and z, or y and z.

We wish to point out here another fact (for use in a later paper on nets of conics) that if (1) is considered as the possible discriminant (equated to zero) of the general conic of a net of conics, then, if $k = 0$, there is such a corresponding net of conics if and only if (1) has a cusp (in which case the net has a double line corresponding to this cusp),* but if $k \neq 0$ there are always such corresponding nets of conics.†

We shall now derive normal forms for the plane cubic curves in $GF[2^n]$. First we suppose $k = 0$ in (1) and the cubic is nondegenerate. Suppose the cubic has a cusp P, which we take as the point (0, 1, 0). For $x = 0$ we take a line PR, where R is a point on the cubic. For $y = 0$ we take the tangent RS to (1) at R; where S is on (1). For $z = 0$ we take PS. So $x = 0$ [when solved simultaneously with (1)] gives $yz^2 = 0$; $y = 0$ gives $x^2z = 0$; $z = 0$ gives $x^2y = 0$; and the cubic (1) has then no terms in x^3, y^3, z^3, xy^2, xz^2, y^2z, xyz. For the tangent at P we take the line

* Such a net is

$$x(x_1^3)+y(bx_2^3+jx_3^3+\sqrt{e}x_2x_3+x_1x_3)+z[(h+\sqrt{e})x_2^3+cx_3^3+x_1x_2] = 0.$$

We first get $a = d = f = g = k = 0$ in (1).

† Such a net is

$$x\left(\frac{a}{k^3}x_1^3+ex_2^3+gx_3^3+kx_2x_3\right)+y\left(\frac{d}{k^3}x_1^3+bx_2^3+jx_3^3+x_1x_3\right)+z\left(\frac{f}{k^3}x_1^3+hx_2^3+cx_3^3+x_1x_2\right)=0.$$

$x = z$, and we get in (1) $j = d \neq 0$, $f \neq 0$.* We then get (after an obvious transformation):

Class *1* of cubics with cusps, with typical cubic:

$$x^2y + x^2z + yz^2 = 0. \tag{13}$$

We note that (1, 0, 0) is an inflection on (13) with tangent $y = z$; also every tangent to the cubic passes through this inflection except the tangent at the cusp. We now test (13) for other inflections. Solving $x = ay + bz$ simultaneously with (13), we get:

$$a^2y^3 + a^2y^2z + (b^2 + 1)yz^2 + b^2z^3 = 0.$$

The value $a = 0$ gives $b = 1$, which leads to the tangent at the cusp. So we must have $a \neq 0$ and

$$\frac{b^2}{a^2} = c^3, \quad \frac{b^2+1}{a^2} = c^2, \quad \frac{a^2}{a^2} = c = 1, \quad \therefore\ b^2 = a^2, \quad b^2+1 = b^2,$$

which are impossible. Hence (13) has only one inflection just as in the ordinary real geometry.

We now suppose $k = 0$ in (1), and that the cubic is nondegenerate and real, but has no cusp and no real inflection. For $x = 0$ we take the tangent PR at a point P, where R is on (1); for $y = 0$ we take the tangent RS at R, where S is on (1); for $z = 0$ we take the line PS, which we first assume cuts the cubic again at a point T, which we take as the point (1, 1, 0). Then we easily obtain:

Set *2*, with a first typical cubic:

$$x^2y + xy^2 + x^2z + \alpha yz^2 = 0, \qquad (\alpha \neq 0). \tag{14}$$

Condition (12) shows that (14) has no cusp. If we solve $z = ax + by$ simultaneously with (14) we get:

$$c^9 + \alpha c^8 + \alpha c^6 + \alpha^3 = 0,$$

which must be irreducible in the $GF[2^n]$ if (14) is to have no real inflections.

Suppose now that in the above discussion the line PS is tangent to the cubic at S; then in (1) we have $a = b = c = d = g = h = k = 0$, $efj \neq 0$.

* This is the only derivation of a typical cubic that we shall give in detail.

We easily reduce (1) to a second typical cubic for Set *2*:

$$\alpha xy^2 + x^2z + yz^2 = 0, \tag{14'}$$

where α is 1 or a non-cube.* Condition (12) shows that (14') has no cusp. If we solve $x = ay + bz$ simultaneously with (14') we get:

$$a^3 + \alpha^2 a^2 + \alpha^3 = 0,$$

which must be irreducible in the $GF[2^n]$ if (14') is to have no real inflections. We know nothing concerning the equivalence or nonequivalence of (14) and (14').

Next we suppose $k = 0$ in (1), and that the cubic has no cusp, but just one real inflection P. For $x = 0$ we take the tangent RS at a point $R(\neq P)$, where S lies on the tangent PS at the inflection P. The line PR is then the polar conic of S with respect to (1); so we have a third tangent to (1) from S, namely ST with point of contact at T (on PR). If we take ST for $y = 0$ and SP for $y = x$ we get:

$$\alpha z^3 + x^2y + xy^2 + xz^2 + yz^2 = 0, \tag{15 Set 3}$$

where $\alpha \neq 0$. The condition (12) shows that (15) has no cusp. If we solve $x = ay + bz$ simultaneously with (15), we have:

$$c^8 + (\alpha + \alpha^3)c^7 + \alpha c^5 + \alpha c^3 + \alpha c + 1 = 0,$$

which must be irreducible in the $GF[2^n]$ if (15) is to have no other real inflection.

We suppose $k = 0$ in (1), and that the cubic has no cusp but has at least two real inflections P and R. For $x = 0$ we take the tangent PS at P, for $y = 0$ the tangent RS at R, for $z = 0$ the line PR (which cuts the cubic in a third point T). We take T as (1, 1, 0) and then $y = x$ is the tangent to (1) at T, because PRT is the polar conic of S. We easily obtain:

$$\alpha z^3 + x^2y + xy^2 = 0, \tag{16 Set 4}$$

where $\alpha = 1$ or α is a definite non-cube in the $GF[2^n]$. The condition (12) shows that (16) has no cusp. The point (1, 1, 0) is a third inflection. We note that the real inflections of (16) lie by threes on straight lines

*If and only if $2^n = 3k + 1$, does $GF[2^n]$ contain non-cubes.

just as in the real domain. If we solve $x = ay + bz$ simultaneously with (16), we have:

$$ac^3 + 1 = 0, \qquad a^2 + a = c^3 a.$$

If $a = 1$, we have six more real inflections on (16) in $GF[2^n]$, $n > 1$. Otherwise we have only the above three real inflections.

We shall now study the nondegenerate imaginary cubic curves (1) that have $k = 0$. We must have $abc \neq 0$, so we suppose $a = 1$. If $e = d^2$ or $g = f^2$, our next argument is superfluous. If $g \neq f^2$, we put:

$$x = x', \quad y = y', \; z = \sqrt{\frac{d^2 + e}{f^2 + g}}\, y' + z',$$

and the new d' and e' are such that $e' = d'^2$. We now put:

$$x = x' + d'y', \qquad y = y', \qquad z = z',$$

and we get (1) with $a = 1$, $b = \delta$ (a non-cube), $d = e = k = 0$, $c \neq 0$. We put

$$x = x' + fz', \quad y = y' + \frac{h}{\delta} z', \quad z = z',$$

and get (1) with $a = 1$, $b = \delta$, $c \neq 0$, $d = e = f = h = k = 0$. If $gj \neq 0$, we put $x = \sqrt{g}\, x'$, $y = g^{3/2}/j\, y'$, $z = z'$, and we easily get:

Set 5, with a first typical cubic:

$$x^3 + \alpha y^3 + \beta z^3 + xz^2 + yz^2 = 0, \tag{17}$$

$\alpha\beta \neq 0$, α a non-cube. If every line $x = \mu y$ cuts (19) in imaginary points, then surely this cubic is imaginary. Solving simultaneously we get:

$$y^3(\mu^3 + \alpha) + (\mu + 1) yz^2 + \beta z^3 = 0,$$

which can be put in the form

$$ax^3 + x + 1 = 0 \tag{18}$$

with $a = \beta^2(\mu^3 + \alpha)/(\mu + 1)^3$. This equation is irreducible if the resultant of (18) and

$$x^{2^{n-1}} + 1 = 0 \tag{19}$$

is not 0; the vanishing of this resultant leads to an equation in μ which must be irreducible if (17) is imaginary.

Similarly if $g = 0$, $j \neq 0$, we get another typical cubic for Set 5:

$$x^3 + \alpha y^3 + \beta z^3 + yz^2 = 0, \tag{17'}$$

where α and β are not cubes. The same discussion* as for (17) applies here if we take $a = \beta^2(\mu^3 + \alpha)$.

If $g = j = 0$ in the preceding discussion, we easily get:

$$x^3 + \alpha y^3 + \beta z^3 = 0, \qquad \text{(20) Set 6}$$

where α and β are not cubes. Proceeding as before, we get the equation

$$y^3 \frac{(\mu^3 + \alpha)}{\beta} + z^3 = 0;$$

so $\frac{\mu^3 + \alpha}{\beta}$ must be a non-cube for every value of μ. If $\frac{\mu^3 + \alpha}{\beta}$ were a perfect cube, it could be put in the form:

$$\frac{\mu^3 + \alpha}{\beta} = w^3 + w^2 + w + 1.$$

If we put $w = 1/x$, then $x = x' + \beta/(\mu^3 + \alpha + \beta)$ and we get (18) with $a = (\mu^6 + \alpha^2 + \beta^2)/(\beta\mu^3 + \alpha\beta)$ and the discussion given for (17) is applicable.

We study next the nondegenerate cubics with $k \neq 0$. First, we suppose the cubic (1) has a crunode P and no real inflections. We take (0, 0, 1) as P, also $x = 0$ and $y = 0$ as the two tangents to (1) at P. We put:

$$x = x', \quad y = y', \quad z = \frac{d}{k}x' + \frac{e}{k}y' + \frac{b}{k}z',$$

and we easily get:

$$\alpha x^3 + y^3 + xyz = 0, \qquad (\alpha \neq 0). \qquad \text{(21) Set 7}$$

Here α must be a non-cube as otherwise $z = \alpha^{2/3}x + \alpha^{1/3}y$ is the tangent at a real inflection; hence† $2^n = 3k + 1$.

We consider the case where $k \neq 0$, (1) has a crunode P and just one real inflection R. We take P as (0, 0, 1), PR as $x = 0$, the tangent to (1) at R as $z = 0$, the two tangents at P as $x = y$ and $y = 0$. We easily get:

$$x^3 + y^2z + xyz = 0. \qquad \text{(22) Class 8}$$

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* We know nothing concerning the equivalence or nonequivalence of (17) and (17').

† We note that in $GF[2^n]$, $n > 1$ every number that is a cube has three real cube roots, so in these fields if α is a cube then (21) has three real inflections.

Solving (22) simultaneously with $z = ax+by$ we get $c^3+c+1 = 0$, which is irreducible only in $GF[2]$.

If (1) has a crunode P and at least two real inflections R and S, we take as $x = 0$ the line PR, as $y = 0$ the line PS, as $z = 0$ the line RST, where T is on the cubic. The tangent at R we take as $x = z$, the tangent at S we take as $y = z$. This gives us finally:

(23) Class *9* $\qquad x^3y+xy^3+x^3z+y^3z+xyz = 0.$

Note that the tangents at the crunode are given by $x^2+y^2+xy = 0$; so this class exists only in $GF[2^n]$, $n>1$. There is one more inflection, the point (1, 1, 0), with tangent $x+y+z = 0$. The three inflections are collinear.

We consider next the cubic (1) with acnode P and no real inflections. We take P as (0, 0, 1); for $x = 0$ we take any line PR, where R is on the cubic; for $z = 0$ we take the tangent RS at R, where S is on the cubic; and take PS as $y = 0$. We easily get:

(24) Set *10* $\qquad x^3y+x^3z+\alpha y^3z+xyz = 0,$

where $\alpha \neq 0$, and $x^3+\alpha y^3+xy = 0$ gives the two tangents at P and so must be irreducible [compare equation (7)]. Solving (24) simultaneously with $z = ax+by$, we get $c^3+\alpha c^2+\alpha^2 = 0$, which must be irreducible in the $GF[2^n]$ if (24) is to have no real inflections.

We suppose now that the cubic (1) has an acnode P and just one real inflection R. We take P as (0, 0, 1), PR as $x = 0$, the line $y = 0$ in such a way that the tangents at P are given by $x^2+\alpha y^2+xy = 0$ (a given irreducible quadratic equation). For $z = 0$ we take the tangent at the inflection R. Then we easily get:

(25) Set *11* $\qquad x^3+x^2z+\alpha y^3z+xyz = 0,$

$\alpha \neq 0$. If we solve (25) simultaneously with $z = ax+by$ we get $\alpha^2c^3+\alpha c+\alpha+1 = 0$, which must be irreducible, since (25) is to have just one real inflection. This set does not exist in $GF[2]$.

We consider the cubic (1) when it has an acnode P and at least two real inflections R, S. In the argument that gave us the cubic (23) we used merely the fact that P was a node (not distinguishing between crunode and acnode). So we have in $GF[2]$ alone:

Class *12*, with typical cubic (23).

We study next the cubics with $k \neq 0$ and with no nodes. We suppose first that (1) has no real inflections. We now prove that on every such

cubic there exist points with polar conics that are real line pairs. We show this by taking a tangent RS at a point R (where S is on the cubic); then since there is at least one real tangent SR to the cubic from S, the polar conic of S is a real line pair SR, SP (where P lies on the cubic). We take as S the point (0, 0, 1), as $x = 0$ the line SR, as $y = 0$ the line SP, as $z = 0$ the line PRT (where T is on the cubic), as (1, 1, 0) the point T, as $x+y+z = 0$ the tangent at T. We easily get:

(25) Set *13* $$x^2y+xy^2+\alpha xz^2+\beta yz^2++xyz = 0,$$

where $\alpha\beta \neq 0$, and $\alpha \neq \beta$ since otherwise (1, 1, 0) is an inflection. The test (11) for nodes gives us equations in x', y', z' that are incompatible, so (26) has no node. We now solve (26) simultaneously with $x = ay+bz$ and get:

$$\alpha^5c^9+(\alpha^5+\beta\alpha^4+\alpha^4)\,c^5+\alpha^3c^7+(\alpha^3+\alpha^2)\,c^6$$
$$+\alpha^2c^4+\alpha c^3+(\alpha^2+\alpha)\,c^2+\alpha c+\alpha+\beta = 0,$$

which must be irreducible in $GF[2^n]$ since (26) is to have no real inflections.

We suppose now that the cubic (1) has no node and has just one real inflection P. We take (0, 1, 0) as P, $x = 0$ as the tangent PR from P (where R is the point of contact). First we suppose R has a real line pair as its polar conic with respect to (1). For $y = 0$ we take a tangent RS from R (where S is on the cubic), for $z = 0$ we take the line PST (where T is on the cubic), for T the point (1, 1, 0), for the tangent at P the line $x = z$. We easily get:

(27) Set *14* $$x^2y+xy^2+\alpha xz^2+y^2z+xyz = 0,$$

$\alpha \neq 0$. We note that $x = y$ is the tangent at (1, 1, 0). The test (11) shows that (27) has no nodes. We solve (27) simultaneously with $z = ax+by$ and we get:

$$c^8+c^6+c^5+c^3+\alpha^2 = 0,$$

which must be irreducible since (27) is to have just the one real inflection.

Now we suppose the point R mentioned above has a conjugate imaginary line pair for its polar conic. We take $y = 0$ through R in such a way that this line pair is $x^2+\alpha y^2+xy = 0$ (a given irreducible quadratic equation). We take, as before, $x = 0$ as the line PR, and also $z = 0$ as the tangent at the inflection P. We easily get:

(28) Set *15* $$\beta x^3+x^2z+xz^2+\alpha y^2z+xyz = 0,$$

$\alpha\beta \neq 0$. The test (11) shows that (28) has no node. We solve (28) simultaneously with $y = ax + bz$, and get:

$$\alpha\beta^2c^4 + \beta c^3 + \alpha = 0, \quad \alpha a^2 + a + 1 + \beta c = 0,$$

and the quartic in c or the quadratic in a must be irreducible since (28) is to have no more real inflections.

Next we suppose the cubic (1) has no node, but has at least two real inflections P, R. For $x = 0$ we take the tangent at P, for $y = 0$ we take the tangent at R. For $z = 0$ we take the line PRS (where S is on the cubic.). For S we take (1, 1, 0). For the tangent at (1, 1, 0) we take the line $x + y + z = 0$. We easily get:

(29) Set *16* $$\alpha z^3 + x^2y + xy^2 + xyz = 0,$$

$\alpha \neq 0$. We note that (1, 1, 0) is a third inflection. So the real inflections of (29) lie by threes on straight lines. Note also that $y = x$ is tangent to the cubic. The test (11) shows that (29) has no node. We solve (29) simultaneously with $x = ay + bz$ and get:

$$\alpha c^3 + c^2 + c + 1 = 0, \quad a^2 + a = \alpha c^3, \quad b = a + \alpha c^2,$$

to determine any other inflections than the three above mentioned. In order that (29) shall have no more real inflections, the above cubic in c or quadratic in a must be irreducible. We note in passing that, as is readily shown, every cubic in $GF[2^n]$ has 0, 1, 3, 7, or 9 real inflections on it, and these lie by threes on straight lines. Also, if the cubic has seven real inflections, they form the vertices of a complete quadrangle and its diagonal triangle.

Let us now study the cubics (1) with $k \neq 0$ that are imaginary and nondegenerate. If we transform (1) by means of:

$$T\colon\ x = x' + b_1y' + c_1z', \quad y = x' + b_2y' + c_2z', \quad z = x' + b_3y' + c_3z',$$

where

$$\begin{vmatrix} 1, & b_1, & c_1 \\ 1, & b_2, & c_2 \\ 1, & b_3, & c_3 \end{vmatrix} \neq 0,$$

then take $a'e' = d'^2$ (where a', e', d' are coefficients in this new cubic), we have:

$$(a + b + c + \cdots + k) \times$$
$$[ab_1^2 + bb_2^2 + cb_3^2 + db_1^2 + eb_2^2 + fb_3^2 + gb_3^2 + hb_3^2 + jb_3^2 + k(b_1b_2 + b_1b_3 + b_2b_3)]$$
$$= a^2b_1^2 + b^2b_2^2 + c^2b_3^2 + d^2b_2^2 + e^2b_1^2 + f^2b_3^2 + g^2b_1^2 + h^2b_3^2 + j^2b_2^2 + k^2(b_3^2 + b_2^2 + b_1^2).$$

If we determine b_3 by $b_1b_2+b_1b_3+b_2b_3=0$, the above equation takes a form like $b_1^2/b_2^2=\alpha/\beta$, which surely has a real solution in b_1 and b_2; so we can always get $a'e'=d'^2$. We divide the cubic by a' and proceed as we did in deriving the cubic (19) and we get (1) with $a=1$, $b=\delta$ (a non-cube), $ck\neq 0$, $d=e=f=h=0$. Also, as with (19), our imaginary non-degenerate cubics with $k\neq 0$ occur only if $2^n=3k+1$. We have as before the three cases $gj\neq 0$, $g=0$ and $j\neq 0$*, $g=j=0$. So we easily get:

Set *17*, with a first typical cubic:

$$x^3+\alpha y^3+\beta z^3+jxz^2+yz^2+xyz=0, \tag{30}$$

$\alpha\neq$ cube, $\alpha\beta\neq 0$. If we solve $x=\mu z$ simultaneously with (30), the discussion given for the cubic (17) is applicable with $a=\alpha\dfrac{(\mu^3+\beta+\gamma\mu)^2}{(1+\mu)^3}$. From (18) we then get equations in μ that must have no common solution. If $\gamma=0$, then β must be a non-cube.

Also we get a second typical cubic for Set 17:

$$x^3+\alpha y^3+\beta z^3+xyz=0, \tag{30'}$$

$\alpha\neq$ cube, $\beta\neq$ cube. Here the same discussion is applicable if we set $a=\dfrac{\alpha(\mu^3+\beta)^3}{\mu^3}$. We know nothing concerning the equivalence or non-equivalence of (30) and (30').

We shall not give the derivation of the typical cubics of the classes of degenerate cubics but we shall merely list them in the table that we now append. We shall list merely typical cubics, omitting all descriptions and almost all the conditions on their coefficients since all this is in the body of the text.

Class	*1*,	$x^2y+x^2z+yz^2$	$=0$.
Set	*2*,	$x^2y+xy^2+x^2z+\alpha yz^2$	$=0$,
	also	$\alpha xy^2+x^2z+yz^2$	$=0$.
Set	*3*,	$\alpha z^3+x^2y+xy^2+xz^2+yz^2$	$=0$.
Set	*4*,	$\alpha z^3+x^2y+xy^2$	$=0$.
Set	*5*,	$x^3+\alpha y^3+\beta z^3+xz^2+yz^2$	$=0$, only in $GF[2^n]$, $2^n=3k+1$,
	also	$x^3+\alpha y^3+\beta z^3+yz^2$	$=0$, only in $GF[2^n]$, $2^n=3k+1$.
Set	*6*,	$x^3+\alpha y^3+\beta z^3$	$=0$, only in $GF[2^n]$, $2^n=3k+1$.

* The case $g\neq 0$ and $j=0$ is reducible to this case.

Set 7,	αx^3+y^3+xyz	$=0$, only in $GF[2^n], 2^n=3k+1$.
Class 8,	x^3+y^2z+xyz	$=0$, only in $GF[2]$.
Class 9,	$x^2y+xy^2+x^2z+y^2z+xyz$	$=0$, only in $GF[2^n], n>1$.
Set 10,	$x^2y+x^2z+\alpha y^2z+xyz$	$=0$.
Set 11,	$x^3+x^2z+\alpha y^2z+xyz$	$=0$, only in $GF[2^n], n>1$.
Class 12,	$x^2y+xy^2+x^2z+y^2z+xyz$	$=0$, only in $GF[2]$.
Set 13,	$x^2y+xy^2+\alpha xz^2+\beta yz^2+xyz$	$=0$.
Set 14,	$x^2y+xy^2+\alpha xz^2+y^2z+xyz$	$=0$.
Set 15,	$\beta x^3+x^2z+xz^2+\alpha y^2z+xyz$	$=0$.
Set 16,	$\alpha z^3+x^2y+xy^2+xyz$	$=0$.
Set 17,	$x^3+\alpha y^3+\beta z^3+\gamma xz^2+yz^2+xyz$	$=0$, only in $GF[2^n], 2^n=3k+1$.
also	$x^3+\alpha y^3+\beta z^3+xyz$	$=0$, only in $GF[2^n], 2^n=3k+1$.
Class 18,	$x(x^2+yz)$	$=0$.
Class 19,	$x(x^2+y^2+\alpha z^2+yz)$	$=0$, for α see (7).
Class 20,	$y(x^2+yz)$	$=0$.
Class 21,	xyz	$=0$.
Class 22,	$z(x^2+\alpha y^2+xy)$	$=0$, for α see (7).
Class 23,	$xy(x+y)$	$=0$.
Class 24,	$x(x^2+\alpha y^2+xy)$	$=0$, for α see (7).
Class 25,	x^2y	$=0$.
Class 26,	x^3	$=0$.
Class 27,	$x^3+\alpha y^3$	$=0$, $\alpha \neq$ cube; only in $GF[2^n]$, $2^n=3k+1$.
Class 28,	$ax^3+xy^2+y^3$	$=0$, for a see (18).

ON CERTAIN DETERMINANT RELATIONS.

By W. H. Metzler.

1. In 1888 Muir gave* the expansion of a determinant of order five, thus

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} - \begin{vmatrix} \cdot & a_2 & a_3 & a_4 & a_5 \\ \cdot & b_2 & b_3 & b_4 & b_5 \\ \cdot & c_2 & c_3 & c_4 & c_5 \\ \cdot & d_2 & d_3 & d_4 & d_5 \\ e_1 & \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & a_1 & a_3 & a_4 & a_5 \\ \cdot & b_1 & b_3 & b_4 & b_5 \\ \cdot & c_1 & c_3 & c_4 & c_5 \\ \cdot & d_1 & d_3 & d_4 & d_5 \\ e_2 & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$- \begin{vmatrix} \cdot & a_1 & a_2 & a_4 & a_5 \\ \cdot & b_1 & b_2 & b_4 & b_5 \\ \cdot & c_1 & c_2 & c_4 & c_5 \\ \cdot & d_1 & d_2 & d_4 & d_5 \\ e_3 & \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & a_1 & a_2 & a_3 & a_5 \\ \cdot & b_1 & b_2 & b_3 & b_5 \\ \cdot & c_1 & c_2 & c_3 & c_5 \\ \cdot & d_1 & d_2 & d_3 & d_5 \\ e_4 & \cdot & \cdot & \cdot & \cdot \end{vmatrix} - \begin{vmatrix} \cdot & a_1 & a_2 & a_3 & a_4 \\ \cdot & b_1 & b_2 & b_3 & b_4 \\ \cdot & c_1 & c_2 & c_3 & c_4 \\ \cdot & d_1 & d_2 & d_3 & d_4 \\ e_5 & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

and showed that instead of the zeros in the first columns we might put A, B, C, D, and instead of the zeros in the last rows we might put $\alpha, \beta, \gamma, \delta$; α, x, y, z; β, x, p, q; γ, y, p, w; δ, z, q, w, respectively and the relation is not thereby altered. In the History† he showed that this was equivalent to combining the frame lines one after another of a Pfaffian of order six with an array of four rows and six columns.

To see what this theorem really is it is better to state it as follows:

If we take with the array

$$\begin{matrix} A & a_1 & a_2 & a_3 & a_4 & a_5 \\ B & b_1 & b_2 & b_3 & b_4 & b_5 \\ C & c_1 & c_2 & c_3 & c_4 & c_5 \\ D & d_1 & d_2 & d_3 & d_4 & d_5 \end{matrix}$$

the rows, one after another, of the zero-axial symmetric determinant

$$\begin{vmatrix} 0 & e_1 & e_2 & e_3 & e_4 & e_5 \\ e_1 & 0 & \alpha & \beta & \gamma & \delta \\ e_2 & \alpha & 0 & x & y & z \\ e_3 & \beta & x & 0 & p & q \\ e_4 & \gamma & y & p & 0 & w \\ e_5 & \delta & z & q & w & 0 \end{vmatrix}$$

* On Vanishing Aggregates of Determinants; Thomas Muir, Proc. Roy. Soc. of Edinb., vol. 15, pp. 96–105.

† Vol. IV, pp. 37–38.

and strike out the column containing the zero, then the sum of the resulting determinants with signs alternately positive and negative is zero.

It will now appear that it is but another statement of Kronecker's theorem* of 1882. Using the double suffix notation and expressing Kronecker's theorem in the umbral form $\Sigma \left|\begin{matrix} 1 & 2 & 3 & 4 & \overline{5} \\ 6 & 7 & 8 & 9 & 10 \end{matrix}\right| = 0$ we see that it involves but six (the last) columns of the determinant $|a_{10,10}|$ and that the six determinants involved are just those obtained by taking with the array consisting of the last six columns of the first four rows of $|a_{10,10}|$, the rows, one after another, of the minor $m \equiv |a_{55}\, a_{66}\, a_{77}\, a_{88}\, a_{99}\, a_{10,10}|$, which is required to be symmetric, and deleting the column containing the principal diagonal element of m.

2. The query naturally arises as to what further generalization may be made by expanding the original determinant, not in terms of the elements of one row and their complementaries, but in terms of the minors formed from k rows and their complementaries and then seeing what new elements may be inserted in place of the zeros without altering the relation.

Thus for a determinant of order five expanded in terms of minors of order two and their complementaries of order three we have

$$
\begin{aligned}
|a_{55}| = {} & \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot \\ a_{21} & a_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{33} & a_{34} & a_{35} \\ \cdot & \cdot & a_{43} & a_{44} & a_{45} \\ \cdot & \cdot & a_{53} & a_{54} & a_{55} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdot & a_{13} & \cdot & \cdot \\ a_{21} & \cdot & a_{23} & \cdot & \cdot \\ \cdot & a_{32} & \cdot & a_{34} & a_{35} \\ \cdot & a_{42} & \cdot & a_{44} & a_{45} \\ \cdot & a_{52} & \cdot & a_{54} & a_{55} \end{vmatrix} \\
(1) \quad + & \begin{vmatrix} a_{11} & \cdot & \cdot & a_{14} & \cdot \\ a_{21} & \cdot & \cdot & a_{24} & \cdot \\ \cdot & a_{32} & a_{33} & \cdot & a_{35} \\ \cdot & a_{42} & a_{43} & \cdot & a_{45} \\ \cdot & a_{52} & a_{53} & \cdot & a_{55} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdot & \cdot & \cdot & a_{15} \\ a_{21} & \cdot & \cdot & \cdot & a_{25} \\ \cdot & a_{32} & a_{33} & a_{34} & \cdot \\ \cdot & a_{42} & a_{43} & a_{44} & \cdot \\ \cdot & a_{52} & a_{53} & a_{54} & \cdot \end{vmatrix} \\
+ & \begin{vmatrix} \cdot & a_{12} & a_{13} & \cdot & \cdot \\ \cdot & a_{22} & a_{23} & \cdot & \cdot \\ a_{31} & \cdot & \cdot & a_{34} & a_{35} \\ a_{41} & \cdot & \cdot & a_{44} & a_{45} \\ a_{51} & \cdot & \cdot & a_{54} & a_{55} \end{vmatrix} + \begin{vmatrix} \cdot & a_{12} & \cdot & a_{14} & \cdot \\ \cdot & a_{22} & \cdot & a_{24} & \cdot \\ a_{31} & \cdot & a_{33} & \cdot & a_{35} \\ a_{41} & \cdot & a_{43} & \cdot & a_{45} \\ a_{51} & \cdot & a_{53} & \cdot & a_{55} \end{vmatrix}
\end{aligned}
$$

* Kronecker, "Die Subdeterminanten symmetrischer Systeme", Berliner Berichte, 1882, pp. 821–824.

$$
(1)\qquad
\begin{array}{l}
\begin{vmatrix} \cdot & a_{12} & \cdot & \cdot & a_{15} \\ \cdot & a_{22} & \cdot & \cdot & a_{25} \\ a_{31} & \cdot & a_{33} & a_{34} & \cdot \\ a_{41} & \cdot & a_{43} & a_{44} & \cdot \\ a_{51} & \cdot & a_{53} & a_{54} & \cdot \end{vmatrix}
+
\begin{vmatrix} \cdot & \cdot & a_{13} & a_{14} & \cdot \\ \cdot & \cdot & a_{23} & a_{24} & \cdot \\ a_{31} & a_{32} & \cdot & \cdot & a_{35} \\ a_{41} & a_{42} & \cdot & \cdot & a_{45} \\ a_{51} & a_{52} & \cdot & \cdot & a_{55} \end{vmatrix} \\
+
\begin{vmatrix} \cdot & \cdot & a_{13} & \cdot & a_{15} \\ \cdot & \cdot & a_{23} & \cdot & a_{25} \\ a_{31} & a_{32} & \cdot & a_{34} & \cdot \\ a_{41} & a_{42} & \cdot & a_{44} & \cdot \\ a_{51} & a_{52} & \cdot & a_{54} & \cdot \end{vmatrix}
+
\begin{vmatrix} \cdot & \cdot & \cdot & a_{14} & a_{15} \\ \cdot & \cdot & \cdot & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot \\ a_{41} & a_{42} & a_{43} & \cdot & \cdot \\ a_{51} & a_{52} & a_{53} & \cdot & \cdot \end{vmatrix}
\end{array}
$$

The problem now is to fill the places occupied by the zeros in the foregoing ten determinants with new numbers so as to leave the relation still true.

If we write

$$
(2)\qquad
\begin{array}{l}
|a_{55}| = \begin{vmatrix} a_{11} & a_{12} & P_1 & P_2 & P_3 \\ a_{21} & a_{22} & Q_1 & Q_2 & Q_3 \\ x_4 & x_5 & a_{33} & a_{34} & a_{35} \\ x_4 & x_5 & a_{43} & a_{44} & a_{45} \\ x_4 & x_5 & a_{53} & a_{54} & a_{55} \end{vmatrix}
+
\begin{vmatrix} a_{11} & P_1 & a_{13} & P_4 & P_5 \\ a_{21} & Q_1 & a_{23} & Q_4 & Q_5 \\ x_3 & a_{32} & x_5 & a_{34} & a_{35} \\ x_3 & a_{42} & x_5 & a_{44} & a_{45} \\ x_3 & a_{52} & x_5 & a_{54} & a_{55} \end{vmatrix} \\
+
\begin{vmatrix} a_{11} & P_2 & P_4 & a_{14} & P_6 \\ a_{21} & Q_2 & Q_4 & a_{24} & Q_6 \\ x_2 & a_{32} & a_{33} & x_5 & a_{35} \\ x_2 & a_{42} & a_{43} & x_5 & a_{45} \\ x_2 & a_{52} & a_{53} & x_5 & a_{55} \end{vmatrix}
+
\begin{vmatrix} a_{11} & P_3 & P_5 & P_6 & a_{15} \\ a_{21} & Q_3 & Q_5 & Q_6 & a_{25} \\ x_1 & a_{32} & a_{33} & a_{34} & x_5 \\ x_1 & a_{42} & a_{43} & a_{44} & x_5 \\ x_1 & a_{52} & a_{53} & a_{54} & x_5 \end{vmatrix} \\
+
\begin{vmatrix} P_1 & a_{12} & a_{13} & P_7 & P_8 \\ Q_1 & a_{22} & a_{23} & Q_7 & Q_8 \\ a_{31} & x_3 & x_4 & a_{34} & a_{35} \\ a_{41} & x_3 & x_4 & a_{44} & a_{45} \\ a_{51} & x_3 & x_4 & a_{54} & a_{55} \end{vmatrix}
+
\begin{vmatrix} P_2 & a_{12} & P_7 & a_{14} & P_9 \\ Q_2 & a_{22} & Q_7 & a_{14} & Q_9 \\ a_{31} & x_2 & a_{33} & x_4 & a_{35} \\ a_{41} & x_2 & a_{43} & x_4 & a_{45} \\ a_{51} & x_2 & a_{53} & x_4 & a_{55} \end{vmatrix} \\
+
\begin{vmatrix} P_3 & a_{12} & P_8 & P_9 & a_{15} \\ Q_3 & a_{22} & Q_8 & Q_9 & a_{25} \\ a_{31} & x_1 & a_{33} & a_{34} & x_4 \\ a_{41} & x_1 & a_{43} & a_{44} & x_4 \\ a_{51} & x_1 & a_{53} & a_{54} & x_4 \end{vmatrix}
+
\begin{vmatrix} P_4 & P_7 & a_{13} & a_{14} & P_{10} \\ Q_4 & Q_7 & a_{23} & a_{24} & Q_{10} \\ a_{31} & a_{32} & x_2 & x_3 & a_{35} \\ a_{41} & a_{42} & x_2 & x_3 & a_{45} \\ a_{51} & a_{52} & x_2 & x_3 & a_{55} \end{vmatrix}
\end{array}
$$

$$(2) \qquad + \begin{vmatrix} P_5 & P_8 & a_{13} & P_{10} & a_{15} \\ Q_5 & Q_8 & a_{23} & Q_{10} & a_{25} \\ a_{31} & a_{32} & x_1 & a_{34} & x_3 \\ a_{41} & a_{42} & x_1 & a_{44} & x_3 \\ a_{51} & a_{52} & x_1 & a_{54} & x_3 \end{vmatrix} + \begin{vmatrix} P_6 & P_9 & P_{10} & a_{14} & a_{15} \\ Q_6 & Q_9 & Q_{10} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & x_1 & x_2 \\ a_{41} & a_{42} & a_{43} & x_1 & x_2 \\ a_{51} & a_{52} & a_{53} & x_1 & x_2 \end{vmatrix}$$

we may show that this is still a true relation by showing that the total coefficient of each of the introduced numbers is zero. The total coefficient of x_1 is readily seen to be

$$+ \begin{vmatrix} P_3 & P_5 & P_6 & a_{15} \\ Q_3 & Q_5 & Q_6 & a_{25} \\ a_{32}-a_{42} & a_{33}-a_{43} & a_{34}-a_{44} & 0 \\ a_{42}-a_{52} & a_{43}-a_{53} & a_{44}-a_{54} & 0 \end{vmatrix} - \begin{vmatrix} P_3 & P_8 & P_9 & a_{15} \\ Q_3 & Q_8 & Q_9 & a_{25} \\ a_{31}-a_{41} & a_{33}-a_{43} & a_{34}-a_{44} & 0 \\ a_{41}-a_{51} & a_{43}-a_{53} & a_{44}-a_{54} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} P_5 & P_8 & P_{10} & a_{15} \\ Q_5 & Q_8 & Q_{10} & a_{25} \\ a_{31}-a_{41} & a_{32}-a_{42} & a_{34}-a_{44} & 0 \\ a_{41}-a_{51} & a_{42}-a_{52} & a_{44}-a_{54} & 0 \end{vmatrix} - \begin{vmatrix} P_6 & P_9 & P_{10} & a_{15} \\ Q_6 & Q_9 & Q_{10} & a_{25} \\ a_{31}-a_{41} & a_{32}-a_{42} & a_{33}-a_{43} & 0 \\ a_{41}-a_{51} & a_{42}-a_{52} & a_{43}-a_{53} & 0 \end{vmatrix}$$

and the coefficient of a_{25} in this is

$$\left[\begin{vmatrix} P_3 & P_5 & P_6 \\ \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_2 & \beta_3 & \beta_4 \end{vmatrix} - \begin{vmatrix} P_3 & P_8 & P_9 \\ \alpha_1 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_3 & \beta_4 \end{vmatrix} + \begin{vmatrix} P_5 & P_8 & P_{10} \\ \alpha_1 & \alpha_2 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_4 \end{vmatrix} - \begin{vmatrix} P_6 & P_9 & P_{10} \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} \right],$$

where

$$\alpha_1 = a_{31}-a_{41}, \quad \alpha_2 = a_{32}-a_{42}, \quad \alpha_3 = a_{33}-a_{43}, \quad \alpha_4 = a_{34}-a_{44};$$
$$\beta_1 = a_{41}-a_{51}, \quad \beta_2 = a_{42}-a_{52}, \quad \beta_3 = a_{43}-a_{53}, \quad \beta_4 = a_{44}-a_{54}.$$

This coefficient of a_{25} is the sum of the results of combining with the array

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{matrix}$$

the rows one after another of the zero-axial symmetric determinant

$$\begin{vmatrix} 0 & P_3 & P_5 & P_6 \\ P_3 & 0 & P_8 & P_9 \\ P_5 & P_8 & 0 & P_{10} \\ P_6 & P_9 & P_{10} & 0 \end{vmatrix}$$

and striking out the columns containing the zeros, and is an example of Kronecker's theorem referred to in § 1 aud is zero. Similarly for the coefficient of a_{15}, and therefore the total coefficient of x_1 is zero. The same process would prove that the coefficient of any x is zero.

The total coefficient of P_1 is

$$\begin{vmatrix} a_{21} & a_{22} & Q_2 & Q_3 \\ x_4 & x_5 & a_{34} & a_{35} \\ x_4 & x_5 & a_{44} & a_{45} \\ x_4 & x_5 & a_{54} & a_{55} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{23} & Q_4 & Q_5 \\ x_3 & x_5 & a_{34} & a_{35} \\ x_3 & x_5 & a_{44} & a_{45} \\ x_3 & x_5 & a_{54} & a_{55} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} & Q_7 & Q_6 \\ x_3 & x_4 & a_{34} & a_{35} \\ x_3 & x_4 & a_{44} & a_{45} \\ x_3 & x_4 & a_{54} & a_{55} \end{vmatrix}$$

which is readily seen to be zero, and similarly for any other P. What is true of the coefficients of the P's is also true of the coefficients of the Q's. It follows therefore that the relation (1) is not altered by writing it in the form (2) where we have introduced $5+2\times10=25$ new numbers.

3. The manner in which the new numbers have been introduced is perhaps best exhibited in diagrammatic form.

Using $|a_{mn}|_{(n-k)\times k}$ to denote the expansion as in (1) of the determinant of the nth order in terms of minors of order k and their complementaries of order $n-k$, the number of terms in $|a_{55}|_{3\times2}$ would be $(5)_2=10$, and if we use *45*123 to denote

$$\begin{vmatrix} a_{11} & a_{12} & P_1 & P_2 & P_3 \\ a_{21} & a_{22} & Q_1 & Q_2 & Q_3 \\ x_4 & x_5 & a_{33} & a_{34} & a_{35} \\ x_4 & x_5 & a_{43} & a_{44} & a_{45} \\ x_4 & x_5 & a_{52} & a_{54} & a_{55} \end{vmatrix}$$

where the italics give the subscripts of the x's and the Roman give the subscripts of the P's and Q's. The the ten determinants on the right of (2) will be expressed in the following diagram

4	5	1	2	3
3	1	5	4	5
2	2	4	5	6
1	3	5	6	5
1	3	4	7	8
2	2	7	4	9
3	1	8	9	4
4	7	2	3	10
5	8	1	10	3
6	9	10	1	2

Plate 1.

In the case of $|a_{66}|_{3\times3}$, $|a_{66}|_{4\times2}$, and $|a_{77}|_{4\times3}$ the right hand sides are given in diagrams (1), (2), (3) of plate II

(1)

10	14	15	1	2	3
9	13	1	15	4	5
8	12	2	4	15	6
7	11	3	5	6	15
6	1	13	14	7	8
5	2	12	7	14	9
4	3	11	8	9	14
3	4	7	12	13	10
2	6	8	11	10	13
1	6	9	10	11	12
1	6	9	10	11	12
2	5	8	11	10	13
3	4	7	12	13	10
4	3	11	8	9	14
5	2	12	7	14	9
6	1	13	14	7	8
7	11	3	5	6	15
8	12	2	4	15	6
9	13	1	15	4	5
10	14	15	1	2	3

(3)

15	20	21	1	2	3	4
14	19	1	21	5	6	7
13	18	2	5	21	8	9
12	17	3	6	8	21	10
11	16	4	7	9	10	21
10	1	19	20	11	12	13
9	2	18	11	20	14	15
8	3	17	12	14	20	16
7	4	16	13	15	16	20
6	5	11	18	19	17	18
5	6	12	17	17	19	19
4	7	13	16	18	19	19
3	8	14	17	17	18	20
2	9	15	18	16	20	18
1	10	16	19	20	16	17
1	10	14	15	21	22	23
2	9	13	21	15	24	25
3	8	12	22	24	15	26
4	7	11	23	25	26	15
5	6	21	13	14	27	28
6	5	22	12	27	14	29
7	4	23	11	28	29	14
8	3	24	27	12	13	30
9	2	25	28	11	30	13
10	1	26	29	30	11	12
11	21	6	9	10	31	32
12	22	5	8	31	10	33
13	23	4	7	32	33	10
14	24	3	31	8	9	34
15	25	2	32	7	34	9
16	26	1	33	34	7	8
17	27	31	3	5	6	35
18	28	32	2	4	35	6
19	29	33	1	35	4	5
20	30	34	35	1	2	3

(2)

5	6	1	2	3	4
4	1	6	5	6	7
3	2	5	6	8	9
2	3	6	8	6	10
1	4	7	9	10	6
1	4	5	11	12	13
2	3	11	5	14	15
3	2	12	14	5	16
4	1	13	15	16	5
5	11	3	4	17	18
6	12	2	17	4	19
7	13	1	18	19	4
8	14	17	2	3	20
9	15	18	1	20	3
10	16	19	20	1	2

Plate II.

4. The law of formation of these is perhaps apparent but the following observations for the general case may be made as a guide.

1) The number of determinants represented in the diagram is $(n)_k$.

2) The number of distinct x's introduced is $(n)_{k-1}$.

3) The number of distinct P's introduced is $(n)_{n-k-1} = (n)_{k+1}$.

4) The total number of new numbers introduced is $(n)_{k-1} + k(n)_{k+1}$.

5) The number of times x_i occurs is $\dfrac{k(n)_k}{(n)_{k-1}} = (n-k+1)$.

6) The number of times P_i occurs is $\dfrac{(n-k)(n)_k}{(n)_{n-k-1}} = k+1$.

7) The x's involved in the total coefficient of any P form by themselves a zero-axial symmetric determinant. Thus in $|a_{77}|_{4\times 3}$ the total coefficient of P_{35} involves

$$\begin{vmatrix} x_3 & x_5 & x_6 & 0 \\ x_2 & x_4 & 0 & x_6 \\ x_1 & 0 & x_4 & x_5 \\ 0 & x_1 & x_2 & x_3 \end{vmatrix}$$

and the coefficient of P_1 involves

$$\begin{vmatrix} x_{15} & x_{20} & x_{21} & 0 \\ x_{14} & x_{19} & 0 & x_{21} \\ x_{13} & 0 & x_{19} & x_{20} \\ 0 & x_{13} & x_{14} & x_{15} \end{vmatrix}.$$

Similarly for the coefficients of the other P's.

8) In the case of $|a_{nn}|_{(n-k)\times k}$, using d's to represent the suffices of the x's, the x's involved in the total coefficent of P_1 are those (using subscripts only) of the determinant

$$\begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1k} & 0 \\ d_{21} & d_{22} & \cdots & 0 & d_{2,k+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & d_{k+1,2} & \cdots & \cdot & d_{k+1,k+1} \end{vmatrix}$$

where $d_{rs} = d_{k+2-s,\,k+2-r}$,

$$d_{1h} = (n-1)_{k-1} + (n-2)_{k-2} \cdots + (n-h)_{k-h}, \qquad (h = 1, 2, \cdots k),$$

$$d_{gh} = d_{1h} - (n-k+g-2)_{g-2}, \qquad \begin{cases} g = 1, 2, \cdots k \\ g+h \leqq k+1 \end{cases},$$

$$= 0 \text{ when } g+h = k+2.$$

9) Similarly the P's involved in the total coefficient of any x form by themselves a similar determinant. Thus the coefficient of x_{19} in $|a_{77}|_{4\times 3}$ involves the P's of the determinant

$$\begin{vmatrix} 0 & P_1 & P_5 & P_6 & P_7 \\ P_1 & 0 & P_{11} & P_{12} & P_{13} \\ P_5 & P_{11} & 0 & P_{17} & P_{18} \\ P_6 & P_{12} & P_{17} & 0 & P_{19} \\ P_7 & P_{13} & P_{18} & P_{19} & 0 \end{vmatrix}.$$

10) In the case of $|a_{nn}|_{(n-k)\times k}$, using c's to represent the suffices of the P's, the P's involved in the total coefficient of x_1 are those (using subscripts only) of the determinant

$$\begin{vmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1,n-k+1} \\ c_{21} & 0 & c_{23} & \cdots & c_{2,n-k+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{n-k+1,1} & c_{n-k+1,2} & c_{n-k+1,3} & \cdots & 0 \end{vmatrix}$$

where

$$c_{rs} = c_{sr},$$

$$c_{rr} = c_{ss} = 0,$$

$$c_{1h} = (n-2)_{n-k-1} + (n-3)_{n-k-2} + \cdots + (n-h)_{n-k-h+1}, \qquad (h = 2, 3, \cdots, n-k+1),$$

$$c_{gh} = c_{g-1,h}\,(n-g)_{n-k-g} \qquad (g = 2, 3, \cdots, k).$$

Similarly for the coefficient of any x.

5. The determinants involving x_1 in $|a_{77}|_{4\times 3}$ are

(3)

$$\begin{vmatrix} P_{20} & P_{30} & P_{34} & P_{35} & a_{15} & a_{16} & a_{17} \\ Q_{20} & Q_{30} & Q_{34} & Q_{35} & a_{25} & a_{26} & a_{27} \\ R_{20} & R_{30} & R_{34} & R_{35} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & x_1 & x_2 & x_3 \\ a_{51} & a_{52} & a_{53} & a_{54} & x_1 & x_2 & x_3 \\ a_{61} & a_{62} & a_{63} & a_{64} & x_1 & x_2 & x_3 \\ a_{71} & a_{72} & a_{73} & a_{74} & x_1 & x_2 & x_3 \end{vmatrix} + \begin{vmatrix} P_{19} & P_{29} & P_{33} & a_{14} & P_{35} & a_{16} & a_{17} \\ Q_{19} & Q_{29} & Q_{33} & a_{24} & Q_{35} & a_{26} & a_{27} \\ R_{19} & R_{29} & R_{33} & a_{34} & R_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & x_1 & a_{45} & x_4 & x_5 \\ a_{51} & a_{52} & a_{53} & x_1 & a_{55} & x_4 & x_5 \\ a_{61} & a_{62} & a_{63} & x_1 & a_{65} & x_4 & x_5 \\ a_{71} & a_{72} & a_{73} & x_1 & a_{75} & x_4 & x_5 \end{vmatrix}$$

$$+ \begin{vmatrix} P_{16} & P_{26} & a_{13} & P_{33} & P_{34} & a_{16} & a_{17} \\ Q_{16} & Q_{26} & a_{23} & Q_{33} & Q_{34} & a_{26} & a_{27} \\ R_{16} & R_{26} & a_{33} & R_{33} & R_{34} & a_{36} & a_{37} \\ a_{41} & a_{42} & x_1 & a_{44} & a_{45} & x_7 & x_8 \\ a_{51} & a_{52} & x_1 & a_{54} & a_{55} & x_7 & x_8 \\ a_{61} & a_{62} & x_1 & a_{64} & a_{65} & x_7 & x_8 \\ a_{71} & a_{72} & x_1 & a_{74} & a_{75} & x_7 & x_8 \end{vmatrix} + \begin{vmatrix} P_{10} & a_{12} & P_{26} & P_{29} & P_{30} & a_{16} & a_{17} \\ Q_{10} & a_{22} & Q_{26} & Q_{29} & Q_{30} & a_{26} & a_{27} \\ R_{10} & a_{32} & R_{26} & R_{29} & R_{30} & a_{36} & a_{37} \\ a_{41} & x_1 & a_{43} & a_{44} & a_{45} & x_{11} & x_{12} \\ a_{51} & x_1 & a_{53} & a_{54} & a_{55} & x_{11} & x_{12} \\ a_{61} & x_1 & a_{63} & a_{64} & a_{65} & x_{11} & x_{12} \\ a_{71} & x_1 & a_{73} & a_{74} & a_{75} & x_{11} & x_{12} \end{vmatrix}$$

$$(3) \qquad + \begin{vmatrix} a_{11} & P_{10} & P_{16} & P_{19} & P_{20} & a_{16} & a_{17} \\ a_{21} & Q_{10} & Q_{16} & Q_{19} & Q_{20} & a_{26} & a_{27} \\ a_{31} & R_{10} & R_{16} & R_{19} & R_{20} & a_{36} & a_{37} \\ x_1 & a_{42} & a_{43} & a_{44} & a_{45} & x_{16} & x_{17} \\ x_1 & a_{52} & a_{53} & a_{54} & a_{55} & x_{16} & x_{17} \\ x_1 & a_{62} & a_{63} & a_{64} & a_{65} & x_{16} & x_{17} \\ x_1 & a_{72} & a_{73} & a_{74} & a_{75} & x_{16} & x_{17} \end{vmatrix}$$

The total coefficient of x_1 in (3) may be written

$$\begin{vmatrix} P_{20} & P_{30} & P_{34} & P_{35} & a_{16} & a_{17} \\ Q_{20} & Q_{30} & Q_{34} & Q_{35} & a_{26} & a_{27} \\ R_{20} & R_{30} & R_{34} & R_{35} & a_{36} & a_{37} \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 & 0 \end{vmatrix} - \begin{vmatrix} P_{19} & P_{29} & P_{33} & P_{35} & a_{16} & a_{17} \\ Q_{19} & Q_{29} & Q_{33} & Q_{35} & a_{26} & a_{27} \\ R_{19} & R_{29} & R_{33} & R_{35} & a_{36} & a_{37} \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_5 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_5 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_5 & 0 & 0 \end{vmatrix}$$

$$(4) + \begin{vmatrix} P_{16} & P_{26} & P_{33} & P_{34} & a_{16} & a_{17} \\ Q_{16} & Q_{26} & Q_{33} & Q_{34} & a_{26} & a_{27} \\ R_{16} & R_{26} & R_{33} & R_{34} & a_{36} & a_{37} \\ \alpha_1 & \alpha_2 & \alpha_4 & \alpha_5 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_4 & \beta_5 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_4 & \gamma_5 & 0 & 0 \end{vmatrix} - \begin{vmatrix} P_{10} & P_{26} & P_{29} & P_{30} & a_{16} & a_{17} \\ Q_{10} & Q_{26} & Q_{29} & Q_{30} & a_{26} & a_{27} \\ R_{10} & R_{26} & R_{29} & R_{30} & a_{36} & a_{37} \\ \alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & 0 & 0 \\ \beta_1 & \beta_3 & \beta_4 & \beta_5 & 0 & 0 \\ \gamma_1 & \gamma_3 & \gamma_4 & \gamma_5 & 0 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} P_{10} & P_{16} & P_{19} & P_{20} & a_{16} & a_{17} \\ Q_{10} & Q_{16} & Q_{19} & Q_{20} & a_{26} & a_{27} \\ R_{10} & R_{16} & R_{19} & R_{20} & a_{36} & a_{37} \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & 0 & 0 \\ \beta_2 & \beta_3 & \beta_4 & \beta_5 & 0 & 0 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & 0 & 0 \end{vmatrix}$$

where

$$\alpha_i = a_{4i} - a_{5i}, \quad \beta_i = a_{5i} - a_{6i}, \quad \gamma_i = a_{6i} - a_{7i}, \qquad (i = 1, 2, 3, 4, 5).$$

In (4) the coefficient of $\begin{vmatrix} a_{26} & a_{27} \\ a_{36} & a_{37} \end{vmatrix}$ is readily seen to be an example of Kronecker's theorem applied to the array

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \end{matrix}$$

and the determinant

$$\begin{vmatrix} P_{20} & P_{30} & P_{34} & P_{35} & 0 \\ P_{19} & P_{29} & P_{33} & 0 & P_{35} \\ P_{16} & P_{26} & 0 & P_{33} & P_{34} \\ P_{10} & 0 & P_{26} & P_{29} & P_{30} \\ 0 & P_{10} & P_{16} & P_{19} & P_{20} \end{vmatrix}.$$

Similarly for the coefficients of $\begin{vmatrix} a_{16} & a_{17} \\ a_{36} & a_{37} \end{vmatrix}$ and $\begin{vmatrix} a_{16} & a_{17} \\ a_{26} & a_{27} \end{vmatrix}$. It follows therefore that the total coefficient of x_1 is zero and what is true of x_1 is true of any x.

The total coefficient of P_1 in $|a_{77}|_{4\times 3}$ is seen to be

$$(5)\quad \begin{vmatrix} a_{22} & a_{23} & a_{24} & Q_{21} & Q_{22} & Q_{23} \\ a_{32} & a_{33} & a_{34} & R_{21} & R_{22} & R_{23} \\ x_{10} & x_{14} & x_{15} & a_{45} & a_{46} & a_{47} \\ x_{10} & x_{14} & x_{15} & a_{55} & a_{56} & a_{57} \\ x_{10} & x_{14} & x_{15} & a_{65} & a_{66} & a_{67} \\ x_{10} & x_{14} & x_{15} & a_{75} & a_{76} & a_{77} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{23} & a_{24} & Q_{11} & Q_{12} & Q_{13} \\ a_{31} & a_{33} & a_{34} & R_{11} & R_{12} & R_{13} \\ x_{10} & x_{19} & x_{23} & a_{45} & a_{46} & a_{47} \\ x_{10} & x_{19} & x_{23} & a_{55} & a_{56} & a_{57} \\ x_{10} & x_{19} & x_{23} & a_{65} & a_{66} & a_{67} \\ x_{10} & x_{19} & x_{23} & a_{75} & a_{76} & a_{77} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{21} & a_{22} & a_{24} & Q_5 & Q_6 & Q_7 \\ a_{31} & a_{32} & a_{34} & R_5 & R_6 & R_7 \\ x_{14} & x_{19} & x_{21} & a_{45} & a_{46} & a_{47} \\ x_{14} & x_{19} & x_{21} & a_{55} & a_{56} & a_{57} \\ x_{14} & x_{19} & x_{21} & a_{65} & a_{66} & a_{67} \\ x_{14} & x_{19} & x_{21} & a_{75} & a_{76} & a_{77} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} & a_{23} & Q_2 & Q_3 & Q_4 \\ a_{31} & a_{32} & a_{33} & R_2 & R_3 & R_4 \\ x_{15} & x_{20} & x_{21} & a_{45} & a_{46} & a_{47} \\ x_{15} & x_{20} & x_{21} & a_{55} & a_{56} & a_{57} \\ x_{15} & x_{20} & x_{21} & a_{65} & a_{66} & a_{67} \\ x_{15} & x_{20} & x_{21} & a_{75} & a_{76} & a_{77} \end{vmatrix}.$$

This is obviously independent of the Q's and R's. The coefficient of $|a_{55}\ a_{66}\ a_{77}|$ in (5) is an example of Kronecker's theorem applied to the array

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{matrix}$$

and the determinant

$$\begin{vmatrix} 0 & x_{10} & x_{14} & x_{15} \\ x_{10} & 0 & x_{19} & x_{20} \\ x_{14} & x_{19} & 0 & x_{21} \\ x_{15} & x_{20} & x_{21} & 0 \end{vmatrix}$$

and is therefore zero.

The same is true of the coefficient of each of the other determinants formed from the array

$$\begin{matrix} a_{45} & a_{46} & a_{47} \\ a_{55} & a_{56} & a_{57} \\ a_{65} & a_{66} & a_{67} \\ a_{75} & a_{76} & a_{77} \end{matrix}$$

and therefore the coefficient of P_1 is zero. What is true of the coefficient of P_1 is true for any other P and what is true of the P's is also true of the Q's and R's.

6. The general theorem, the proof of which would proceed on precisely the same lines, is that, if we expand the determinant $|a_{nn}|_{(n-k)\times k}$ as in (1) and then fill the vacant places by new numbers according to the plan indicated by the diagrams in the plates and the observations 1)–10) of § 4, the relation remains true.

7. Expanding $|a_{44}|$ we have

$$(6)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdot & \cdot & \cdot \\ \cdot & a_{22} & a_{23} & a_{24} \\ \cdot & a_{32} & a_{33} & a_{34} \\ \cdot & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} \cdot & a_{12} & \cdot & \cdot \\ a_{21} & \cdot & a_{23} & a_{24} \\ a_{31} & \cdot & a_{33} & a_{34} \\ a_{41} & \cdot & a_{43} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} \cdot & \cdot & a_{13} & \cdot \\ a_{21} & a_{22} & \cdot & a_{24} \\ a_{31} & a_{32} & \cdot & a_{34} \\ a_{41} & a_{42} & \cdot & a_{44} \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot & a_{14} \\ a_{21} & a_{22} & a_{23} & \cdot \\ a_{31} & a_{32} & a_{33} & \cdot \\ a_{41} & a_{42} & a_{43} & \cdot \end{vmatrix}.$$

In place of the zeros we may insert as follows and still have a true relation.

$$(7)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & s & t & u \\ x & a_{22} & a_{23} & a_{24} \\ y & a_{32} & a_{33} & a_{34} \\ z & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} s & a_{12} & r & q \\ a_{21} & x & a_{23} & a_{24} \\ a_{31} & y & a_{33} & a_{34} \\ a_{41} & z & a_{43} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} t & r & a_{13} & p \\ a_{21} & a_{22} & x & a_{24} \\ a_{31} & a_{32} & y & a_{34} \\ a_{41} & a_{42} & z & a_{44} \end{vmatrix} + \begin{vmatrix} u & q & p & a_{14} \\ a_{21} & a_{22} & a_{23} & x \\ a_{31} & a_{32} & a_{33} & y \\ a_{41} & a_{42} & a_{43} & z \end{vmatrix}.$$

That (7) is true as being an example of Kronecker's theorem may be seen on taking the sum of the determinants resulting from taking with the array

$$\begin{matrix} x & a_{21} & a_{22} & a_{23} & a_{24} \\ y & a_{31} & a_{32} & a_{33} & a_{34} \\ z & a_{41} & a_{42} & a_{43} & a_{44} \end{matrix}$$

the rows, one after another, of the determinant

$$\begin{vmatrix} 0 & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{11} & 0 & s & t & u \\ a_{12} & s & 0 & p & q \\ a_{13} & t & p & 0 & r \\ a_{14} & u & q & r & 0 \end{vmatrix}$$

and striking out the column containing the zero in each case.

Again, expanding $|a_{44}|$ in terms of the elements of the first column and their complementaries and filling in similarly as in (7) we get

$$(8)\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & x' & y' & z' \\ l & a_{22} & a_{23} & a_{24} \\ m & a_{32} & a_{33} & a_{34} \\ n & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} l & a_{12} & a_{13} & a_{14} \\ a_{21} & x' & y' & z' \\ \gamma & a_{32} & a_{33} & a_{34} \\ \beta & a_{42} & a_{43} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} m & a_{12} & a_{13} & a_{14} \\ \gamma & a_{22} & a_{23} & a_{24} \\ a_{31} & x' & y' & z' \\ \alpha & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} n & a_{12} & a_{13} & a_{14} \\ \beta & a_{22} & a_{23} & a_{24} \\ \alpha & a_{32} & a_{33} & a_{34} \\ a_{41} & x' & y' & z' \end{vmatrix}.$$

Equating the right hand sides of (7) and (8) we get

$$(9)\quad \begin{vmatrix} a_{11} & s & t & u \\ x & a_{22} & a_{23} & a_{24} \\ y & a_{32} & a_{33} & a_{34} \\ z & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} s & a_{12} & r & q \\ a_{21} & x & a_{23} & a_{24} \\ a_{31} & y & a_{33} & a_{34} \\ a_{41} & z & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} t & r & a_{13} & p \\ a_{21} & a_{22} & x & a_{24} \\ a_{31} & a_{32} & y & a_{34} \\ a_{41} & a_{42} & z & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} u & q & p & a_{14} \\ a_{21} & a_{22} & a_{23} & x \\ a_{31} & a_{32} & a_{33} & y \\ a_{41} & a_{42} & a_{43} & z \end{vmatrix} = \begin{vmatrix} a_{11} & x' & y' & z' \\ l & a_{22} & a_{23} & a_{24} \\ m & a_{32} & a_{33} & a_{34} \\ n & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} l & a_{12} & a_{13} & a_{14} \\ a_{21} & x' & y' & z' \\ \gamma & a_{32} & a_{33} & a_{34} \\ \beta & a_{42} & a_{43} & a_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} m & a_{12} & a_{13} & a_{14} \\ \gamma & a_{22} & a_{23} & a_{24} \\ a_{31} & x' & y' & z' \\ \alpha & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} n & a_{12} & a_{13} & a_{14} \\ \beta & a_{22} & a_{23} & a_{24} \\ \alpha & a_{32} & a_{33} & a_{34} \\ a_{41} & x' & y' & z' \end{vmatrix}$$

a relation between eight determinants of order four containing 34 letters, which should be compared with Muir's similar relation* (vii) containing 31 letters.

If in (9) $x', y', z', = s, t, u,$ and $l, m, n = x, y, z$ respectively then the first determinant on the left cancels with the first on the right and we have

$$\begin{vmatrix} s & a_{12} & r & q \\ a_{21} & x & a_{23} & a_{24} \\ a_{31} & y & a_{33} & a_{34} \\ a_{41} & z & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} t & r & a_{13} & p \\ a_{21} & a_{22} & x & a_{24} \\ a_{31} & a_{32} & y & a_{34} \\ a_{41} & a_{42} & z & a_{44} \end{vmatrix} + \begin{vmatrix} u & q & p & a_{14} \\ a_{21} & a_{22} & a_{23} & x \\ a_{31} & a_{32} & a_{33} & y \\ a_{41} & a_{42} & a_{43} & z \end{vmatrix}$$

(10)

$$= \begin{vmatrix} x & a_{12} & a_{13} & a_{14} \\ a_{21} & s & t & u \\ \gamma & a_{32} & a_{33} & a_{34} \\ \alpha & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} y & a_{12} & a_{13} & a_{14} \\ \gamma & a_{22} & a_{23} & a_{24} \\ a_{31} & s & t & u \\ \alpha & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} z & a_{12} & a_{13} & a_{14} \\ \beta & a_{22} & a_{22} & a_{24} \\ \alpha & a_{32} & a_{33} & a_{34} \\ a_{41} & s & t & u \end{vmatrix}$$

a relation involving six determinants and 27 letters which should be compared with that obtained from equating the two Kronecker relations

$$\begin{vmatrix} 1 & 2 & 3 & \overline{4} \\ \underline{5} & 6 & 7 & 8 \end{vmatrix} = 0 = \begin{vmatrix} \overline{1} & 2 & 3 & 4 \\ \underline{5} & 6 & 7 & 8 \end{vmatrix}$$

which involves six determinants and 28 letters.

If in (9) we put $x', y', z' = x, y, z$ and $l, m, n = s, t, u$ respectively and make $|\, a_{22}\ a_{33}\ a_{44}\,|$ symmetric then the first term on the left cancels with the first term on the right and we have

$$\begin{vmatrix} s & a_{12} & r & q \\ a_{21} & x & a_{23} & a_{24} \\ a_{31} & y & a_{33} & a_{34} \\ a_{41} & z & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} t & r & a_{13} & p \\ a_{21} & a_{22} & x & a_{24} \\ a_{31} & a_{32} & y & a_{34} \\ a_{41} & a_{42} & z & a_{44} \end{vmatrix} + \begin{vmatrix} u & q & p & a_{14} \\ a_{21} & a_{22} & a_{23} & x \\ a_{31} & a_{32} & a_{33} & y \\ a_{41} & a_{42} & a_{43} & z \end{vmatrix}$$

(11)

$$= \begin{vmatrix} s & a_{12} & a_{13} & a_{14} \\ a_{21} & x & y & z \\ \gamma & a_{32} & a_{33} & a_{34} \\ \beta & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} t & a_{12} & a_{13} & a_{14} \\ \gamma & a_{22} & a_{23} & a_{24} \\ a_{31} & x & y & z \\ \alpha & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} u & a_{12} & a_{13} & a_{14} \\ \beta & a_{22} & a_{23} & a_{24} \\ \alpha & a_{32} & a_{33} & a_{34} \\ a_{41} & x & y & z \end{vmatrix}$$

which is a relation involving 6 determinants and 24 letters.

* Muir: Cayley's Linear Relation between Minors of a Special Three-Row Array, Mess. Math., new ser., No. 482, June 1911.

In general if we expand the determinant $|a_{nn}|$ in terms of minors of order k formed from the first k rows and their complementaries, filling in the zero places as in § 4 and then expand in terms of minors of order k formed from the first k columns and their complementaries, filling in as before with different letters, we may equate these two expansions and have a relation between $2(n)_k$ determinants. If we introduce the same letters in the first term of the second expansion as were introduced in the first term of the first expansion except that they be placed in the conjugate positions, then these two will cancel provided $|a_{11} \cdots a_{kk}|$ and $|a_{k+1,\,k+1} \cdots a_{nn}|$ are both symmetric and we have a relation between $2(n)_k - 2$ determinants.

NEW YORK STATE COLLEGE FOR TEACHERS.

March 10th, 1926.

ON THE DECOMPOSITION OF TENSORS.

By J. W. Alexander.

Let $A_{i_1 i_2 \cdots i_p}$ be a covariant tensor in n-space. Then, the number of linearly independent contravariant tensors $X^{s_1 s_2 \cdots s_q}$ satisfying the equations

$$A_{s_1 \cdots s_q t_{q+1} \cdots t_p} X^{s_1 s_2 \cdots s_q} = 0 \tag{1}$$

is independent of the coördinate system in terms of which the components of the tensor $A_{i_1 i_2 \cdots i_p}$ are expressed; hence, *the number of linearly independent equations in the system* (1) *is an invariant* $\varrho_{12 \cdots q}$ *of the tensor* $A_{i_1 i_2 \cdots i_p}$. Other invariants $\varrho_{\alpha_1 \alpha_2 \cdots \alpha_q}$ similar to $\varrho_{12 \cdots q}$ are obtainable by calculating the ranks of systems of equations similar to (1) but such that the subscripts of $A_{i_1 i_2 \cdots i_p}$ that are paired with superscripts of $X^{s_1 s_2 \cdots s_q}$ are the ones occuring in the α_1-th, α_2-th, $\cdots$, α_q-th positions respectively. There are also invariants obtainable by calculating the ranks of composite systems formed by combining two or more groups of equations similar to (1).

To interpret the invariant $\varrho_{12 \cdots q}$, let

$$X^{s_1 s_2 \cdots s_q}_{\alpha}, \qquad (\alpha = 1, 2, \cdots, \mu), \tag{2}$$

be a complete set of linearly independent solutions of the equations (1), and let

$$B^{\beta}_{s_1 s_2 \cdots s_q}, \qquad (\beta = 1, 2, \cdots, \nu), \tag{3}$$

be a complete set of linearly independent solutions of the equations

$$B_{s_1 s_2 \cdots s_q} X^{s_1 s_2 \cdots s_q}_{\alpha} = 0, \qquad (\alpha = 1, 2, \cdots, \mu). \tag{4}$$

Then the system (1) evidently admits the same solutions $X^{s_1 s_2 \cdots s_q}$ as the system

$$B^{\beta}_{s_1 s_2 \cdots s_q} X^{s_1 s_2 \cdots s_q} = 0, \qquad (\beta = 1, 2, \cdots, \nu). \tag{5}$$

It therefore follows that the coefficients $A_{s_1 \cdots s_q t_{q+1} \cdots t_p}$ of (1) are linearly expressible in the coefficients $B^{\beta}_{s_1 s_2 \cdots s_q}$ of (4); or, in other words, that the tensor $A_{s_1 \cdots s_q t_{q+1} \cdots t_p}$ may be expressed in the form

$$A_{s_1 \cdots s_q t_{q+1} \cdots t_p} = B^{\beta}_{s_1 \cdots s_q} C_{\beta, t_{q+1} \cdots t_p}, \tag{6}$$

where $C_{\beta, t_{q+1} \cdots t_p}$ $(\beta = 1, 2, \cdots, \nu)$, are ν tensors of order $p-q$. Conversely, if the tensor $A_{i_1 i_2 \cdots i_p}$ is expressible in the form (6), every solution of (5) is a solution of (1); therefore, the invariant $\varrho_{12 \cdots q}$ cannot exceed ν. To summarize:

If the invariant $\varrho_{12 \cdots q}$ of a tensor $A_{i_1 i_2 \cdots i_p}$ is equal to ν, the tensor is expressible in the form (6), *where β runs from* 1 *to ν. Moreover, the tensor cannot be expressed in a form similar to* (6) *with β taking on less than ν values.*

As a corollary to this, we have the following factorization theorem:

A necessary and sufficient condition that a tensor $A_{i_1 i_2 \cdots i_p}$ have a factor $B_{s_1 s_2 \cdots s_q}$ such that

$$A_{s_1 \cdots s_q t_{q+1} \cdots t_p} = B_{s_1 \cdots s_q} C_{t_{q+1} \cdots t_p}$$

is that its invariant $\varrho_{12 \cdots q}$ be equal to unity.

Among the invariants expressing the ranks of composite sets of equations similar to (1) the invariant ϱ equal to the number of linearly independent equations in the system

$$\begin{aligned} A_{s i_2 \cdots i_p} X^s &= 0 \\ A_{i_1 s \cdots i_p} X^s &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ A_{i_1 i_2 \cdots s} X^s &= 0 \end{aligned} \tag{7}$$

is of especial significance. Let

$$X^s_\alpha, \qquad (\alpha = \varrho+1, \ \varrho+2, \cdots, n), \tag{8}$$

be a complete set of linearly independent solutions of (7) and let

$$B^\beta_s, \qquad (\beta = 1, 2, \cdots, \varrho), \tag{9}$$

be a complete set of linearly independent solutions of

$$B_s X^s_\alpha = 0, \quad (\alpha = \varrho+1, \ \varrho+2, \cdots, n). \tag{9.1}$$

Then, the set of solutions (8) of (9.1) is also a complete set of solutions of

$$B^\beta_s X^s = 0, \qquad (\beta = 1, 2, \cdots, \varrho). \tag{10}$$

Let us adjoin to the contravariant vectors (9) an additional set of $n-\varrho$ vectors B^β_s, $\beta > \varrho$, in such a manner that the enlarged set

$$B^\beta_s, \qquad (\beta, s = 1, 2, \cdots, n), \tag{11}$$

is linearly independent. We may then choose as the solutions (8) of (9.1) and (10) the $n-\rho$ covariant vectors X_α^s defined by the conditions

$$B_s^\beta X_\alpha^s = \delta_\alpha^\beta, \quad (\alpha > \rho;\ \beta = 1, 2, \cdots, n), \tag{12}$$

where δ_β^α is the Kronecker delta which has the value 0 or 1 according as α is different from or equal to β.

Now, if (11) is any set of n linearly independent covariant vectors, a covariant tensor $A_{i_1 i_2 \cdots i_p}$ may always be expressed as a form

$$A_{i_1 i_2 \cdots i_p} = \Phi_{\alpha_1 \alpha_2 \cdots \alpha_p} B_{i_1}^{\alpha_1} B_{i_2}^{\alpha_2} \cdots B_{i_p}^{\alpha_p}, \qquad (\alpha_s, i_s = 1, 2, \cdots, n), \tag{13}$$

in the vectors (11) with scalar coefficients. Let us compound both sides of this expression with the vector $X_\alpha^{i_1}$ determined by (12). We find

$$\begin{aligned} 0 &= \Phi_{\alpha_1 \alpha_2 \cdots \alpha_p} (B_{i_1}^{\alpha_1} X_\alpha^{i_1}) B_{i_2}^{\alpha_2} \cdots B_{i_p}^{\alpha_p} \\ &= \Phi_{\alpha\, \alpha_2 \cdots \alpha_p} B_{i_2}^{\alpha_2} \cdots B_{i_p}^{\alpha_p}. \end{aligned} \qquad (\alpha > \rho).$$

In other words, in the right hand member of (13) the variable α_1 need only be summed from 1 to ρ. A similar argument shows that the other variables α_s need only be summed from 1 to ρ. Hence, we have the theorem:

If ρ is the number of linearly independent equations (7), *the tensor $A_{i_1 i_2 \cdots i_p}$ may be expressed as a form*

$$A_{i_1 i_2 \cdots i_p} = \Phi_{\alpha_1 \alpha_2 \cdots \alpha_p} B_{i_1}^{\alpha_1} B_{i_2}^{\alpha_2} \cdots B_{i_p}^{\alpha_p}, \qquad (\alpha_s = 1, 2, \cdots, \rho), \tag{14}$$

in ρ vectors B_i^α with scalar coefficients. Moreover, it cannot be expressed as a similar form in less than ρ vectors.

The proof of the last statement in the theorem follows at once from the fact that every solution X^s of

$$B_s^\alpha X^s = 0$$

is a solution of (7), as we see at once by compounding (14) with X^{i_s}. Hence, if there were less than ρ vectors B_i^α in the form (14) the rank of (7) would be less than ρ, since the system (7) would admit of more than $n-\rho$ independent solutions.

Similar theorems may be proved about contravariant and mixed tensors.

PRINCETON, N. J.

FLOW IN A MÖBIUS STRIP.

By Bernard Osgood Koopman.

Let a rectangle of length a and breadth b be cut from a piece of tin foil, and let the opposite sides, of length b, be joined, so as to form a Möbius strip. Suppose that two or more wires connected to a galvanic cell are placed in contact with the foil; it is our problem to study the nature of the steady flow of electricity which takes place in the strip.

The foil in the neighborhood of any interior point of the strip may be bent into a plane neighborhood without stretching or tearing. Such a deformation will not alter the nature of the flow, which depends only on the internal *metric* of the surface. Hence the potential will go into a harmonic function in the plane neighborhood. The situation is, then, quite similar to that in the familiar problems in the theory of the potential, the only difference lying in the peculiar nature of the manifold from the point of view of analysis situs.

We shall begin by setting the points of the strip into a one to one correspondence with the points of the rectangle in the xy-plane which has for corners the points $(0,0)$, $(a,0)$, (a,b), $(0,b)$. Let the Möbius strip be cut by a line of length b so that it may be developed into a plane rectangle, and let the result be placed upon the rectangle of the xy-plane, thus yielding the desired correspondence. We must exclude one line, say $x = 0$, from the rectangle. Suppose for simplicity that the strip has been dissected by a line not passing through any electrical contact.

The potential in the Möbius strip goes over by the above transformation into a function $u(x,y)$ having the following properties:

(1) $u(x,y)$ is single-valued* and harmonic in the rectangle and upon its boundary, except at the points of electrical contact, where it has logarithmic or polar singularities of know position, intensity, and moment.

(2) $\dfrac{\partial u}{\partial y} = 0$ for $y = 0$ and for $y = b$, (the edges being lines of flow).

(3) $u(0,y) = u(a, b-y)$; $\qquad \dfrac{\partial u(0,y)}{\partial x} = \dfrac{\partial u(a,b-y)}{\partial x}$.

These properties determine the function $u(x,y)$ except for an additive constant. For if $u'(x,y)$ were a second function having the same properties, the difference $\zeta(x,y) = u'(x,y) - u(x,y)$ would be harmonic throughout the rectangle (having only removable singularities at the points of elec-

* For, with the set of electrical contacts we are assuming, the line integral of the E. M. F. about a general closed curve must be zero.

trical contact), and would satisfy the boundary conditions (2) and (3). Consider Green's Theorem for a plane region S with the boundary C:

$$\int_S\int\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial V}{\partial y}\right)dx\,dy+\int_S\int U\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)dx\,dy=\int_C U\frac{\partial V}{\partial n}ds,$$

where the functions U, V are single-valued and have continuous second partial derivatives on S and C. If we set $U = V = \zeta$ we get, since ζ is harmonic,

$$\int_S\int\left[\left(\frac{\partial \zeta}{\partial x}\right)^2+\left(\frac{\partial \zeta}{\partial y}\right)^2\right]dx\,dy=\int_C \zeta\frac{\partial \zeta}{\partial n}ds.$$

We wish to show that the integrand on the left vanishes identically. For this is sufficient, as in the classical cases, to take as the curve C the boundary of the rectangle: the line integral around it is found to reduce to zero. Since the integrand on the right is continuous and never negative, it must vanish identically. Hence $\zeta = u' - u$ is a constant, as we wished to show.

It remains, then, to exhibit a function $u(x,y)$ which has the properties (1), (2), (3). We may arrive at such a function through the following physical picture: Let a sheet of tin foil be spread over the entire xy-plane, and let infinitely many electrical contacts be made with it, so that the singularities are as follows:

1. In the rectangle $(0,0)$, $(a,0)$, (a,b), $(0,b)$, the singularities correspond to the original electrical contacts in the Möbius strip, let us say at the points

$$(x_k, y_k), \qquad k = 1, 2, \cdots, n.$$
$$0 < x_k < a, \qquad 0 \leqq y_k \leqq b.$$

2. In the rectangle $(a,0)$, $(2a,0)$, $(2a,b)$, (a,b), the singularities are obtained by turning the preceding rectangle together with its singularities upside down, and then bringing it into coincidence with the present rectangle. The transformation is

$$\begin{cases} x' = x - a & a \leqq x' \leqq 2a, \\ y' = b - y & 0 \leqq y' \leqq b\ . \end{cases}$$

3. In the rectangle $(0,b)$, $(2a,b)$, $(2a,2b)$, $(0,2b)$, the singularities are obtained by reflecting those in the lower rectangle in the line $y = b$, according to the method of images. The transformation is

$$\begin{cases} x' = x & 0 \leqq x' \leqq 2a, \\ y' = 2b - y & b \leqq y' \leqq 2b. \end{cases}$$

4. The remaining singularities are obtained by translating the rectangle $(0,0)$, $(2a,0)$, $(2a,2b)$, $(0,2b)$, with its singularities, by multiples of $2a$ to the right and left, and of $2b$ up and down, our rectangle thus playing the role of the fundamental parallelogram of an elliptic function:

$$\begin{cases} x' = x+2am \\ y' = y+2bn, \end{cases} \qquad \begin{cases} m = 0, \pm 1, \pm 2, \cdots, \\ n = 0, \pm 1, \pm 2, \cdots. \end{cases}$$

It is evident from physics and from considerations of symmetry that a flow takes place in the xy-plane in the above circumstances, which, in the rectangle $(0,0)$, $(a,0)$, (a,b), $(0,b)$, is identical with that which takes place in the Möbius strip. For the potentials in the two cases each have the properties (1), (2), (3), and hence can differ at most by a constant. The analytic representation of the potential, as well as the proof that it has the properties (1), (2), (3), is a problem which is solved in the classical theory.*

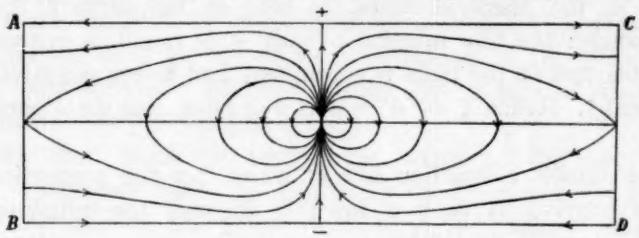

Fig. 1. *Möbius strip.* Join $\overrightarrow{AB}$ with $\overrightarrow{DC}$.

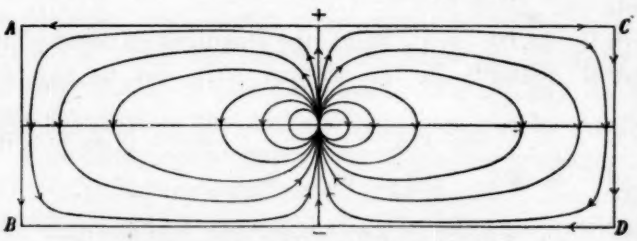

Fig. 2. *Cylinder.* Join $\overrightarrow{AB}$ with $\overrightarrow{CD}$.

In order to illustrate the nature of the phenomenon, we have drawn the lines of flow schematically in a simple case: the positive and negative poles are in the limiting position, so that we have a doublet of vertical axis. The method of drawing follows the solution we have just given: we introduce singularities in the neighboring rectangles, and make use of them. The drawing is supposed to be cut out and joined so as to form a Möbius strip. For purposes of comparison, we have made the corresponding drawing for the cylindrical strip.

* If $v(x,y)$ is the conjugate of $u(x,y)$, the derivative of the analytic function $u+iv$ with respect to $x+iy$ will be seen to be an elliptic function of periods $2a$ and $2bi$. Thus the analytic solution of our problem is furnished by the Elliptic Theory. Use is made of the fact that the total inflow of electricity into the rectangle is zero, and of the invariance of the singularities under the transformations of nos. 2, 3, 4 above.

HARVARD UNIVERSITY,
February, 1926.

A CLASS OF RECIPROCAL FUNCTIONS.*

By Einar Hille.

Part 1. Introduction.

1. The present paper is concerned with the functional equation

$$(F) \qquad \varphi(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(z-u)^2} f(u)\, du,$$

and especially with some aspects of the relationship between this equation and the theory of Hermitian series.

The equation is a special case of one which plays a fundamental role in the theory of the linear flow of heat and in mathematical statistics. It is well known that the Hermitian polynomials are closely connected with the equation of heat conduction and they have been much utilized in the problem of representing statistical frequency functions.

2. The equation (F) and the related equation in which the lower limit of integration is 0 instead of $-\infty$, are closely connected with the theory of Fourier's integral and the reciprocal functions introduced by Cauchy. Solutions of (F) can be readily obtained in view of this relationship. Thus, if (F) possesses a solution $f(u)$ which is of bounded variation on every finite interval and such that the integral in (F) is absolutely convergent when $a < x < b$ $(z = x + iy)$, then

$$(F_1) \qquad \frac{1}{2}[f(u+0)+f(u-0)] = \frac{1}{i\sqrt{\pi}} \lim_{\omega \to +\infty} \int_{c-i\omega}^{c+i\omega} e^{(u-z)^2} \varphi(z)\, dz$$

where $a < c < b$.†

An obvious generalization of (F) consists in replacing the kernel $\exp[-(z-u)^2]$ by a more general function of $(z-u)$ or of $|z-u|$. Equations of this type have been considered by Runge‡ and by Crijns.‡ Runge's solution would in the special case of (F) reduce to

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* Presented to the American Mathematical Society, September 11, 1925, and (Part 6), February 27, 1926.

† See H. Hamburger: Über eine Riemannsche Formel aus der Theorie der Dirichletschen Reihen, Math. Zeitschrift 6, (1920), 1–10. Also E. Stridsberg: Några aritmetiska undersökningar rörande fakulteter och vissa allmännare koefficientsviter, Not 3, Arkiv för Mat., Astr. och Fysik, 15, 22, (1921), 1–126 especially 82–84.

‡ C. Runge: Über eine besondere Art von Integralgleichungen, Math. Annalen, 75 (1914), 130–132. Runge assumes the given function as well as the kernel to be identically zero

$$(F_2) \qquad f(u) \sim \sqrt{2}\, e^{-u^2} \sum_{n=0}^{\infty} c_n 2^{\frac{n}{2}} H_n(u),$$

if

$$\varphi(\sqrt{2}\, u) \sim \quad e^{-u^2} \sum_{n=0}^{\infty} c_n H_n(u).$$

This solution is valid at least if the series in (F_2) is absolutely convergent but its range of validity is much more general. Crijns attacked these equations with the aid of Fourier's integral. In our special case his solution would reduce to

$$(F_3) \qquad f(u) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{s^2 - 2usi} \int_{-\infty}^{+\infty} \varphi(t) e^{2tsi}\, dt\, ds.$$

A direct attack on (F) along totally different lines has been given by Eddington* who found the following formal solution

$$(F_4) \qquad f(u) = \exp\left[-\frac{1}{4}\frac{d^2}{du^2}\right] \varphi(u).$$

Eddington is not concerned with questions of convergence. It can be shown, however, that the series does converge for all values of u (even complex) and represents an entire function in u of genus two at most, provided $\varphi(u)$ is itself an entire function such that $|\varphi(u)| < Ae^{B|u|^2}$ where A and B are positive constants and $B < 1$. If $\varphi(u)$ does not satisfy these conditions, the series in (F_4) is usually divergent.†

If the series in (F_4) is divergent, it may still be summable to a solution of (F). This question has been investigated by C. W. Oseen.‡ The problem is of interest in physics as series of the form $\exp\left[\frac{t}{a}\frac{d^2}{dx^2}\right]\varphi(x)$ appear when we try to solve the equation of heat conduction by power series in t.

for large values of $|u|$ in order to avoid convergence questions. L. Crijns: Zwei Integralgleichungen, Nieuw Archief etc., (2) 13 (1921), 292–294. Crijns' reasoning is largely formal. Sufficient conditions for the validity of (F_3) as well as a generalization of the formula can be obtained with the aid of the method given by T. Carleman on pp. 158–165 of his memoir: Sur les équations intégrales singulières à noyau réel et symétrique, Uppsala Universitets Årsskrift 1923. The pages quoted deal with integral equations of the second kind where the kernel is of the form $h_1(|x-y|)+h_2(x+y)$, the base interval being $(-\infty, +\infty)$.

* On a formula for correcting statistics for the effect of a known probable error of observation, Monthly Notices R. Astr. Soc., 73 (1914), 359–360.

† A striking example of this possibility is furnished by taking $\varphi(z) = \exp(-z^2)$.

‡ Sur une application des méthodes sommatoires de MM. Borel et Mittag-Leffler, Arkiv för Mat. etc., 12, (16), (1917), 1–13.

3. It would be interesting to investigate the range of validity of these solutions in more detail, a problem which, however, is outside of the scope of the present paper. Our main concern is not to prove the existence of a solution of (F) under very general assumptions on $\varphi(z)$, but rather to study the properties of $\varphi(z)$ when $f(u)$ is given and to show how $f(u)$ can be expressed in terms of the corresponding $\varphi(z)$.

In this study the Hermitian polynomials come in at almost every step in view of the following formal relations of reciprocity:

$$(R_1) \qquad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(z-u)^2} \left[\sum_{n=0}^{\infty} a_n H_n(u) \right] du = \sum_{n=0}^{\infty} a_n (2z)^n,$$

$$(R_2) \qquad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iu+y)^2} \left[\sum_{n=0}^{\infty} a_n (2iy)^n \right] dy = \sum_{n=0}^{\infty} a_n H_n(u).$$

Hence we have formally

$$(F_5) \qquad f(u) \sim \sum_{n=0}^{\infty} a_n H_n(u), \text{ if } \varphi(z) = \sum_{n=0}^{\infty} a_n (2z)^n,$$

$$(F_5^{-1}) \qquad \varphi(z) = \sum_{n=0}^{\infty} a_n (2z)^n, \text{ if } f(u) \sim \sum_{n=0}^{\infty} a_n H_n(u).$$

It is possible to impose such conditions on $\varphi(z)$ and $f(u)$ respectively that these relations have a meaning. Our investigation will be largely centered around these formulas of reciprocity, the sense of which we propose to clarify.

4. In part 2 of the present paper we give an outline of the theory of Hermitian polynomials adapted to our particular needs. Most of the formal results are to be found scattered in the literature but too frequently without adequate proofs. This part of the paper is completed by an extensive bibliography.

The investigation proper starts in part 3 where we study the functions $\varphi(z)$ which correspond to a given class of functions $f(u)$, which is chosen in such a way that the integral in (F) shall be convergent on the imaginary axis of the z-plane. In part 4 we prove the existence of the integral

$$f(u; s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iu+y)^2} \varphi(isy)\, dy$$

for every function $\varphi(z)$ defined in part 3. $f(u; s)$ is an entire function of u for every fixed value of s on the open interval $(-1, +1)$. If the

generating function $f(u)$ is suitably restricted, $f(u;s)$ is an analytic function of s, the Maclaurin series of which is closely related to the Hermitian series of $f(u)$. In fact, we have $f(u;s) = \sum_{n=0}^{\infty} f_n H_n(u) s^n$, when $|s|$ is sufficiently small.

The inversion problem is solved for this class of functions $\varphi(z)$ in part 5, where we prove that

$$(F_6) \qquad \lim_{s \to +1} f(u;s) = e^{u^2} \lim_{h \to 0} \frac{1}{2h} \int_{u-h}^{u+h} e^{-t^2} f(t)\, dt,$$

when the limit at the right exists, provided $e^{-t^2}|f(t)|$ is integrable over $(-\infty, +\infty)$ and $f(u)$ is bounded in $(u-\varepsilon, u+\varepsilon)$.

In part 6 we apply these results to the convergence theory of Hermitian series. It can be concluded that

$$\lim_{s \to +1} \sum_{n=0}^{\infty} f_n H_n(u) s^n = f(u)$$

almost everywhere, if $e^{-\gamma t^2}|f(t)|$ is integrable over $(-\infty, +\infty)$ whenever $\gamma > \frac{1}{2}$, and $f(u)$ is bounded when u is bounded, the series on the left being convergent if $|s| < 1$. This Abel summation theorem can be utilized in two different ways in the expansion theory. First, it reduces our problem to a question of convergence: if the series converges on a set of positive measure, it converges to the function almost everywhere. Secondly, it enables us to utilize the important Tauberian theorems in the expansion theory. We prove three expansion theorems (Theorems VI–VIII) with the aid of such notions.

In a recent number of the Mathematische Zeitschrift Prof. G. Szegö presents results on Hermitian series which are in some respects final. This paper became available to me first after the present paper was practically complete; thus it has not been possible for me to refer to it in the text. In the last paragraph, however, I quote Szegö's main result and indicate how it can be extended by imposing a less restrictive condition.

Part 2. On Hermitian polynomials.

5. We proceed to give an outline of the theory of Hermitian polynomials on which the subsequent discussion is based. For the literature on the subject we refer to the Bibliography on pp. 441–443. Papers listed in this Bibliography are usually referred to in the text or in foot notes by the author's name, the number in the list and inclusive pages, thus: Mehler, (22) 173–175.

6. The polynomials of Hermite can be defined by the series:

$$e^{2zu-u^2} = \sum_{n=0}^{\infty} H_n(z)\frac{u^n}{n!}, \tag{1}$$

which converges for all finite values of z and u. Hence we have:

$$H_n(z) = \frac{n!}{2\pi i}\int_C e^{2zu-u^2}u^{-n-1}\,du, \tag{2}$$

where the contour C surrounds the origin, and also

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}(e^{-z^2}). \tag{3}$$

Let C be the circle $|u| = r$. Then

$$|H_n(z)| < n!\, e^{2r|z|+r^2} r^{-n}.$$

If $|z|$ and n are fixed, the right hand side of this inequality will be a minimum when

$$r = R = \frac{1}{2}\left[\sqrt{|z|^2+2n}-n\right]. \tag{4}$$

Consequently

$$|H_n(z)| < n!\, e^{|z|R+n/2} R^{-n}.* \tag{5}$$

7. With the aid of (2) and (3) we easily verify the following functional equations for $H_n(z)$, namely

$$H_n'(z) = 2n\,H_{n-1}(z), \tag{6}$$

$$H_n''(z) - 2z\,H_n'(z) + 2n\,H_n(z) = 0, \tag{7}$$

$$H_n(z) - 2z\,H_{n-1}(z) + 2(n-1)\,H_{n-2}(z) = 0. \tag{8}$$

Anyone of these relations can serve as the basis of a theory of Hermitian polynomials and leads immediately to important generalizations, e. g. the

* Here z is arbitrary complex. If z is real and $|z| \leqq 1.25\sqrt{2n}$, better estimates can be found, see below formulas (28) and (29). Estimates of $|H_n(z)|$, usually for real values of z, occur in quite a few of the papers quoted in the bibliography. Reference should be made to Adamoff, (4) 141; Watson, (33); Galbrun, (10) 25, and Cramér, (7) 402.

theory of parabolic cylinder functions, if n is permitted to take on arbitrary values, real or complex.

8. With the aid of Taylor's series and of (6) we find*

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(y)\,(2x)^k. \tag{9}$$

In this formula we set $y = 0$, using the value of $H_m(0)$ which is easily read off from (2), namely

$$H_{2k+1}(0) = 0, \qquad H_{2k}(0) = (-1)^k \frac{(2k)!}{k!}, \tag{10}$$

and obtain

$$H_n(x) = \sum_{2k \leq n} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2x)^{n-2k}. \tag{11}$$

Using D as symbol for the operator $\frac{d}{dx}$, we can write (11) as follows:

$$H_n(x) = e^{-\frac{1}{4}D^2}[(2x)^n], \tag{12}$$

and conversely

$$(2x)^n = e^{\frac{1}{4}D^2}[H_n(x)] = \sum_{2k \leq n} \binom{n}{2k} \frac{(2k)!}{k!} H_{n-2k}(x). \tag{13}$$

The latter form of (13) occurs in Appell, (6) 132. These formulas are of some interest in connection with the formal relations (F_4), (F_5) and (F_5^{-1}) of the Introduction.

9. It is easy to verify that the function

$$E_n(z) = (\sqrt{\pi}\, 2^n n!)^{-\frac{1}{2}} H_n(z) e^{-\frac{z^2}{2}} \tag{14}$$

satisfies the differential equation

$$E_n''(z) + (2n+1-z^2)\, E_n(z) = 0, \tag{15}$$

and that

$$\begin{aligned}[\overline{E_n(z)}\, E_n'(z)]_{z_1}^{z_2} - \int_{x_1}^{x_2} |E_n'(z)|^2\, dx \\ + \int_{x_1}^{x_2} (2n+1-x^2+y_1^2-2xy_1 i)\, |E_n(z)|^2\, dx = 0.\end{aligned} \tag{16}$$

* A more elegant "addition theorem" occurs in Runge, (26) 131.

Here $\overline{E_n(z)}$ is the conjugate of $E_n(z)$; further, $z = x + iy_1$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_1$. Now suppose that z_1 is a zero of $E_n(z)\ E_n'(z)$ such that $x_1 \geqq 0$, and let $x_2 \to +\infty$. Evidently the integrals remain convergent and the integrated part vanishes. In order that the left hand side of (16) shall actually be zero, it is obviously necessary that $y_1 = 0$ and that $x_1 < \sqrt{2n+1}$. Thus the zeros of $E_n(z)\ E_n'(z)$ are real and confined to the interval $(-\sqrt{2n+1},\ +\sqrt{2n+1})$.*

It follows from this discussion that $E_n(x) > 0$ and $E_n'(x) < 0$ when $x \geqq \sqrt{2n+1}$. But

$$\frac{E_n'(x)}{E_n(x)} = x - \frac{H_{n+1}(x)}{H_n(x)} \tag{17}$$

in view of (6) and (8). It follows that $H_{n+1}(x) > 0$ when $x \geqq \sqrt{2n+1}$, i. e. $H_n(x)$ and $E_n(x)$ are > 0 when $x \geqq \sqrt{2n-1}$.

It is easily verified that $xH_1(x) - H_2(x) < 0$ when $x \geqq \sqrt{2}$. Now suppose that for a particular k we know that $xH_k(x) - H_{k+1}(x) < 0$ when $x \geqq \sqrt{2k}$. Then if $x \geqq \sqrt{2(k+1)}$

$$\begin{aligned} xH_{k+1}(x) - H_{k+2}(x) &= 2(k+1)\,H_k(x) - xH_{k+1}(x) \\ &< \left[\frac{2(k+1)}{x} - x\right] H_{k+1}(x) < 0. \end{aligned}$$

Hence for all values of n

$$E_n'(x) < 0, \text{ when } x \geqq \sqrt{2n}. \tag{18}$$

Now using (18) in conjunction with (17) we see that $H_{n+1}(x) > 0$ when $x \geqq \sqrt{2n}$, or, finally,

$$E_n(x) > 0, \text{ when } x \geqq \sqrt{2n-2}. \tag{19}$$

We shall amplify this result by proving that there is at least one zero of $H_n(x)$ which exceeds

$$x_n = h_n - 2\pi^{\frac{2}{3}} h_n^{-\frac{1}{3}}, \qquad h_n = \sqrt{2n+1},\ n \geqq 1. \tag{20}$$

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* The preceding analysis is based upon some older work of mine, see Hille, (14) 48. — That the zeros of $H_n(z)$ are real was first proved by Sturm, (28) 424–426. Many other proofs have been given. The one above has the advantage of giving at the same time a good upper bound for the zeros. The question of finding such an upper bound has been studied by Hermite, (13) 295; Kiehl, (18) 15; Markoff, (20) 435; and Watson, (33) 121. Watson found that Whittaker's function $D_n(u)$ is positive and has no extrema when $u \geqq 2\sqrt{n}$. It follows from his results that the limit $\sqrt{2n+1}$ found above can be lowered to $\sqrt{2n}$. We shall improve upon this result below. [Prof. G. Pólya has kindly informed me that estimates better than (19) were found by Laguerre, (19a) 127–129. Added in proof.]

This result is trivial when $n<4$, since x_n is negative for such n. Now take $n \geqq 4$ and consider an interval $(x_n, \bar{x}_n)$ of length $L = \frac{3}{2}\pi^{\frac{2}{3}} h_n^{-\frac{1}{3}}$ whose left end-point is x_n. The minimum m of $(h_n^2 - x^2)$ on this interval corresponds to $x = \bar{x}_n$, and

$$L\sqrt{m} = \frac{3}{2}\pi^{\frac{2}{3}} h_n^{-\frac{1}{3}} \sqrt{\pi^{\frac{2}{3}} h_n^{\frac{2}{3}} - \frac{1}{4}\pi^{\frac{4}{3}} h_n^{-\frac{2}{3}}} = \frac{3\pi}{2}\sqrt{1 - \frac{1}{4}\pi^{\frac{2}{3}} h_n^{-\frac{4}{3}}} > \pi,$$

if $h_n \geqq 1$. Hence, in view of well known oscillation theorems, there is at least one zero to every solution of (15) on $(x_n, \bar{x}_n)$, and, in particular, there will be one zero of $E_n(x)$ on this interval.

10. We shall use the results of the discussion in the preceding paragraph for the purpose of estimating the size of $|E_n(x)|$ for real values of x.

$E_n'(x)$ has $(n+1)$ real zeros between $-\sqrt{2n}$ and $+\sqrt{2n}$; let the non-negative zeros be $\xi_1, \xi_2, \cdots, \xi_N$ in increasing order of magnitude, N being $\left[\frac{n}{2}\right]+1$. The first thing to notice is that

$$|E_n(\xi_1)| < |E_n(\xi_2)| < \cdots < |E_n(\xi_N)|, \tag{21}$$

whence

$$|E_n(x)| \leqq |E_n(\xi_N)| \tag{22}$$

for every real x and every positive integral n. This follows from the fact that the derivative of

$$F_n(x) = E_n^2(x) + (2n+1-x^2)^{-1}[E_n'(x)]^2 \tag{23}$$

is positive when $0 < x < \sqrt{2n+1}$ except when $x = \xi_\nu$ and for such a value of x, $F_n(\xi_\nu) = E_n^2(\xi_\nu)$. Hence the quantities $|E_n(\xi_\nu)|$ form an increasing sequence, which proves (21).*

Next we consider the function

$$\begin{aligned}\Theta_n(x) &= (h_n^2 - x^2)^{\frac{1}{2}} E_n^2(x) - x(h_n^2 - x^2)^{-\frac{3}{2}} E_n(x)\, E_n'(x) \\ &+ (h_n^2 - x^2)^{-\frac{1}{2}}\left[1 - \frac{1}{2}(h_n^2 - x^2)^{-3}(h_n^2 + 2x^2)\right][E_n'(x)]^2,\end{aligned} \tag{24}$$

* This argument is due to Watson, (33) 147–148. The expression (24) is modelled upon a similar quadratic form employed by Watson in his paper Bessel Functions and Kapteyn series, Proc. London Math. Soc., ser. 2, vol. 16 (1917) pp. 150–174, especially pp. 170–171. In fact, Watson remarks, (33) 148, that the methods which he has applied to Bessel functions can be used in finding limits for the maximum of $|D_n(u)|$.

where as above $h_n = \sqrt{2n+1}$. The derivative of this function is simply

$$-\frac{1}{2}x(h_n^2-x^2)^{-\frac{9}{2}}(11h_n^2+10x^2)\,[E_n'(x)]^2,$$

which expression is never positive when $0 \leqq x < h_n$. Hence $\Theta_n(x)$ is decreasing in this interval, or

$$\Theta_n(0) \geqq \Theta_n(\xi_\nu) = (h_n^2-\xi_\nu^2)^{\frac{1}{2}}\,E_n^2(\xi_\nu).$$

Hence $|E_n(\xi_\nu)| \leqq (h_n^2-\xi_\nu^2)^{-\frac{1}{4}}\,[\Theta_n(0)]^{\frac{1}{2}}$, and a fortiori

$$|E_n(x)| \leqq (h_n^2-\xi_\nu^2)^{-\frac{1}{4}}\,[\Theta_n(0)]^2, \quad \text{if } -\xi_\nu \leqq x \leqq +\xi_\nu. \tag{25}$$

But

$$\Theta_{2m}(0) = h_{2m}\,E_{2m}^2(0) = \frac{1}{\pi}\sqrt{4m+1}\,\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} < C^2,$$

and

$$\Theta_{2m+1}(0) = h_{2m+1}^{-1}\left(1-\frac{1}{2}h_{2m+1}^{-4}\right)[E_{2m+1}'(0)]^2 < C^2,$$

where C is independent of m.

Further, $h_n^2-\xi_N^2>1$ in view of (18). Hence we have for all real values of x independently of n

$$|E_n(x)| < C, \tag{26}$$

and, if $|x| \leqq \varrho\sqrt{2n+1}$, $\varrho<1$,

$$|E_n(x)| < C_\varrho\, n^{-\frac{1}{4}}. \tag{27}$$

The corresponding relations for the Hermitian polynomials are

$$|H_n(x)| < K\;2^{\frac{n}{2}}\sqrt{n!}\;e^{\frac{x^2}{2}} \qquad (K = \sqrt[4]{\pi}\,C) \tag{28}$$

for all values of x and n, and

$$|H_n(x)| < K_\varrho\,2^{\frac{n}{2}}\sqrt{n!}\;n^{-\frac{1}{4}}\,e^{\frac{x^2}{2}}, \tag{29}$$

when $|x|<\varrho\sqrt{2n+1}$. It follows from (5) by numerical calculation that (29) is also true when $|x| \geqq 1.25\sqrt{2n+1}$.

Formula (28) is not the best estimate which can be obtained. The whole question depends upon the actual order of magnitude of $h_n^2 - \xi_N^2$. We know in view of (18) and (20) that $1 < h_n^2 - \xi_N^2 < 4\pi^{\frac{2}{3}} h_n^{\frac{2}{3}}$. It is perfectly conceivable that the true lower limit is of the same order of magnitude as the upper limit in which case we could replace (28) by the sharper estimate

$$|H_n(x)| < K\, 2^{\frac{n}{2}} \sqrt{n!}\; n^{-\frac{1}{12}}\, e^{\frac{x^2}{2}}. \tag{30}$$

If on the other hand, $h_n^2 - \xi_N^2 = o\left(h_n^{\frac{2}{3}}\right)$, the last maximum falls in an interval where Watson's asymptotic expansion of $D_n(2\sqrt{n} + \omega)$ applies, which again leads to the same estimate as (30). Thus (30) is actually true and is moreover the "best possible" estimate. In the applications which we have to make we shall only use the weaker formula (28) for which we have given a complete proof.*

11. The Hermitian polynomials have the following properties of orthogonality:

$$h_{mn} = \int_{-\infty}^{+\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = \begin{cases} 0 & , \text{ if } m \neq n, \\ \sqrt{\pi}\, 2^m m!, & \text{if } m = n. \end{cases} \tag{31}$$

Thus the functions $E_n(x)$ are normalized orthogonal functions for the interval $(-\infty, +\infty)$.

If $m = n = 0$, (31) reduces to a well known integral whose value is $\sqrt{\pi}$. If $m = 0$, $n > 0$, we can substitute the expression in (3) for $H_n(x)$ and carry out the integration, obtaining 0 as the result.

Now suppose $m > 0$. We multiply both sides of (1) by $e^{-x^2} H_m(x)\, dx$, after changing z into x, and integrate with respect to x from $-\infty$ to $+\infty$ obtaining

$$\int_{-\infty}^{+\infty} e^{-(u-x)^2} H_m(x)\, dx = \int_{-\infty}^{+\infty} e^{-x^2} \left[\sum_{n=0}^{\infty} H_m(x)\, H_n(x)\, \frac{u^n}{n!} \right] dx.$$

The series converges uniformly with respect to x on any finite x-interval, u being fixed. Furthermore, if we replace every term by its absolute value and integrate term-wise, the resulting series is convergent for every finite value of u, as is readily seen by estimating the individual integrals with the aid of (28). Hence we have the right to interchange the order of integration and summation,† thus obtaining

* Formulas (28) and (29) occur without proof in Cramér, (7) 402. I am indebted to Dr. Cramér for the privilege of reading his paper in proof.

† See e. g. Bromwich: Theory of infinite series, p. 453. This theorem, which we shall use frequently in the following, will be referred to as the B-test, and the series formed by the integrals of the absolute values of the terms in the original series will be called the corresponding B-series for short.

$$\int_{-\infty}^{+\infty} e^{-(u-x)^2} H_m(x)\,dx = \sum_{n=0}^{\infty} h_{mn}\frac{u^n}{n!}.$$

On the other hand, it follows from (9) that

$$\int_{-\infty}^{+\infty} e^{-(u-x)^2} H_m(x)\,dx = \int_{-\infty}^{+\infty} e^{-t^2} H_m(u+t)\,dt$$

$$= \sum_{k=0}^{m} \binom{m}{k} (2u)^k \int_{-\infty}^{+\infty} e^{-t^2} H_{m-k}(t)\,dt = \sqrt{\pi}\,(2u)^m.$$

Thus we have, identically in u,

$$\sqrt{\pi}\,(2u)^m = \sum_{n=0}^{\infty} h_{mn}\frac{u^n}{n!},$$

from which (31) is an immediate consequence.

12. Using the sign of equivalence in the sense introduced by Hurwitz, we write

$$f(x) \sim \sum_{n=0}^{\infty} f_n H_n(x), \tag{32}$$

if

$$f_n = \frac{1}{h_{nn}} \int_{-\infty}^{+\infty} e^{-t^2} H_n(t)\, f(t)\,dt, \tag{33}$$

it being understood that these integrals exist. We call (32) the Hermitian series of $f(x)$. In recent years the related orthogonal series

$$F(x) = e^{-\frac{x^2}{2}} f(x) \sim \sum_{n=0}^{\infty} f_n \sqrt{h_{nn}}\, E_n(x), \tag{34}$$

and the exponential series

$$\Phi(x) = e^{-x^2} f(x) \sim \sum_{n=0}^{\infty} (-1)^n\, n!\, f_n \frac{1}{n!}\frac{d^n}{dx^n}(e^{-x^2}), \tag{35}$$

have received more attention than (32).* All these series have advantages of their own; we use (32) in the following because it suits our particular problem best.

* All three series are frequently given in terms of a different variable, usually $t = x/\sqrt{2}$. These differences in the choice of variable, in the definition of the Hermitian polynomials and in the type of series considered have caused some confusion among the writers on mathematical statistics, see Cramér, (7) 401.

The convergence theory of these series has been studied a good deal though for obvious reasons not so thoroughly as e. g. Fourier or Bessel series. There is, however, scarcely a valid proof for an expansion theorem to be found in the literature before 1906–07, though more or less serious attempts at a proof have been made earlier by Hermite (13); Kiehl, (18) 35—42, and Halphen (12) 483–488.* Rigorous proofs with more or less restrictive assumptions on the function to be expanded have been given by V. Myller-Lebedeff, (23) 400;† Adamoff, (5) 159–160; Stekloff, (27) §§ 19—20; Weyl, (35) 306; Neumann, (24) 93; Galbrun, (10) 40; and Cramér, (7) 405 and 407. The work of these later writers is chiefly based upon the theory of integral equations or on Dirichlet's method for Fourier series. Cramér's first method, however, which is very short and elegant, is based on Parseval's theorem (= the closure relation); the results which he obtains are among the best so far known.‡

The summation theory of Hermitian series appears to have been neglected. In the present paper we cover this gap as far as Abel summation is concerned which we take as the basis for a new attack on the convergence theory.

So far we have had real variables in mind exclusively. The series (32) and (34) have been studied for complex values of the variable by Watson, (32) 417–421, and Volk, (31) 312–314.§ The coefficients of the expansion are not given by (33) in the work of these writers, but it is easy to reduce the integrals they give to the form (33).

13. In deriving the properties of orthogonality we obtained the formula

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(z-u)^2} H_n(u)\,du = (2z)^n \tag{36}$$

as a by-product. This formula is the integral relation corresponding to the differential relation (13). To (12) corresponds‖

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(iw+y)^2}(2iy)^n\,dy = H_n(w). \tag{37}$$

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* A fairly recent unsuccessful attempt is due to Kapteyn, (15) 1198–99.

† This is the paper usually quoted or misquoted by the writers on mathematical statistics.

‡ See Szegö, (29) 114–115, and Rotach, (39) 7.

§ The work by Nielsen in (25) on this question is marred by his fallacious notion that an Hermitian series ought to converge in the whole plane, if it converges anywhere.

‖ The equivalence of differential and integral relations of this type has played a great role in the proofs for Waring's theorem due to Hausdorff and Stridsberg. Formula (37) is due to Mehler, (22) 174, and Kiehl, (18) 19. Formula (36) seems to be due to Kapteyn, (16) 1203.

This relation is easily derived from (13) by replacing x by y, multiplying both sides of the equation by $e^{-(iw+y)^2}\,dy$ and integrating with respect to y from $-\infty$ to $+\infty$. The different integrals which occur in the process are all of the same type as (36).

We shall say that $H_n(w)$ and $(2z)^n$ are reciprocal functions in the sense expressed by formulas (36) and (37). These relations of course lie at the bottom of the formulas of reciprocity (R_1) and (R_2) of the Introduction.

Suppose that x is real; then it follows from (37) that

$$|H_n(x)| \leqq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} |e^{-(ix+y)^2}(2iy)^n|\,dy = \frac{1}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{+\infty} e^{-y^2} |2y|^n\,dy,$$

(38)

$$|H_n(x)| \leqq \frac{1}{\sqrt{\pi}} 2^n \Gamma\left(\frac{n+1}{2}\right) e^{x^2}.$$

This estimate which seems to be due to Galbrun, (10) 25, is frequently more convenient than (28) and (29). The factors involving n are of the same order of magnitude as the corresponding factors in (29).

14. In equation (1) we set $z = x$ and replace u by isu, where $|s|<1$. Multiplying both sides of the resulting equation by $e^{-(iy+u)^2}\,du$, integrating term-wise with respect to u from $-\infty$ to $+\infty$, and using (37), we obtain

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iy+u)^2+2ixsu+s^2u^2}\,du = \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!} s^n \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iy+u)^2} (2iu)^n\,du,$$

or

$$(39)\quad (1-s^2)^{-\frac{1}{2}} \exp\left[-\frac{s^2(x^2+y^2)-2sxy}{1-s^2}\right] = \sum_{n=0}^{\infty} H_n(x)\,H_n(y)\,\frac{s^n}{2^n n!}.$$

We denote the left hand side by $E(x, y; s)$. The relation holds when $|s|<1$.* The term-wise integration is easily justified. In fact, assuming x and y to be real and using (38) we find the following majorant for the corresponding B-series, namely:

$$\frac{1}{\sqrt{\pi}} e^{x^2+y^2} \sum_{n=0}^{\infty} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma(n+1)} |2s|^n,$$

convergent when $|s|<1$. The coefficient of $|s|^n$ tends to zero like $n^{-\frac{1}{2}}$; hence the coefficient of s^n in (39) also tends to zero with $\frac{1}{n}$. But $E(x, y; s)$

* The formula is due to Mehler, (22) 173. It has been rediscovered by Kapteyn, (15) 1198, who sums the series by means of a relation equivalent to our formula (37), assuming the series to converge. He applies the same method to the series obtained by putting $s = 2i$ in (39) and separating reals and imaginaries, without noticing that these series are divergent.

is an analytic function of s when x and y are fixed, and holomorphic if $|s|<1$, the only singular points being $s=\pm 1$. Hence, by Fatou's theorem, the power series in (39) converges on the unit-circle in the s-plane except at $s=\pm 1$, provided x and y are real. If x and y are not real, we can still prove the convergence of the series when $|s|<1$. An interesting special case is obtained by putting $y=0$, $s=i$ in (39), namely:

$$\frac{1}{\sqrt{2}}e^{\frac{x^2}{2}}=\sum_{n=0}^{\infty}\frac{H_{2n}(x)}{2^{2n}n!}, \tag{40}$$

the series being convergent for all real values of x.

If x and y are real and distinct, $E(x,y;s)\to 0$ when $s\to+1$ radially. Thus if (39) is convergent when $s=1$ for any such pair of values of x and y, its sum must be zero. On the other hand, (39) can never be convergent when $s=+1$ if x is real and equal to y because $\lim_{s\to+1} E(x,x;s)=+\infty$. For the partial sums of (39) in the case $s=+1$ we can obtain with the aid of (8) the "Christoffel formula"

$$\sum_{m=0}^{n}\frac{H_m(x)\,H_m(y)}{2^m m!}=\frac{H_{n+1}(x)\,H_n(y)-H_n(x)\,H_{n+1}(y)}{2^n n!\,(x-y)}. \tag{41}$$

It follows from (29) or (38) that the right hand side of (41) is bounded, x and y being fixed real and $x\neq y$. If $y=0$ and $s=1$, it can be shown that (39) is not convergent when x is real and $\neq 0$, but the series is summable (C,k) to the sum zero whenever $k>0$. The right hand side of (41) is the singular kernel encountered by the writers who use Dirichlet's methods in studying the convergence of Hermitian series. In fact, if (41) be multiplied by $\frac{1}{\sqrt{\pi}}e^{-y^2}f(y)$ and integrated with respect to y from $-\infty$ to $+\infty$, the nth partial sum of the formal Hermitian series of $f(x)$ results.

15. We shall need the value of the integral

$$h_n(a)=\int_{-\infty}^{+\infty}e^{-au^2}[H_n(u)]^2\,du, \qquad (a>0) \tag{42}$$

which reduces to h_{nn} when $a=1$. Assuming $s<\frac{a}{2-a}$ we have

$$\sum_{n=0}^{\infty}h_n(a)\frac{s^n}{2^n n!}=\int_{-\infty}^{+\infty}e^{-au^2}E(u,u;s)\,du$$

$$=\sqrt{\frac{\pi}{a}}(1-s)^{-\frac{1}{2}}\left(1-\frac{2-a}{a}s\right)^{-\frac{1}{2}},$$

and

$$(43) \qquad h_n(a) = (-1)^n 2^n n! \sqrt{\frac{\pi}{a}} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{n-k} \left(\frac{2-a}{a}\right)^{n-k}.$$

The term-wise integration involved in this derivation is fairly easy to justify with the aid of the B-test for sufficiently small values of $|s|$. The infinite series is of course convergent when $|s|$ is less than the smaller of the two numbers 1 and $\frac{a}{2-a}$.

Since $(-1)^n \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{n-k} = 1$, we conclude that

$$(44) \qquad h_n(a) < \sqrt{\frac{\pi}{a}}\, 2^n n! \left(\frac{2-a}{a}\right)^n, \qquad \text{if } 0 < a < 1.$$

Another useful formula follows from (39) by putting $y = 0$, namely

$$\int_{-\infty}^{+\infty} e^{-au^2} H_{2n}(u)\, du = \sqrt{\frac{\pi}{a}} \left(\frac{1-a}{a}\right)^n \frac{(2n)!}{n!}.$$

16. The following Bibliography is not claimed to be complete, but the author hopes that the omissions will prove to be few. Papers on parabolic cylinder functions are listed only if they are quoted in the text. Only a few papers on Laguerre's series are quoted; such papers frequently have a bearing on Hermitian series. An asterisk before a number in the list indicates that the paper has not been available and is known to me through quotations.

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ADDENDA.*

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Part 3. The passage from $f(u)$ to $\varphi(z)$.

17. We now proceed to the study of a particular class of functions $f(u)$ and the corresponding functions $\varphi(z)$. We have

$$\varphi(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(z-u)^2} f(u)\, du, \tag{F}$$

where we can write the integral as follows:

$$e^{-z^2} \int_{-\infty}^{+\infty} e^{-u^2+2uz} f(u)\, du = e^{-z^2} D(z).$$

$D(z)$ is an integral of the Laplace-Dirichlet type; it is convergent in a strip $a < x < b$ $(z = x + iy)$ and absolutely convergent in another strip $c < x < d$, where $a \leqq c \leqq d \leqq b$. If $c = d$ the strip of absolute convergence reduces to a single vertical line or is entirely missing. If, on the other hand, $a = b$ the points of convergence may form an arbitrary point set on the line $x = a$. Finally, it may happen that the integral is not convergent anywhere in the z-plane, nevertheless the integral may be summable by some method of summation in a more or less extensive region of the plane.

* See also the note on p. 464.

18. We shall assume that $D(z)$ actually has a line of convergence which we can assume to be the imaginary axis. If this is not so from the start, a suitable linear transformation will carry the line of convergence over into the new imaginary axis.

We shall consider a class F of functions $f(u)$ with the following properties:

(1) $f(u)$ is integrable in the sense of Lebesgue over every finite interval.

$$D(iy) = \lim_{\substack{\omega_1 \to +\infty \\ \omega_2 \to -\infty}} \int_{\omega_2}^{\omega_1} e^{-u^2+2iuy} f(u)\, du \tag{2}$$

shall exist uniformly with respect to y.

The subsequent discussion is based upon this family F, though most of the time we shall be concerned with some of the sub-families of F. We define three of these as follows:

I. F_α is the sub-family of F for which the integral

$$C_\alpha = \int_{-\infty}^{+\infty} e^{-u^2+2\alpha|u|}\, |f(u)|\, du \tag{45}$$

exists for a fixed $\alpha \geqq 0$. By F^* we understand the class of functions which belong to all families F_α, $\alpha \geqq 0$.

II. Q_β is the sub-family for which the integral

$$K_\beta = \int_{-\infty}^{+\infty} e^{-\beta u^2}\, |f(u)|^2\, du \tag{46}$$

exists for a fixed β, $1 \leqq \beta < 2$. By Q^* we understand the class of functions which belong to all families Q_β with $\beta > 1$.

III. J_γ is the sub-family for which the integral

$$I_\gamma = \int_{-\infty}^{+\infty} e^{-\gamma u^2}\, |f(u)|\, du \tag{47}$$

exists for a fixed γ, $\frac{1}{2} \leqq \gamma < 1$. By J^* we understand the class of functions which belong to all families J_γ with $\gamma > \frac{1}{2}$.

19. We shall prove the following:

THEOREM I: *If $f(u)$ belongs to F, formula* (F) *defines $\varphi(z)$ when z is purely imaginary. The function $\varphi(iy)$ is continuous in y and*

$$\varphi(iy) = e^{y^2}\, \eta(y), \tag{48}$$

where $\eta(y) \to 0$ when $|y| \to +\infty$.

If $f(u)$ belongs to F_α, $\alpha > 0$, formula (F) *defines $\varphi(z)$ as a holomorphic function of z in the strip $-\alpha < x < +\alpha$. In the circle $|z| < \alpha$ we have*

$$\varphi(z) = \sum_{n=0}^{\infty} f_n (2z)^n, \tag{49}$$

where f_n is defined by (33). *In particular, if $f(u)$ belongs to F^*, $\varphi(z)$ is is an entire function of z.*

It follows from the definition of F that $\varphi(z)$ is defined for all purely imaginary values of z by (F). Further consider the integral

$$\int_{\omega_2}^{\omega_1} e^{-u^2+2iuy} f(u)\, dy.$$

This integral tends to zero when $|y| \to +\infty$, in view of the Riemann-Lebesgue theorem. In view of our assumption (2) § 18 the integral converges uniformly with respect to y when $\omega_1 \to +\infty$ and $\omega_2 \to -\infty$. Formula (48) is a consequence of these two facts.

Now suppose that $f(u)$ belongs to F_α and $-\alpha \leqq x \leqq \alpha$. Then

$$|\varphi(x+iy)| \leqq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} |e^{-(x+iy-u)^2} f(u)|\, du$$

$$= \frac{1}{\sqrt{\pi}} e^{y^2-x^2} \int_{-\infty}^{+\infty} e^{-u^2+2xu} |f(u)|\, du < \frac{C_\alpha}{\sqrt{\pi}} e^{y^2-x^2}.$$

If $f(u)$ belongs to J_γ we can estimate the last integral as follows:

$$R = \int_{-\infty}^{+\infty} e^{-u^2+2xu} |f(u)|\, du = \int_{-\infty}^{+\infty} e^{-(1-\gamma)u^2+2xu} e^{-\gamma u^2} |f(u)|\, du$$

$$\leqq I_\gamma e^{\frac{x^2}{1-\gamma}}.$$

For a function belonging to Q_β we obtain instead†

$$R^2 \leqq \int_{-\infty}^{+\infty} e^{-\beta u^2} |f(u)|^2 du \cdot \int_{-\infty}^{+\infty} e^{-(2-\beta)u^2+4xu}\, du = K_\beta \left(\frac{\pi}{2-\beta}\right)^{\frac{1}{2}} e^{\frac{4x^2}{2-\beta}}.$$

Consequently

$$|\varphi(x+iy)| \leqq \frac{I_\gamma}{\sqrt{\pi}} \exp\left[\frac{\gamma}{1-\gamma} x^2 + y^2\right], \tag{50}$$

or

$$|\varphi(x+iy)| \leqq \left[\frac{K_\beta^2}{\pi(2-\beta)}\right]^{\frac{1}{4}} \exp\left[\frac{\beta}{2-\beta} x^2 + y^2\right], \tag{51}$$

according as $f(u)$ belongs to J_γ or Q_β.

† Every $f(u)$ in Q_β belongs to any J_γ with $2\gamma > \beta$. In particular, the class Q^* belongs to the class J^*. No use of this fact is made in the present paper.

That $\varphi(z)$ is holomorphic in the strip $-\alpha<x<\alpha$, when $f(u)$ belongs to F_α follows from well-known theorems. We notice that $\varphi(z)$ is continuous at every finite point on the boundary of the strip.

We pass over to the series expansion. Assuming $f(u)$ to belong to F_α, we have in view of (1),

$$\varphi(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(z-u)^2} f(u)\,du = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2}\left[\sum_{n=0}^{\infty}\frac{H_n(u)}{n!}z^n\right] f(u)\,du$$
$$= \sum_{n=0}^{k} f_n(2z)^n + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2}\left[\sum_{n=k+1}^{\infty}\frac{H_n(u)}{n!}z^n\right] f(u)\,du.$$

Using (2) we find that, when $|z|<\alpha$,

$$\sum_{n=k+1}^{\infty}\frac{H_n(u)}{n!}z^n = \sum_{n=k+1}^{\infty} z^n \frac{1}{2\pi i}\int_{|s|=\alpha} e^{2us-s^2} s^{-n-1}\,ds$$
$$= \sum_{n=k+1}^{\infty}\frac{1}{2\pi i}\int_{|s|=\alpha}\left(\frac{z}{s}\right)^n e^{2us-s^2}\frac{ds}{s}$$
$$= \frac{1}{2\pi i}\int_{|s|=\alpha}\left[\sum_{n=k+1}^{\infty}\left(\frac{z}{s}\right)^n\right] e^{2us-s^2}\frac{ds}{s}$$
$$= \frac{1}{2\pi i}\int_{|s|=\alpha}\left(\frac{z}{s}\right)^{k+1}\frac{e^{2us-s^2}}{s-z}\,ds.$$

Hence

$$\left|\sum_{n=k+1}^{\infty}\frac{H_n(u)}{n!}z^n\right| \leqq \left|\frac{z}{\alpha}\right|^{k+1}\frac{\alpha}{\alpha-|z|}e^{2\alpha|u|+\alpha^2},$$

and

$$\int_{-\infty}^{+\infty} e^{-u^2}\left|\sum_{n=k+1}^{\infty}\frac{H_n(u)}{n!}z^n\right| |f(u)|\,du$$
$$\leqq \alpha\left|\frac{z}{\alpha}\right|^{k+1}\frac{e^{\alpha^2}}{\alpha-|z|}\int_{-\infty}^{+\infty} e^{-u^2+2\alpha|u|}\,|f(u)|\,du,$$

which tends to zero as $k\to+\infty$, provided $|z|<\alpha$. Hence (49) holds when $|z|<\alpha$.

The following example shows that the radius of convergence which we have found for the series (49) cannot in general be replaced by any larger value. We take

$$f(u) = e^{u^2-2a|u|}, \qquad a>0.$$

This function belongs to $F_{a-\varepsilon}$ provided $\varepsilon>0$. A simple calculation gives

$$f_{2n+1} = 0,$$
$$\sqrt{\pi}\,a(2a)^{2n} f_{2n} = \sum_{k=0}^{n}(-1)^k\frac{a^{2k}}{k!}\to e^{-a^2}.$$

Consequently the corresponding series $\sum f_n(2z)^n$ is convergent when $|z|<a$, which value agrees with that given by Theorem I.

20. We can now give an interpretation of (R_1), the first formula of reciprocity given in § 3.* From Theorem I we obtain the following

COROLLARY 1: *If $f(u)$ belongs to F_α, $\alpha>0$, and*

$$f(u)\sim\sum_{n=0}^{\infty} f_n H_n(u), \tag{52}$$

then

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(z-u)^2} f(u)\,du = \sum_{n=0}^{\infty} f_n (2z)^n, \tag{53}$$

where the series is convergent when $|z|<\alpha$.

For the second formula of reciprocity (R_2) we can at present give the following formulation:

If

$$\varphi(z) = \sum_{n=0}^{\infty} f_n (2z)^n$$

is an entire function such that

$$\sum_{n=0}^{\infty} |f_n|\, 2^n\, \Gamma\left(\frac{n+1}{2}\right) \tag{54}$$

is convergent, then

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(iu+y)^2}\varphi(iy)\,dy = \sum_{n=0}^{\infty} f_n H_n(u), \tag{55}$$

where the series to the right is absolutely convergent for all real finite values of u.

The term-wise integration necessary for the derivation of (55) is justified with the aid of the B-test as usual; in view of (38) and (54) the corresponding B-series is convergent. It seems plausible that the convergence of the series in (54) is a necessary condition for the absolute convergence of the Hermitian series on an interval.

If we give up condition (54) formula (55) may still have a meaning. Take e. g. $\varphi(z) = e^{\frac{z^2}{4}}$, then (55) becomes

$$\frac{1}{\sqrt{2}}\, e^{\frac{u^2}{2}} = \sum_{n=0}^{\infty} \frac{1}{4^n n!} H_{2n}(u),$$

* The formula as such seems to be due to Deruyts, (9) 15, who, however, did no investigate its validity. [See also Kameda, (42) 29–31. Added in proof.]

which is the same as formula (40) above. Here the series is convergent for all finite values of u but never absolutely convergent.

It is natural, however, to expect that (55) has a significance even if the Hermitian series is not convergent, but it seems to be rather difficult to prove that the series is equivalent to the integral, if we assume, for instance, that $\varphi(z)$ is holomorphic in a strip $-\alpha<x<+\alpha$ and that the integral in (55) is absolutely convergent for all real values of u.

Part 4. The function $f(w;s)$.

21. Formula (R_2) of the Introduction suggests forming

$$\int_{-\infty}^{+\infty} e^{-(iw+y)^2}\varphi(iy)\,dy,$$

with the function $\varphi(z)$ defined by (F). This integral is, however, in general divergent. To overcome this difficulty we sum the integral by introducing a parameter s in the integrand, forming:

$$f(w;s)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(iw+y)^2}\varphi(isy)\,dy. \tag{56}$$

If s is real and $-1<s<+1$, the integral is absolutely convergent.

THEOREM II: *For every fixed value of s, $-1<s<+1$, $f(w;s)$ is an entire function of $w=u+iv$ such that*

$$f(w;s)=\sum_{n=0}^{\infty} h(s,n)(-2iw)^n, \tag{57}$$

where

$$\sqrt{\pi}\,2^n n!\,h(s,n)=\int_{-\infty}^{+\infty} e^{-y^2}H_n(y)\,\varphi(isy)\,dy. \tag{58}$$

If $f(u)$ belongs to F_α,

$$|f(w;s)|\leqq\frac{C_\alpha}{\sqrt{\pi}}(1-s^2)^{-\frac{1}{2}}\exp\left[u^2+\frac{s^2}{1-s^2}v^2\right]. \tag{59}$$

If $f(u)$ belongs to Q_β, $f(w;s)$ will exist also for complex values of $s=\sigma+it$ in the interior of the ellipse

$$E(s,\beta)\equiv\sigma^2+\frac{\beta}{2-\beta}t^2=1,$$

and is a holomorphic function of s in this region satisfying

$$|f(w;s)|\leqq\left[\frac{K_\beta^2}{\pi(2-\beta)}\right]^{\frac{1}{4}}[1-E(s,\beta)]^{-\frac{1}{2}}\exp\left[u^2+\frac{E(s,\beta)}{1-E(s,\beta)}v^2\right]. \tag{60}$$

If $f(u)$ belongs to J_γ instead, $f(w; s)$ is holomorphic inside of

$$e(s,\gamma) \equiv \sigma^2 + \frac{\gamma}{1-\gamma}\, t^2 = 1,$$

and

$$(61) \qquad |f(w;s)| \leqq \frac{I_\gamma}{\sqrt{\pi}} [1-e(s,\gamma)]^{-\frac{1}{2}} \exp\left[u^2 + \frac{e(s,\gamma)}{1-e(s,\gamma)}\, v^2\right].$$

Proof: Theorem II is essentially a special case of Theorem I. To see this fact we have only to write (56) as follows:

$$f(w;s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(-iw-y)^2}\, \varphi(isy)\, dy.$$

This expression is the same as formula (F) provided we replace u by y, z by $-iw$ and $f(u)$ by $\varphi(isy)$. Considered as a function of the real variable y, $\varphi(isy)$ belongs to any family Q_β with $\beta > 2s^2$, s being real, $|s|<1$. Thus formulas (57) and (58) correspond exactly to (49) and (33). Further, if $f(u)$ belongs to F_α, we have

$$|\varphi(isy)| \leqq \frac{C_\alpha}{\sqrt{\pi}}\, e^{s^2y^2},$$

and

$$|f(w;s)| \leqq \frac{1}{\sqrt{\pi}}\, e^{u^2-v^2} \int_{-\infty}^{+\infty} e^{-y^2+2vy}\, |\varphi(isy)|\, dy$$

$$\leqq \frac{C_\alpha}{\pi}\, e^{u^2-v^2} \int_{-\infty}^{+\infty} e^{-(1-s^2)y^2+2vy}\, dy.$$

If the integral be evaluated, formula (59) results. So far we have had to assume s to be real. This assumption is not necessary if we assume $f(u)$ to belong to Q_β or to J_γ, since in this case $\varphi(z)$ is an entire function of genus two. In view of (51)

$$|\varphi(i(\sigma+it)y)| \leqq \left(\frac{K_\beta^2}{\pi(2-\beta)}\right)^{\frac{1}{4}} e^{E(s,\beta)y^2},$$

if $f(u)$ belongs to Q_β. If we substitute this estimate of $|\varphi(isy)|$ into the integral above, we find that the integral exists when $E(s,\beta)<1$, and with this assumption formula (60) is obtained. Formula (61) is obtained in a similar manner with the aid of (50). If $f(u)$ belongs to Q^* or to J^*, the ellipse of holomorphism becomes the unit-circle in the s-plane.

22. The expansion of $f(w;s)$ in terms of powers of w is of little interest compared with the power series in s which exists when $f(u)$ belongs to Q_β or to J_γ.

THEOREM III: *If $f(u)$ belongs to Q_β,*

$$f(w;s) = \sum_{n=0}^{\infty} f_n H_n(w)\, s^n, \tag{62}$$

where the series converges when $|s| < \sqrt{\frac{2-\beta}{\beta}}$. If $w = u$ is real, the series converges on the circle of convergence at every point where $f(u;s)$ is holomorphic in s. The expansion (62) holds when $|s| < \sqrt{\frac{1-\gamma}{\gamma}}$ if $f(u)$ belongs to J_γ. If $\gamma = \frac{1}{2}$ and $w = u$ is real, the series converges on $|s| = 1$ wherever $f(u;s)$ is holomorphic.

Proof: Let us first assume that $f(u)$ belongs to Q_β. We begin the proof by obtaining an estimate of the order of magnitude of the Fourier-Hermite coefficients f_n. We have, using (42) and (44),

$$\left[\int_{-\infty}^{+\infty} e^{-u^2} H_n(u)\, f(u)\, du\right]^2 \leqq \int_{-\infty}^{+\infty} e^{-\beta u^2} |f(u)|^2\, du \int_{-\infty}^{+\infty} e^{-(2-\beta)u^2} H_n^2(u)\, du$$

$$= K_\beta h_n(2-\beta) < K_\beta \sqrt{\frac{\pi}{2-\beta}}\, 2^n\, n! \left(\frac{\beta}{2-\beta}\right)^n,$$

and

$$|f_n| < K_\beta^{\frac{1}{2}} [\pi(2-\beta)]^{-\frac{1}{4}}\, 2^{-\frac{n}{2}}\, (n!)^{-\frac{1}{2}} \left(\frac{\beta}{2-\beta}\right)^{\frac{n}{2}}. \tag{63}$$

We now proceed to the derivation of (62). In view of (49),

$$f(w;s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iw+y)^2} \varphi(isy)\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iw+y)^2} \left(\sum_{n=0}^{\infty} f_n s^n (2iy)^n\right) dy.$$

If we have the right to integrate term-wise, formula (62) is a mere con-consequence of (37). The question of term-wise integration can again be settled with the aid of the B-test. Suppose for the moment that w is real and equals u. Using (38) and (63) we find that the corresponding B-series has the following majorant:

$$K_\beta^{\frac{1}{2}}\, \pi^{-\frac{3}{4}}\, (2-\beta)^{-\frac{1}{2}}\, e^{u^2} \sum_{n=0}^{\infty} 2^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\Gamma(n+1)}} \left(\frac{\beta}{2-\beta}\right)^{\frac{n}{2}} |s|^n. \tag{64}$$

This series converges when $|s| < \sqrt{\frac{2-\beta}{\beta}}$, hence the series in (62) converges in the same region and represents $f(u; s)$. Further we notice that the coefficient of $\left(\frac{\beta}{2-\beta}\right)^{\frac{n}{2}} |s|^n$ in (64) tends to zero as $n^{-\frac{1}{4}}$, whence a fortiori

$$n^{\frac{1}{4}} f_n H_n(u) \left(\frac{2-\beta}{\beta}\right)^{\frac{n}{2}} = O(1). \tag{65}$$

Hence we conclude with the aid of Fatou's theorem that the series in (62) converges on the circle of convergence at any point where $f(u; s)$, considered as a function of s, is holomorphic. If $\beta > 1$ this fact implies convergence everywhere on the circle with the possible exception of the points $\pm\sqrt{\frac{2-\beta}{\beta}}\, i$ which lie on the boundary of the known elliptical region of holomorphism of $f(u; s)$. No such sweeping conclusion can be drawn if $\beta = 1$, since $E(s, 1) = 1$ is the unit-circle in the s-plane.

If w is complex we can still obtain the same radius of convergence of the power series and the convergence is uniform with respect to w in any finite part of the w-plane, but we can no longer draw any conclusions concerning the behavior of the series on the boundary.

We now proceed to the case in which $f(u)$ belongs to J_γ. The essential thing is to prove that the corresponding B-series has a positive radius of convergence. In general we cannot obtain such a simple estimate of the coefficients as above. A crude estimate shows, however, that the radius of convergence of (62) is at least equal to $r_1 = e^{-\pi\left(\gamma-\frac{1}{2}\right)}$. This estimate cannot be improved if $\gamma = \frac{1}{2}$. Now let $\gamma > \frac{1}{2}$. Then $f(w; s)$ is a holomorphic function of s in the circle $|s| < r_2 = \sqrt{\frac{1-\gamma}{\gamma}}$, and can be expanded in a power series $\sum_{n=0}^{\infty} h_n(w)\, s^n$ convergent for such values of s. The coefficients of this series are clearly entire functions of w. When $|s| < r_1$ the series coincides with (62). Hence $h_n(w) \equiv f_n H_n(w)$ for every n, and (62) converges when $|s| < r_2$. Let us return to the case $\gamma = \frac{1}{2}$. There we have, in view of (28) and (29),

$$\left|\int_{-\infty}^{+\infty} e^{-w^2} H_n(u) f(u)\, du\right| \leqq \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} |H_n(u)|\, e^{-\frac{u^2}{2}} |f(u)|\, du$$

$$= \varepsilon(n)\, 2^{\frac{n}{2}} (n!)^{\frac{1}{2}} I_{\frac{1}{2}}$$

where $\varepsilon(n)\to 0$ when $n\to+\infty$. Hence we have, if $|u|<\varrho\sqrt{2n+1}$, $\varrho<1$,

$$|f_n H_n(u)| = o\left(n^{-\frac{1}{4}}\right), \tag{66}$$

if $f(u)$ belongs to $J_{\frac{1}{2}}$. In view of this estimate we can apply Fatou's theorem on the boundary in the special case when $\gamma = \frac{1}{2}$. This completes the proof of Theorem III.

23. Let us return for a moment to the family Q_β and consider the special case $\beta = 1$. The expansion (62) can be proved in a much simpler way for functions belonging to Q_1. For such functions the integral

$$\int_{-\infty}^{+\infty} e^{-u^2}\left|f(u)-\sum_{n=0}^{m} f_n H_n(u)\right|^2 du$$

exists and is never negative, m being arbitrary. Hence

$$\int_{-\infty}^{+\infty} e^{-u^2}|f(u)|^2 du \geqq \sqrt{\pi}\sum_{n=0}^{\infty} 2^n n!|f_n|^2. \tag{67}$$

As a matter of fact, the equality sign holds, but we shall not use this fact which lies fairly deep.*

Then we have, in view of (38) and (67),

$$\left|\sum_{n=0}^{\infty} f_n H_n(u) s^n\right| \leqq \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}|f_n| e^{u^2} 2^n \Gamma\left(\frac{n+1}{2}\right)|s|^n$$

$$\leqq \frac{1}{\sqrt{\pi}} e^{u^2}\left\{\sum_{n=0}^{\infty}|f_n|^2 2^n n! \sum_{n=0}^{\infty}\frac{2^n \Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma(n+1)}|s|^{2n}\right\}^{\frac{1}{2}}.$$

The first series is convergent since $f(u)$ belongs to Q_1, the second one converges when $|s|<1$, and the coefficients tend to zero as $n^{-\frac{1}{2}}$. The rest of the proof is as above.

Part 5. The passage from $f(u; s)$ to $f(u)$.

24. We pass over to the inversion problem for the functions $\varphi(z)$ which, however, we solve only for the case in which the generating function $f(u)$ belongs to F_0.

* See Weyl, (35) 305. The general question of the validity of Parseval's theorem for infinite intervals has been studied by M. Riesz: Sur le problème des moments et le theorème de Parseval correspondant, Acta Litt. ac Scient., Sectio Scient. Math., vol. 1 (1922-23) pp. 209-225.

THEOREM IV: *Let $f(u)$ belong to F_0 and let $u = u_0$ be such that* (i) *$f(u)$ is bounded in $(u_0 - \varepsilon, u_0 + \varepsilon)$ when ε is sufficiently small, and* (ii)

$$e^{u_0^2} \lim_{h \to 0} \frac{1}{2h} \int_{u_0 - h}^{u_0 + h} e^{-t^2} f(t)\, dt \tag{68}$$

exists and equals $F(u_0)$, say. Then

$$\lim_{s \to +1} f(u_0; s) = F(u_0). \tag{69}$$

The limit exists uniformly on any finite closed interval interior to an interval of continuity of $f(u)$.

In view of the properties of the Lebesgue integral, we have the following

COROLLARY 2: *If $f(u)$ is bounded when u is bounded, then*

$$\lim_{s \to +1} f(u; s) = f(u) \tag{70}$$

almost everywhere; in particular, whenever $f(u)$ is continuous or equals the derivative of its indefinite integral. Further, the limit in (68) *exists at every point of discontinuity of the first kind; at such a point*

$$\lim_{s \to +1} f(u; s) = \frac{1}{2}[f(u - 0) + f(u + 0)]. \tag{71}$$

Proof. We have evidently

$$f(w; s) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-(iw+y)^2} \int_{-\infty}^{+\infty} e^{-(isy-t)^2} f(t)\, dt\, dy, \tag{72}$$

where all the variables with the exception of w are supposed to be real. For a fixed positive $s < 1$ this repeated integral is absolutely convergent and we have the right to interchange the order of the two integrations. Thus after some simplifications

$$\begin{aligned} f(w; s) &= \frac{1}{\pi} e^{w^2} \int_{-\infty}^{+\infty} e^{-t^2} f(t) \int_{-\infty}^{+\infty} e^{-(1-s^2)y^2 - 2i(w-st)y}\, dy\, dt \\ &= e^{w^2} \int_{-\infty}^{+\infty} e^{-t^2} f(t)\, K(w, t, s)\, dt, \end{aligned} \tag{73}$$

where

$$K(w, t, s) = [\pi(1 - s^2)]^{-\frac{1}{2}} \exp\left[-\frac{(w - st)^2}{1 - s^2}\right], \tag{74}$$

or

$$f(w; s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} f(t)\, E(w, t; s)\, dt. \tag{75}$$

Here $E(w, t; s)$ is defined by formula (39). We shall work with formula (73) which we write as follows, assuming w to be real and equal to u,

$$e^{-u^2} f(u; s) = \int_{-\infty}^{+\infty} e^{-t^2} f(t)\, K(u, t, s)\, dt. \tag{76}$$

This expression can be interpreted as a *singular integral* with the kernel $K(u, t, s)$. We shall need some of the results of the theory of singular integrals.† We introduce with Hahn an auxiliary kernel $K(u, t, s, \varepsilon)$ defined when ε is positive as being equal to $K(u, t, s)$ except on the interval $u-\varepsilon \leqq t \leqq u+\varepsilon$ where it is identically zero. It is easy to verify that this kernel has the following two properties:

(A) To every fixed $\varepsilon > 0$ corresponds a positive M_ε such that $|K(u, t, s, \varepsilon)| < M_\varepsilon$ for every s, $0 \leqq s < 1$, and every real t and this uniformly with respect to u in any finite u-interval.

(B) $\lim_{s \to +1} \int_\alpha^\beta K(u, t, s, \varepsilon)\, dt = 0$ for all finite α and β.

It follows from these two conditions‡ that

$$\lim_{s \to +1} \left[g(u; s) - \int_{u-\varepsilon}^{u+\varepsilon} g(t)\, K(u, t, s)\, dt \right] = 0, \tag{77}$$

if

$$\int_{-\infty}^{+\infty} |g(t)|\, dt$$

exists. Here we have written for the sake of brevity

$$g(u; s) = e^{-u^2} f(u; s), \qquad g(u) = e^{-u^2} f(u). \tag{78}$$

We introduce another auxiliary kernel

$$K^*(u, t, s) = [\pi(1-s^2)]^{-\frac{1}{2}} \exp\left[-\frac{(u-t)^2}{1-s^2} \right], \tag{79}$$

and write

$$\begin{aligned} \int_{u-\varepsilon}^{u+\varepsilon} g(t)\, K(u, t, s)\, dt &= \int_{u-\varepsilon}^{u+\varepsilon} g(t)\, K^*(u, t, s)\, dt \\ &\quad + \int_{u-\varepsilon}^{u+\varepsilon} g(t)\, [K(u, t, s) - K^*(u, t, s)]\, dt. \end{aligned} \tag{80}$$

† We base the following discussion upon H. Hahn's memoir: Über die Darstellung gegebener Funktionen durch singuläre Integrale, Denkschriften d. K. Akademie d. Wissenschaften, Wien, Math.-Naturwissenschaftliche Klasse, vol. 93 (1917), pp. 585–655 and 657–692.—The basis of the theory was laid by Hobson and Lebesgue.

‡ See Hahn, l. c. p. 588, Theorem Ia, and p. 599, Theorem VIa.

But $K^*(u, t, s)$ is merely an unusual form of the well-known Weierstrassian kernel

$$\sqrt{\frac{k}{\pi}}\, e^{-k(u-t)^2}.$$

Hence we have

$$\lim_{s\to+1}\int_{u-\varepsilon}^{u+\varepsilon} g(t)\, K^*(u, t, s)\, dt = \lim_{h\to 0}\frac{1}{2h}\int_{-h}^{+h} g(u+t)\, dt, \tag{81}$$

if the latter limit exists.† Moreover the convergence is uniform on any finite closed interval interior to an interval of continuity.

Thus in order to prove Theorem IV we have merely to show that the second integral on the right hand side of (80) tends to zero uniformly on every fixed finite u-interval where $g(u)$ is bounded.

Let M be the maximum of $|g(u)|$ on $(u-\varepsilon, u+\varepsilon)$. Then

$$\left|\int_{u-\varepsilon}^{u+\varepsilon} g(t)\,[K(u, t, s) - K^*(u, t, s)]\, dt\right| \leqq M\int_{u-\varepsilon}^{u+\varepsilon} |K(u, t, s) - K^*(u, t, s)|\, dt.$$

It is enough to prove that the right hand side tends to zero uniformly with $(1-s)$ when $0 \leqq u \leqq U$. If $\varepsilon < s < 1$, we have

$$\int_{u-\varepsilon}^{u+\varepsilon} |K-K^*|\, dt < \int_{-\varepsilon}^{(1+s)u} |K-K^*|\, dt$$

$$= \left[\frac{1}{s}J_1 - J_2\right] + \left[J_3 - \frac{1}{s}J_4\right] + \left[\frac{1}{s}J_5 - J_6\right],$$

where

$$J_\nu = \frac{1}{\sqrt{\pi}}\int_{a_\nu}^{b_\nu} e^{-r^2}\, dr;$$

$$a_1 = -\frac{u+\varepsilon s}{\sqrt{1-s^2}}; \quad a_2 = -\frac{u+\varepsilon}{\sqrt{1-s^2}}; \quad a_3 = b_1;$$

$$a_4 = b_1; \quad a_5 = -b_3; \quad a_6 = b_3;$$

$$b_1 = -\frac{u}{\sqrt{1-s^2}}; \quad b_2 = b_1; \quad b_3 = \frac{u}{1+s}\sqrt{\frac{1-s}{1+s}};$$

$$b_4 = -b_3; \quad b_5 = -sb_1 - u\sqrt{1-s^2}; \quad b_6 = -sb_1.$$

A simple calculation shows that each of the three brackets $\left[J_\nu - \frac{1}{s}J_\mu\right]$ tends to zero when $s \to +1$, uniformly with respect to u when $0 \leqq u \leqq U$. This completes the proof of Theorem IV.

† See Hahn, l. c. p. 634. For the notion of the generalized derivative of the indefinite integral see p. 612.

25. It follows from the preceding discussion that, if (F) has a solution belonging to F_0, then

$$f(u) = \lim_{s \to +1} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iu+y)^2} \varphi(isy)\, dy \tag{82}$$

almost everywhere. This solution should be compared with formulas $(F_1)-(F_5)$ of the Introduction. We shall not be further concerned with the solutions of (F).

26. If we set $s = 1$ in (72) we obtain formally

$$f(u;1) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-(iu+y)^2} \int_{-\infty}^{+\infty} e^{-(iy-t)^2} f(t)\, dt\, dy,$$

or

$$e^{-u^2} f(u;1) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-2iuy} \int_{-\infty}^{+\infty} e^{2ity} e^{-t^2} f(t)\, dt\, dy. \tag{83}$$

The right hand side of (83) is Fourier's integral for $e^{-u^2} f(u)$. If it is convergent at a point $u = u_0$ where $f(u)$ is continuous, its value is $e^{-u_0^2} f(u_0)$. Thus, at such a point $\lim_{s \to +1} f(u_0; s) = f(u_0; 1) = f(u_0)$. Fourier's integral is, however, not valid for the functious $e^{-u^2} f(u)$ if we do not impose any restriction upon $f(u)$ in addition to assuming that $f(u)$ belongs to F_0.*

The device of introducing convergence factors and limiting processes in Fourier's integral is an old one which goes back to Cauchy and Poisson though systematic investigations are of much later date.† It is very likely that the particular device used above is to be found somewhere in the literature.

27. The analysis of § 24 can be extended to the complex plane. Suppose that $f(w)$ is an analytic function of w holomorphic in a strip $-b_1 < v < +b_2$, $w = u + iv$, and further that

$$\int_{-\infty}^{+\infty} e^{-u^2} |f(u+iv)|\, du \tag{84}$$

exists when $-b_1 < v < +b_2$. Finally, suppose that there exists a doubly infinite sequence of values u_n such that

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* For the validity of Fourier's integral theorem see Weyl, (34) 69–70. Cf. also Carleman: Équations intégrales singulières, pp. 157–158.

† See e. g. Sommerfeld: Die willkürlichen Funktionen in der mathematischen Physik, Inaugural-Dissertation, Königsberg (1891), especially the first chapter.

(85) $$e^{-u_n^2}|f(u_n+iv)|<M,\ \lim u_n=\pm\infty,$$

where M is independent of v and n.

Formula (73) represents $f(w;s)$. Now give w a fixed value u_0+iv_0 in the strip and replace the real axis as path of integration by the line $v=v_0$. This is permissible; in fact, we can replace the segment from u_{-n} to u_n by the polygonal line from u_{-n} over $u_{-n}+iv_0$ and u_n+iv_0 to u_n without changing the value of the integral and, in view of (85), we can let $n\to+\infty$. On the other hand, the kernels $K(u,t,s)$, $K^*(u,t,s)$ and $K(u,t,s,\varepsilon)$ do not lose any of the properties which we have utilized in § 24 if we replace the real variables u and t by the complex variables $w=u+iv_0$ and $T=t+iv_0$ respectively. Thus we are able to conclude that

(86) $$\lim_{s\to+1} f(w;s)=f(w)$$

for every finite value of w in the strip. Moreover, the approach of $f(w;s)$ to its limit is evidently uniform, not merely on any finite part of every fixed parallel to the real axis, but also in every closed finite portion of the strip.

Part 6. Applications to Hermitian series.

28. We shall now apply the results of parts 3—5 to the summation and convergence theory of Hermitian series.

Let $f(u)$ belong to either Q_β or to J_γ. The series

$$\sum_{n=0}^{\infty} f_n H_n(u)\sim f(u)$$

is in general divergent, but the associated power series

$$\sum_{n=0}^{\infty} f_n\, H_n\,(u)\, s^n = f\,(u;s)$$

is convergent when $|s|<\sqrt{\frac{2-\beta}{\beta}}$ or $\sqrt{\frac{1-\gamma}{\gamma}}$, and $f(u;s)$ is holomorphic in $E(s,\beta)<1$ or $e(s,\gamma)<1$ respectively. Further for approach along the real axis $\lim_{s\to+1} f(u;s)=f(u)$ almost everywhere. We have here a method of summing the formal Hermitian series of $f(u)$ based upon analytic continuation of $f(u;s)$ along the real axis in the s-plane.

Similar results hold in the case of a function $f(w)$, holomorphic in a strip $-b_1<v<+b_2$, provided $f(u+iv_0)$ considered as a function of u

belongs to Q_β for every fixed v_0, $-b_1 < v_0 < +b_2$, and $f(w)$ satisfies (85) or some similar condition.

THEOREM V: *If $f(u)$ belongs to J^* (or to Q^*), and*

$$e^{u^2} \lim_{h \to 0} \frac{1}{2h} \int_{u-h}^{u+h} e^{-t^2} f(t)\, dt = F(u)$$

exists for all finite values of u and, in addition, $f(u)$ is bounded on every finite interval, then the formal Hermitian series of $f(u)$ is summable Abel to the sum $F(u)$, i. e.

$$\lim_{s \to +1} \sum_{n=0}^{\infty} f_n H_n(u)\, s^n = F(u). \tag{87}$$

In particular, if the formal series is convergent when $u = u_0$, its sum equals $F(u_0)$.

If $f(u)$ belongs to J_1 or to Q_1, and if $f(u; s)$ is a holomorphic function of s at $s = +1$,† then the formal series is convergent when $u = u_0$.

This theorem is a corollary of Theorems III and IV. All but the last part of the theorem extends to complex variables provided $f(w)$ satisfies restrictions of the type mentioned above. Theorem V is of the same degree of generality as the corresponding theorem on Poisson summation for ordinary Fourier series as found by Fatou, Lebesgue and others.

29. It is now very natural to ask if and how the preceding summation theorem can be utilized in the convergence theory of Hermitian series. For this purpose we shall use a Tauberian theorem due to Littlewood,‡ namely:

If $\sum a_n$ is summable Abel to the sum A and if $a_n = O(1/n)$, then the series is convergent to the sum A.

We know already that the formal Hermitian series of a function $f(u)$ which is bounded when u is bounded and belongs to J^* (or to Q^*) is actually summable Abel to the sum $f(u)$ almost everywhere. It remains for us to give examples of functions with these properties such that

† This condition can be weakened. See M. Riesz: Sur les séries de Dirichlet et les séries entières, C. R., vol. 149 (1909), pp. 909–912, and P. Dienes: Sur la sommabilité de la série de Taylor, ibid., vol. 153 (1911), p. 802.

‡ J. E. Littlewood: The converse of Abel's theorem on power series, Proc. London Math. Soc., ser. 2, vol. 9 (1911), pp. 434–448, especially p. 438. See also G. H. Hardy and J. E. Littlewood: Abel's theorem and its converse, in the same periodical, vol. 18 (1918–19), pp. 205–235. Both papers contain applications of the theorem quoted to the theory of ordinary Fourier series. As far as I know Tauberian theorems have not previously been applied to other expansions except for a paper of mine: On Laguerre's series, Third Note, Proc. Nat. Acad. Sci, vol. 12. no. 5 (1926), pp. 348–352.

$f_n H_n(u) = O(1/n)$ for a fixed value of u or uniformly on a fixed finite interval. In the present paragraph we shall study such a class of functions; another method of constructing functions of this type is outlined in § 31.

THEOREM VI: *Let $f(u)$ satisfy the following conditions:* (i) *$f(u)$ belongs to J^* (or to Q^*),* (ii) *$f(u)$ is of bounded variation on every fixed finite interval, and* (iii) *the following Stieltjes integral exists, namely*

$$\left\{\int_{-\infty}^{-a} + \int_{a}^{+\infty}\right\} e^{-\frac{t^2}{2}} |t|^{\frac{1}{2}} |df(t)|. \tag{88}$$

Then Littlewood's Tauberian theorem applies and the series $\sum_{n=0}^{\infty} f_n H_n(u)$ converges for every finite value of u to the sum $\frac{1}{2}[f(u-0)+f(u+0)]$. The convergence is uniform on every finite interval interior to an interval of continuity of $f(u)$.

Remark 1: In formula (88) $|t|^{\frac{1}{2}}$ could be replaced by $|t|^{\frac{1}{3}}$ if we use formula (30) instead of (29) in the proof below.

Remark 2: Theorem VI is less general than the theorem found by Cramér, (7), as the generalization of Galbrun's expansion theorem (really Adamoff's theorem).

Proof: Let $\varphi(z)$ and $f(u;s)$ have their usual meaning and write

$$\varphi^*(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(z-t)^2}\, df(t), \tag{89}$$

$$f^*(u;s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(iu+y)^2} \varphi^*(isy)\, dy. \tag{90}$$

These functions are related to $\varphi(z)$ and $f(u;s)$ as follows:

$$\varphi^*(z) = \varphi'(z); \qquad f^*(u;s) = \frac{1}{s}\frac{\partial}{\partial u} f(u;s). \tag{91}$$

We shall use and prove only the first of these relations. We notice that the Stieltjes integral in (89) exists. Further the integral

$$-\frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (z-t)\, e^{-(z-t)^2} f(t)\, dt$$

exists and is uniformly convergent with respect to z when $|z| \leqq R$, say. Hence the value of the integral is $\varphi'(z)$. Further, we have the right to

integrate by parts in (89) between any two points of continuity of $f(u)$, thus obtaining

$$\frac{1}{\sqrt{\pi}}\int_a^b e^{-(z-t)^2}\,df(t) = \frac{1}{\sqrt{\pi}}\left[e^{-(z-t)^2}f(t)\right]_a^b - \frac{2}{\sqrt{\pi}}\int_a^b (z-t)\,e^{-(z-t)^2}f(t)\,dt.$$

If $b\to+\infty$, the integrals converge to finite limits uniformly with respect to z, $|z|\leqq R$, hence $\lim_{t\to+\infty}\exp(-(z-t)^2)\,f(t)$ exists uniformly for these values of z; the value of the limit is zero since $\varphi(z)$ exists for all values of z by assumption. The same type of reasoning applies, if we let $a\to-\infty$. Thus the left hand side of the equation tends to $\varphi^*(z)$, the right one to $\varphi'(z)$; hence $\varphi^*(z)=\varphi'(z)$.

Consequently, if we write

$$\varphi^*(z) = \sum_{n=0}^{\infty} f_n^*(2z)^n,$$

then we have in view of (33) and (49)

$$f_n^* = 2(n+1)f_{n+1} \qquad (n = 0, 1, \cdots) \tag{92}$$

$$\sqrt{\pi}\,2^n\,n!\,f_n^* = \int_{-\infty}^{+\infty} e^{-t^2}H_n(t)\,df(t). \tag{93}$$

The next step is to estimate the magnitude of f_n^*. Putting $a=\varrho\sqrt{2n+1}$, where $\varrho<1$, and $b=1.25\sqrt{2n+1}$, we have

$$\begin{aligned}\sqrt{\pi}\,2^n\,n!\,f_n^* &= \left[\int_{-\infty}^{-b}+\int_{-b}^{-a}+\int_{-a}^{+a}+\int_{+a}^{+b}+\int_{+b}^{+\infty}\right]e^{-t^2}H_n(t)\,df(t)\\ &= J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

Using (28) we find that

$$|J_3| < K_\varrho\,2^{\frac{n}{2}}(n!)^{\frac{1}{2}}\,n^{-\frac{1}{4}}\int_{-a}^{+a} e^{-\frac{t^2}{2}}\,|df(t)|,$$

$$|J_5| < K_\varrho\,2^{\frac{n}{2}}(n!)^{\frac{1}{2}}\,n^{-\frac{1}{4}}\int_{+b}^{+\infty} e^{-\frac{t^2}{2}}\,|df(t)|.$$

The latter integral exists in view of (88). J_1 is estimated like J_5. Finally, using (29) we find that

$$\begin{aligned}|J_4| &\leqq K 2^{\frac{n}{2}}(n!)^{\frac{1}{2}}\int_a^b e^{-\frac{t^2}{2}}\,|df(t)|\\ &< K\varrho^{-\frac{1}{2}}2^{\frac{n}{2}}(n!)^{\frac{1}{2}}(2n+1)^{-\frac{1}{4}}\int_a^b e^{-\frac{t^2}{2}}\sqrt{t}\,|df(t)|.\end{aligned}$$

The integral J_2 is handled in the same manner. Hence

$$|f_n^*| < A\,2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}n^{-\frac{1}{4}}. \tag{94}$$

where A is a suitably chosen constant.

Now let us restrict u to a fixed interval $(-u_0, +u_0)$. Then if n is so large that $u_0 < \varrho\sqrt{2n+1}$, we have

$$|f_n H_n(u)| = \frac{1}{2n}|f_{n-1}^* H_n(u)|$$

$$< \frac{1}{2n}A\,2^{-\frac{n-1}{2}}[(n-1)!]^{-\frac{1}{2}}(n-1)^{-\frac{1}{4}}K\,2^{\frac{n}{2}}(n!)^{\frac{1}{2}}n^{-\frac{1}{4}}e^{\frac{u^2}{2}} < \frac{B}{n}e^{\frac{u_0^2}{2}},$$

where B is independent of n and of u.

We know, in view of corollary 2 in conjunction with Theorem V, that the series $\sum_{n=0}^{\infty} f_n H_n(u)$ is summable Abel to the sum $\frac{1}{2}[f(u-0)+f(u+0)]$ for every finite value of u. We have just proved that the terms of the series are $O(1/n)$. Hence Littlewood's Tauberian theorem is applicable and the series is actually convergent. The uniform convergence on any finite interval interior to an interval of continuity follows from the fact, that the series is uniformly summable on such an interval and, consequently, uniformly convergent if it is convergent throughout the interval. This completes the proof of Theorem VI.

30. The machinery developed for the proof of Theorem VI can be utilized for the proof of the following

THEOREM VII. *Let $f(u)$ satisfy the following conditions*: (i) *$f(u)$ belongs to Q^**, (ii) *$f(u)$ is absolutely continuous on every finite interval, and* (iii) *the integral*

$$\int_{-\infty}^{+\infty} e^{-t^2}|f'(t)|^2\,dt \tag{95}$$

exists. Then $\sum_{n=0}^{\infty} f_n H_n(u)$ converges absolutely and uniformly to the sum $f(u)$ on every fixed finite interval.

Remark: The prototype of this theorem occurs in Weyl, (35) 306, and Cramér (7) 405. The subsequent proof has but little in common with the older proofs.

Proof: Formula (92) is valid with our new, more restrictive assumptions. In virtue of (67) and (95) the series

$$\sum_{n=0}^{\infty} 2^n n!\,|f_n^*|^2 = 2\sum_{n=1}^{\infty} 2^n n\,n!\,|f_n|^2$$

converges. Hence

$$\left[\sum_{n=1}^{m} |f_n H_n(u)|\right]^2 \leqq \sum_{n=1}^{\infty} 2^n n!\, |f_n|^2 \sum_{n=1}^{\infty} \frac{|H_n(u)|^2}{2^n n!}.$$

If $|u| \leqq u_0$, the terms of the second series are less than a constant times $n^{-\frac{3}{2}}$. Hence the series $\sum_{n=0}^{\infty} f_n H_n(u)$ is absolutely and uniformly convergent on $-u_0 \leqq u \leqq +u_0$. The series being summable Abel to the sum $f(u)$ and convergent, must converge to the sum $f(u)$.

31. We proceed to outline the construction of a class of functions whose Hermitian expansions can be proved to be convergent with the aid of Littlewood's Tauberian theorem.

We use the name *primitive saltus function* to designate a function defined as follows

$$s(u, a) = \begin{cases} 0, & \text{if } u < a, \\ \frac{1}{2}, & \text{if } u = a, \\ 1, & \text{if } u > a. \end{cases}$$

It is easy to verify that

$$(96) \quad s(u, a) = \frac{1}{\sqrt{\pi}} \int_a^{\infty} e^{-t^2} dt + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} H_{n-1}(a)\, e^{-a^2} \frac{H_n(u)}{2^n n!}.$$

That the series converges to the function follows from Theorem VI or by direct estimate of the order of magnitude of the terms.

Now consider a general saltus function $s(u)$ belonging to J^* which has jumps $s(a_\nu + 0) - s(a_\nu - 0) = c_\nu$ on a preassigned denumerable set (a_ν). For the sake of simplicity we assume that $s(u) = 0$ when $u < 0$. We assume further that the function $s(u)$ can be represented by the series

$$(97) \qquad s(u) = \sum_{\nu=1}^{\infty} c_\nu\, s(u, a_\nu),$$

and that the Hermitian series of $f(u)$ is simply

$$(98) \quad \frac{1}{\sqrt{\pi}} \sum_{\nu=1}^{\infty} c_\nu \int_{a_\nu}^{\infty} e^{-t^2} dt + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} c_\nu H_{n-1}(a_\nu)\, e^{-a_\nu^2} \frac{H_n(u)}{2^n n!}.$$

We suppose that all the summations with respect to ν lead to convergent series; moreover that

$$99) \qquad \left| \sum_{\nu=1}^{\infty} c_\nu H_{n-1}(a_\nu)\, e^{-a_\nu^2} \right| < C(n-1)^{-\frac{1}{4}}\, 2^{\frac{n-1}{2}} [(n-1)!]^{\frac{1}{2}},$$

for every $n>1$. In such a case the series (98) converges everywhere to $\frac{1}{2}[s(u-0)+s(u+0)]$. It follows from (28) and (29), by the same type of argument as was used in deriving formula (94), that (99) is certainly satisfied if

$$\sum_{\nu=1}^{\infty} |c_\nu| \sqrt{a_\nu}\, e^{-\frac{a_\nu^2}{2}} \tag{100}$$

converges; in fact, in this case $s(u)$ satisfies the conditions of Theorem VI. If $s(u)$ is not equal to zero when $u<0$, we consider the function $s(-u)$ for positive values of u separately and assume that $s(-u)$ satisfies conditions corresponding to (97)–(99).

So far we have merely been concerned with saltus functions. The preceding analysis applies, however, to certain limit functions of saltus functions as well. With the aid of such considerations we arrive at the following

THEOREM VIII: *Let $f(u)$ belong to J^* and be bounded when u is bounded. Suppose further that we can find a set of saltus functions $s_m(u)$ with the following properties*: (i) *$s_m(u)$ satisfies conditions* (97)–(99) *for every m with a fixed value of C in* (99), *the same for all values of n and m, and* (ii)

$$\int_{-\infty}^{+\infty} e^{-t^2} H_n(t)\, s_m(t)\, dt = \sqrt{\pi}\; 2^n n!\; [f_n+\eta_{nm}], \tag{101}$$

where $\eta_{nm}\to 0$ uniformly with respect to n when $m\to+\infty$. Then $\sum_{n=0}^{\infty} f_n H_n(u)$ converges to $f(u)$ almost everywhere.

The problem of characterizing the functions satisfying the conditions of this theorem by means of descriptive properties must be deferred to some other occasion.

32. In a recent paper Szegö, (29) 114–115, has proved the following theorem:

Suppose that $f(u)$ has a proper integral over every finite interval and that $f(u)=O(|u|^M)$ $(M>0)$ when $|u|\to+\infty$. Then $\sum_{n=0}^{\infty} f_n H_n(u)$ is summable (C,ε) to the sum $f(u)$ when $\varepsilon>0$ and $f(u)$ is continuous. Further, the n-th partial sum of the series is

$$S_n(u)=\frac{1}{\pi}\int_{u-h}^{u+h} f(t)\,\frac{\sin\sqrt{2n}\,(u-t)}{u-t}\,dt+o(1) \tag{102}$$

where $0<h<|u|$. Thus $S_n(u)$ behaves with respect to convergence, summability etc. as the $[\sqrt{2n}\,]$-th partial sum of the Fourier series of a function which agrees with $f(u)$ in the neighborhood of the point in question.

This elegant theorem gives a very satisfactory solution of the expansion problem for the Hermitian series. It is deficient in one respect: the condition that $f(u)$ shall be $O(|u|^M)$ is unnecessarily restrictive. This defect is, however, amendable: the condition can be replaced by the requirement that the integral

$$\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} |f(t)|\, dt$$

shall exist. Szegö obtains his theorem as a corollary of a corresponding theorem for Laguerre's series. I shall return to the question in connection with the convergence theory of the latter series.

PRINCETON UNIVERSITY,
April, 1926.

Note.

After my paper had already been sent to the printers, my attention was directed to several papers on Hermitian polynomials and series. It was possible to refer to most of these in the Bibliography; thus nos. 19a, 37, 39 and 41 are such later interpolations. I am indebted to Prof. J. Tamarkin for the reference to Uspensky (40), to Prof. G. Pólya for the reference to Laguerre (19a) and to Professors M. Plancherel and G. Szegö for the reference to Rotach (39). Dr. Rotach's thesis contains elegant and important equi-convergence theorems for Hermite's and Laguerre's series which in certain respects complete those due to Szegö. Finally, I have to refer to:

42. Kameda, Toyojiro: Theory of generating functions and its application to the theory of probability. Journal Fac. Sci. Imp. Univ. Tôkyo, Section 1, 1 (1) (1925) 1–62 (27–36). [Revised translation from German original in Proc. Tôkyo Math.-Phys. Soc., (2) 8 (1915) 262–295, 336–360.]

This paper shows some degree of contact with my own, especially in the point of departure and to some extent also in the treatment of Hermitian series. [Added in proof, September, 1926.]

SOME EXTENSIONS OF THE GENERALIZED KRONECKER SYMBOL.

By J. J. Nassau.

1. Professor Murnaghan in a recent paper* has considered the use of the generalized Kronecker symbol

$$\begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}$$

in representing a determinant. In this paper I shall consider some extensions and applications of the symbol in the field of determinants, permanents and their minors, multiple-suffixed determinants, and some analytical functions recently considered by Macmahon.†

2. By means of the generalized Kronecker symbol a determinant is defined by‡

$$|a_{1n}| = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}. \tag{2}$$

Inasmuch as a permanent differs from a determinant only in having all its terms positive, it can readily be expressed by means of this symbol; i. e:

$$\overset{+}{|}a_{1n}\overset{+}{|} = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}. \tag{2.1}$$

Since the two sets of subscripts are identical this symbol indicates but one summation.

All negative values of the symbol $\begin{bmatrix} 1 & 2 & \cdots & n \\ r_1 & r_2 & \cdots & r_n \end{bmatrix}$ will be annulled by being multiplied by the same symbol. Some other notations might be used to advantage, e. g.,

$$\overset{+}{|}a_{1n}\overset{+}{|} = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}^2 a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \tag{2.2}$$

or

$$= \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}.$$

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* The generalized Kronecker symbol and its applications to the theory of determinants, Amer. Math. Monthly, vol. 32, p. 233. This paper will be denoted by (A) hereafter.

† Researches in the theory of determinants, Trans. Cambr. Phil. Soc., vol. 23, pp. 89–135.

‡ Paper (A), equation (3.1).

From this simple extension it is observed that relations in determinants expressible by means of the Kronecker symbol yield corresponding relations in permanents.

2.1. The introduction of the product idea to annul negative signs suggests the following:

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix} = n!$$

and

$$\begin{bmatrix} 1 & 2 & \cdots & r \\ \alpha_1 & \alpha_2 & \cdots & \alpha_r \end{bmatrix} \cdot \begin{bmatrix} r+1 & \cdots & n \\ \alpha_{r+1} & \cdots & \alpha_n \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix} = r!\,(n-r)!$$

Here the Greek letters appear as subscripts and superscripts in the symbol and hence they denote summation. (Paper (A) p. 235).

3. Consider the expression

$$(3) \qquad \begin{bmatrix} 1 & 2 & \cdots & n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix}^2 a_1^{\varrho_1} a_2^{\varrho_2} \cdots a_n^{\varrho_n} \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix} b_{\alpha_1}^{\varrho_1} b_{\alpha_2}^{\varrho_2} \cdots b_{\alpha_n}^{\varrho_n}.$$

From equation (2) this is equal to

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix}^2 a_1^{\varrho_1} a_2^{\varrho_2} \cdots a_n^{\varrho_n} \begin{vmatrix} b_1^{\varrho_1} & b_1^{\varrho_2} & \cdots & b_1^{\varrho_n} \\ b_2^{\varrho_1} & b_2^{\varrho_2} & \cdots & b_2^{\varrho_n} \\ \cdot & \cdot & \cdots & \cdot \\ b_n^{\varrho_1} & b_n^{\varrho_2} & \cdots & b_n^{\varrho_n} \end{vmatrix}$$

$$= \begin{bmatrix} 1 & 2 & \cdots & n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix} a_1^{\varrho_1} a_2^{\varrho_2} \cdots a_n^{\varrho_n} \begin{bmatrix} 1 & 2 & \cdots & n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix} \begin{vmatrix} b_1^{\varrho_1} & b_1^{\varrho_2} & \cdots & b_1^{\varrho_n} \\ b_2^{\varrho_1} & b_2^{\varrho_2} & \cdots & b_2^{\varrho_n} \\ \cdot & \cdot & \cdots & \cdot \\ b_n^{\varrho_1} & b_n^{\varrho_2} & \cdots & b_n^{\varrho_n} \end{vmatrix}.$$

Assigning the set of values $(r_1 r_2 \cdots r_n)$ for $(\varrho_1 \varrho_2 \cdots \varrho_n)$, the first part of this product becomes a term of $|a_{1n}|$. The second part is

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ r_1 & r_2 & \cdots & r_n \end{bmatrix} |b_{1r_1} b_{2r_2} \cdots b_{nr_n}|,$$

which is equal to $|b_{1n}|$, since the symbol $\begin{bmatrix} 1 & 2 & \cdots & n \\ r_1 & r_2 & \cdots & r_n \end{bmatrix}$ is $+1$ when an even number of interchanges is required to rearrange the columns of the determinant to the order of $|b_{11} b_{22} \cdots b_{nn}|$, and is -1 when an odd number is required. Hence the expression in (3) is equal to $|a_{1n}| \cdot |b_{1n}|$.

By rearranging the factors of (3) and making use of equation (2.2), we have

$$|a_{1n}|\,|b_{1n}| = \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix}\begin{bmatrix}1 & 2 & \cdots & n\\ \varrho_1 & \varrho_2 & \cdots & \varrho_n\end{bmatrix}^2 (a_1^{\varrho_1} b_{\alpha_1}^{\varrho_1})\cdot(a_2^{\varrho_2} b_{\alpha_2}^{\varrho_2})\cdots(a_n^{\varrho_n} b_{\alpha_n}^{\varrho_n})$$

$$= \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix}\cdot\begin{vmatrix}a_{11} b_{\alpha_1 1} & a_{12} b_{\alpha_1 2} & \cdots & a_{1n} b_{\alpha_1 n}\\ a_{21} b_{\alpha_2 1} & a_{22} b_{\alpha_2 2} & \cdots & a_{2n} b_{\alpha_2 n}\\ \cdot & \cdot & \cdots & \cdot\\ a_{n1} b_{\alpha_n 1} & a_{n2} b_{\alpha_n 2} & \cdots & a_{nn} b_{\alpha_n n}\end{vmatrix}.$$

The sign preceding each term of the sum is + or − according as there has been, in its formation, an even or an odd number of interchanges in the rows of $|b_{11}\ \ b_{22}\ \cdots\ b_{nn}|$. Therefore: *The product of two determinants of the nth order is expressible as an aggregate of n! permanents of the same order.**

3.1. Similarly, consider the product

$$|a_{1n}|\cdot|b_{1n}| = \begin{bmatrix}1 & 2 & \cdots & n\\ \varrho_1 & \varrho_2 & \cdots & \varrho_n\end{bmatrix}^2 a_1^{\varrho_1} a_2^{\varrho_2}\cdots a_n^{\varrho_n}\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix} b_{1\alpha_1} b_{2\alpha_2}\cdots b_{n\alpha_n}$$

$$= \begin{bmatrix}1 & 2 & \cdots & n\\ \varrho_1 & \varrho_2 & \cdots & \varrho_n\end{bmatrix}^2\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix}(a_1^{\varrho_1} b_{1\alpha_1})(a_2^{\varrho_2} b_{2\alpha_2})\cdots(a_n^{\varrho_n} b_{n\alpha_n})$$

$$= \begin{bmatrix}1 & 2 & \cdots & n\\ \varrho_1 & \varrho_2 & \cdots & \varrho_n\end{bmatrix}^2\begin{vmatrix}a_1^{\varrho_1} b_{11} & a_1^{\varrho_2} b_{12} & \cdots & a_1^{\varrho_n} b_{1n}\\ a_2^{\varrho_1} b_{21} & a_2^{\varrho_2} b_{22} & \cdots & a_2^{\varrho_n} b_{2n}\\ \cdot & \cdot & \cdots & \cdot\\ a_n^{\varrho_1} b_{n1} & a_n^{\varrho_2} b_{n2} & \cdots & a_n^{\varrho_n} b_{nn}\end{vmatrix}$$

which is the sum of $n!$ determinants. Hence the theorem:† *The product of a permanent and a determinant both of the nth order is expressible as the sum of n! determinants of the same order.*

The mode of formation of the determinants is clearly seen from the expression last given.

A corresponding theorem and proof can be given for the product $|a_{1n}|\cdot|b_{1n}|$.

4. In this section the generalized symbol will be employed in expressing some relations between minors of determinants and permanents.

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* Muir, On a class of permanent symmetric functions, Proc. Roy. Soc. Edin., vol. 11 (1881–82), pp. 409–418.

† Muir, loc. cit.

If in equation (2) a_{nr} is set equal to 1, we have

$$(4)\qquad \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix} a_{\alpha_1 1}\, a_{\alpha_2 2} \cdots a_{\alpha_{n-1} n-1} = \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & 1\\ a_{21} & \cdots & a_{2,n-1} & 1\\ . & \cdots & . & .\\ a_{n1} & \cdots & a_{n,n-1} & 1\end{vmatrix}.$$

It is also clear that

$$(4.1)\qquad \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix} a_{\alpha_1 1}\, a_{\alpha_2 2} \cdots a_{\alpha_r r} = 0 \text{ when } r < n-1.$$

Making use of equation (4), the value of

$$\begin{bmatrix}1 & 2 & \cdots & n\\ \varrho_1 & \varrho_2 & \cdots & \varrho_n\end{bmatrix} \cdot \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix} a_{\alpha_1}^{\varrho_1} a_{\alpha_2}^{\varrho_2} \cdots a_{\alpha_{n-1}}^{\varrho_{n-1}}$$

$$(4.2)\qquad = \begin{bmatrix}1 & 2 & \cdots & n\\ \varrho_1 & \varrho_2 & \cdots & \varrho_n\end{bmatrix} \cdot \begin{vmatrix} a_1^{\varrho_1} & a_1^{\varrho_2} & \cdots & a_1^{\varrho_{n-1}} & 1\\ a_2^{\varrho_1} & a_2^{\varrho_2} & \cdots & a_2^{\varrho_{n-1}} & 1\\ . & . & \cdots & . & .\\ a_n^{\varrho_1} & a_n^{\varrho_2} & \cdots & a_n^{\varrho_{n-1}} & 1\end{vmatrix}.$$

There are $n!$ terms in this sum, one of which is

$$(4.3)\qquad \begin{bmatrix}1 & 2 & \cdots & n\\ s_1 & s_2 & \cdots & s_n\end{bmatrix} \cdot \begin{vmatrix} a_{1s_1} & a_{1s_2} & \cdots & a_{1s_{n-1}} & 1\\ a_{2s_1} & a_{2s_2} & \cdots & a_{2s_{n-1}} & 1\\ . & . & \cdots & . & .\\ a_{ns_1} & a_{ns_2} & \cdots & a_{ns_{n-1}} & 1\end{vmatrix}.$$

By interchanging $(s_1 \cdots s_{n-1})$ we obtain $(n-1)!$ like determinants each representing the signed sum of n primary minors of $|a_{1n}|$. Hence the sum in (4.2) involves n different determinants of the type shown in (4.3). Therefore $\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ \varrho_1 & \varrho_2 & \cdots & \varrho_n\end{bmatrix} a_{\alpha_1}^{\varrho_1} \cdots a_{\alpha_{n-1}}^{\varrho_{n-1}} = (n-1)!$ times the sum of the signed primary minors of $|a_{1n}|$.*

The expression corresponding to equation (4.1) for the permanent is

$$(4.4)\qquad \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ 1 & 2 & \cdots & n\end{bmatrix}^2 a_{\alpha_1 1}\, a_{\alpha_2 2} \cdots a_{\alpha_r r}.$$

Assigning the set of values $(s_1\, s_2 \cdots s_n)$ to $(\alpha_1\, \alpha_2 \cdots \alpha_n)$, we have

$$\begin{bmatrix}s_1 & s_2 & \cdots & s_n\\ 1 & 2 & \cdots & n\end{bmatrix}^2 a_{s_1 1}\, a_{s_2 2} \cdots a_{s_r r}.$$

* Muir, The generating function of certain special determinants, Proc. Roy. Soc. Edin., vol. 24, pp. 387–392.

There are as many such terms as there interchanges in the set $(s_{r+1} \cdots s_n)$ and hence the permanent in (4.4) is equal to $(n-r)!$ times the sum of the permanents of order r formed from the matrix

$$\left\| \begin{matrix} a_{11} & a_{21} & \cdots & a_{n1} \\ . & . & \cdots & . \\ a_{1r} & a_{2r} & \cdots & a_{nr} \end{matrix} \right\|.$$

5. The generalized Kronecker symbol is well adapted to the expressing of determinants with multiple suffixes and theorems related to them are derived with great ease. To illustrate this the case of the cubic determinant will be taken

$$|a_{1nn}| = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} \varrho_1 & \varrho_2 & \cdots & \varrho_n \\ 1 & 2 & \cdots & n \end{bmatrix} a_{1\varrho_1}^{\alpha_1} a_{2\varrho_2}^{\alpha_2} \cdots a_{n\varrho_n}^{\alpha_n}$$

$$= \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \begin{vmatrix} a_{11}^{\alpha_1} & a_{12}^{\alpha_1} & \cdots & a_{1n}^{\alpha_1} \\ a_{21}^{\alpha_2} & a_{22}^{\alpha_2} & \cdots & a_{2n}^{\alpha_2} \\ . & . & \cdots & . \\ a_{n1}^{\alpha_n} & a_{n2}^{\alpha_n} & \cdots & a_{nn}^{\alpha_n} \end{vmatrix}.$$

This shows that the cubic determinant of order n is equal to the algebraic sum of $n!$ ordinary determinants of the same order.

The cubic determinant whose planes consist of the determinant $|a_{1n}|$ is expressed by

$$\begin{bmatrix} \varrho_1 & \varrho_2 & \cdots & \varrho_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_{\varrho_1}^{\alpha_1} a_{\varrho_2}^{\alpha_2} \cdots a_{\varrho_n}^{\alpha_n}.$$

Referring to equation (3.3) of paper (A) this becomes equal to $n!\,|a_{1n}|$.

The product of two determinants, each of order n, may be expressed as a cubic determinant. That is,

$$|a_{1n}|\,|b_{1n}| = \begin{bmatrix} 1 & 2 & \cdots & n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix} a_1^{\varrho_1} a_2^{\varrho_2} \cdots a_n^{\varrho_n} \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix} b_{1\alpha_1} b_{2\alpha_2} \cdots b_{n\alpha_n}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix} (a_1^{\varrho_1} b_{1\alpha_1}) (a_2^{\varrho_2} b_{2\alpha_2}) \cdots (a_n^{\varrho_n} b_{n\alpha_n}).$$

Also

$$|a_{1nn}|\,|b_{1n}| = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix} a_{1\alpha_1}^{\varrho_1} a_{2\alpha_2}^{\varrho_2} \cdots a_{n\alpha_n}^{\varrho_n} \begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{bmatrix} b_1^{\sigma_1} b_2^{\sigma_2} \cdots b_n^{\sigma_n}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \varrho_1 & \varrho_2 & \cdots & \varrho_n \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{bmatrix} (a_{1\alpha_1}^{\varrho_1} b_1^{\sigma_1}) (a_{2\alpha_2}^{\varrho_2} b_2^{\sigma_2}) \cdots (a_{n\alpha_n}^{\varrho_n} b_n^{\sigma_n}).$$

Hence the product of a cubic determinant whose elements are a_{ijk} and an ordinary determinant whose elements are b_{il} may be expressed as a determinant of fourth class whose elements are given by $c_{ijkl} = a_{ijk}\, b_{il}$.

6. Without any elaboration in the notation already used, it is possible to represent alternants and simple symmetric functions by means of the generalized symbol. In the case of the alternant we have

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix} a_{\alpha_1}^{p_1} a_{\alpha_2}^{p_2} \cdots a_{\alpha_n}^{p_n}.$$

Here the superscripts are replaced by exponents, as they will be throughout this section. Referring to equation (4), the expression for the difference-product $\zeta^{1/2}(a_1\, a_2 \cdots a_n)$ is

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix} a_{\alpha_2} a_{\alpha_3}^2 \cdots a_{\alpha_n}^{n-1}.$$

For the case of the simple symmetric function, we have

$$\sum a_1\, a_2^2 \cdots a_{n-1}^{n-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix}^2 a_{\alpha_1} a_{\alpha_2}^2 \cdots a_{\alpha_{n-1}}^{n-1}.$$

Consider the more general case

$$\sum a_1^{p_1} a_2^{p_2} \cdots a_r^{p_r} = \frac{1}{(n-r)!} \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix}^2 a_{\alpha_1}^{p_1} a_{\alpha_2}^{p_2} \cdots a_{\alpha_r}^{p_r}.$$

This is clear from the discussion at the end of section 4. If $p_1 = p_2 = \cdots = p_s$, the interchange of the assigned values of $\alpha_1, \alpha_2, \cdots, \alpha_s$ yields $s!$ like terms. For example

$$\sum a_1\, a_2\, a_3^3 = \frac{1}{2!} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 1 & 2 & 3 & 4 \end{bmatrix}^2 a_{\alpha_1} a_{\alpha_2} a_{\alpha_3}^3.$$

7. The aim of the above discussion has been to exhibit some of the possibilities of the Kronecker symbol as extended in paper (A), and at the same time to adhere to the original definition given in that paper. In this section a slight alteration of the original definition will be introduced to make possible the representation of a generalized analytical function considered recently by Macmahon, in his paper referred to above.

The labels $(r_1\, r_2 \cdots r_n)$ and $(s_1\, s_2 \cdots s_n)$ in the symbol $\begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}$ will be made capable of receiving any value out of the n integers $1, 2, \cdots, n$; that is, allowing repetitions. Otherwise the definition remains the same.

For example, $\begin{bmatrix} 1 & 1 & 2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} = a_{11}\, a_{21}\, a_{32} - a_{11}\, a_{22}\, a_{31} + a_{12}\, a_{21}\, a_{31}$.

The general case given on page 96 of that paper is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & \cdots & 2 & \cdots & s \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{p_1} & \alpha_{p_1+1} & \cdots & \alpha_{p_1+p_2} & \cdots & \alpha_n \end{bmatrix} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}, \text{ where } \sum_1^s p_r = n.$$

CASE SCHOOL OF APPLIED SCIENCE.

March 6, 1926.

ON THE INDETERMINATE EQUATION $t^2 - p^2 Du^2 = 1$.

By Derrick Henry Lehmer.

It is the purpose of the present paper to discuss a method of solving in integers the equation

$$t^2 - p^2 Du^2 = 1 \qquad p > 1 \tag{1}$$

without having recourse to the usual method of expanding the square root of $p^2 D$ in a continued fraction. We shall assume in what follows that the fundamental solution (T_1, U_1) of the equation

$$T^2 - DU^2 = 1 \tag{2}$$

is known from tables which extend to $D = 1700$. The solution of (1) depends on discovering a fundamental or multiple solution of (2) for which the U is divisible by p. Having found such a solution (T_n, U_n) it is easily seen that $(T_n, U_n/p)$ is a solution of (1). We shall first prove that it is always possible to do this.

Theorem 1. *There are an infinite number of U's satisfying the equation $T^2 - DU^2 = 1$ which are divisible by any integer p.*

Lagrange was the first to prove that there exist an infinite number of solutions of the Pellian equation. In particular there exist an infinite number of solutions of our equation

$$t^2 - p^2 Du^2 = 1.$$

For each one of these solutions we may put

$$t = T, \qquad pu = U$$

and obtain a solution of the equation

$$T^2 - DU^2 = 1.$$

But U is divisible by p. Hence the theorem is true.

We now consider the problem of finding among the solutions of equation (2) those U's which contain the factor p. Let

$$\varphi = \operatorname{arc\,cosh} T = \operatorname{arc\,sinh} (U\sqrt{D})$$

so that

$$T_n = \cosh n\varphi, \quad U_n \sqrt{D} = \sinh n\varphi.$$

Then we have

$$T_{-n} = T_n, \qquad U_{-n} = -U_n. \tag{3}$$

We also have

$$\begin{aligned} T_{m+n} &= \cosh(m\varphi + n\varphi) = T_m T_n + D U_m U_n \\ \sqrt{D}\, U_{m+n} &= \sinh(m\varphi + n\varphi) = (U_m T_n + U_n T_m)\sqrt{D}. \end{aligned} \tag{4}$$

For $m = n$ we have

$$\begin{aligned} T_{2n} &= T_n^2 + D U_n^2 = 2T_n^2 - 1 \\ U_{2n} &= 2 T_n U_n. \end{aligned} \tag{5}$$

For $m = 1$ we have

$$\begin{aligned} T_{n+1} &= 2 T_1 T_n - T_{n-1} \\ U_{n+1} &= 2 T_1 U_n - U_{n-1}. \end{aligned} \tag{6}$$

The recursion formulas (6) are the ones employed in finding the successive multiple solutions of the Pell equation when the fundamental solution (T_1, U_1) is known. T_0 and U_0 are of course 1 and 0 respectively.

We shall first dispose of the case where $p = 2$ which occurs most frequently in practice. We wish to know for what values of n the number U_n is even. We see from (5) that U_2 is always even and (6) tells us that if U_1 is even then all the U's are even. If U_1 is odd then the U's are alternately odd and even. Hence the fundamental solution of the equation

$$t^2 - 4Du^2 = 1$$

is either $(T_1, U_1/2)$ or $(2T_1^2 - 1, T_1 U_1)$ where (T_1, U_1) is the fundamental solution of the equation

$$T^2 - DU^2 = 1.$$

We shall now confine ourselves to the case in which p is an odd prime. That we can do this without loss of generality is evident from the following consideration. Suppose for the moment that p is made up of the prime factors $\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n$ where some or all of the α's may be identical. That is we wish to solve the equation

$$t^2 - \alpha_1^2 \alpha_2^2 \alpha_3^2 \cdots \alpha_n^2 Du^2 = 1.$$

We first find the fundamental solution of

$$t^2 - \alpha_1^2 Du^2 = 1.$$

Then letting $\alpha_1^2 D = D_1$ we solve the equation

$$t^2 - \alpha_2^2 D_1 u^2 = 1.$$

We then let $a_2^2 D_1 = D_2$ and so on until we have included all the a's. Hence we need only discuss the case in which p is an odd prime.

By expanding the left side of the well known relation

$$T_p + U_p\sqrt{D} = (T_1 + U_1\sqrt{D})^p$$

it can be shown* that

$$(7) \qquad \begin{aligned} T_p &\equiv T_1 \pmod p \\ U_p &\equiv \varepsilon U_1 \pmod p \end{aligned}$$

where $\varepsilon = \pm 1$ or 0 according as p is a quadratic residue of D, a non-residue of D, or a divisor of D. Making use of these facts we can prove the following theorem.

THEOREM 2. *$U_{p-\varepsilon}$ is divisible by p.*

For we have by (3) and (4)

$$\begin{aligned} U_{p-\varepsilon} &= U_p T_\varepsilon - U_\varepsilon T_p \\ &\equiv \varepsilon U_1 T_\varepsilon - U_\varepsilon T_1 \pmod p. \end{aligned}$$

If $\varepsilon = +1$ this last expression is manifestly zero. If $\varepsilon = -1$, it is zero by (3). If $\varepsilon = 0$, then it is zero by (7) and also since $U_0 = 0$. Since these are all the possible cases, the theorem is true.

We therefore have a sure method of finding a U which is divisible by p. But in solving the equation

$$t^2 - p^2 D u^2 = 1,$$

we are often interested in getting the fundamental solution and how can we be sure that $U_{p-\varepsilon}$ is the smallest U which contains the factor p? The method that at once suggests itself is that of calculating the successive U's and testing each for divisibility by p. But since formulas (4) and (5) allow us to skip many of the U's, this method would be unnecessarily laborious if we failed to find any $U < U_{p-\varepsilon}$ divisible by p. It is therefore important to know for a given p which U's are possible multiples of p.

Let us first consider the case where p divides D. Then since

$$\begin{aligned} T_1^2 &= 1 + D U_1^2 \equiv 1 \pmod p \\ T_1 &\equiv \pm 1. \end{aligned}$$

If $T_1 \equiv +1$ then (6) gives:

$$\begin{aligned} U_2 &\equiv 2 U_1, \\ U_3 &\equiv 2 U_2 - U_1 \equiv 3 U_1, \\ U_4 &\equiv 2 U_3 - U_2 \equiv 4 U_1. \end{aligned}$$

* Mathews, *Theory of Numbers*, p. 93.

And by induction $U_n \equiv n U_1$.

If $T_1 \equiv -1$ we have like manner

$$U_n \equiv (-)^{n+1} n U_1.$$

In either case $U_p \equiv p U_1$.

Hence we have the following theorem.

THEOREM 3. *If p divides D and also U_1 it divides every U. If p divides D but not U_1 then the smallest U that is divisible by p is U_p.*

Hence the fundamental solution of (1), if p divides D, is either $(T_1, U_1/p)$ or else $(T_p, U_p/p)$.

We shall now consider the case in which p does not divide D. By using formulas (6) and (7) we may deduce that

$$\begin{aligned} T_{p+1} &\equiv T_{\varepsilon+1} \pmod{p}, \\ U_{p+1} &\equiv U_{\varepsilon+1}. \end{aligned}$$

And in general it can be shown that

$$\begin{aligned} T_{np+h} &\equiv T_{n\varepsilon+h}, \\ U_{np+h} &\equiv U_{n\varepsilon+h}, \end{aligned} \tag{8}$$

hence the residues of (T_n, U_n) modulo p recur, the number of terms in each period being a factor of $p-\varepsilon$. We therefore have the following theorem.

THEOREM 4. *If U_m is the first U to contain the factor p, then m is a factor of $p-\varepsilon$.*

This follows from Theorem 2. Although we cannot say off hand which divisor of $p-\varepsilon$ is the one we seek, the number of possible cases is materially reduced. In this connection we are reminded strongly of solving

$$a^{\xi} \equiv 1 \pmod{n}$$

where we only know that, n being prime, ξ is a divisor of $n-1$. We may however limit m still further by proving the following theorem.

THEOREM 5. *If U_m is the first U to contain the factor p, then m is a factor of $(p-\varepsilon)/2$.*

Let $p-\varepsilon = 2\varkappa$. Then we have by (5) and Theorem 2,

$$U_{p-\varepsilon} = U_{2\varkappa} = 2 U_\varkappa T_\varkappa \equiv 0 \pmod{p}.$$

Hence either U_x or T_x is divisible by p. We shall now show that the latter alternative is impossible. If in (8) we put $n = 1$ and $h = -\varepsilon$ we have at once

$$T_{2x} = T_{p-\varepsilon} \equiv T_{\varepsilon-\varepsilon} \equiv T_0 \equiv 1 \pmod{p}.$$

But if T_x were divisible by p, we would have

$$T_{2x} = 2T_x^2 - 1 \equiv -1 \pmod{p}.$$

which is impossible. Therefore U_x must be a multiple of p. The theorem then follows by (8).

This theorem yields the following well known corollary which is useful in the complete factorisation of T_n and U_n. As we are not concerned at present with this problem we shall omit the proof.

COROLLARY. *The primitive divisors of T_n and U_n belong to the form* $2^{nx} \pm 1$.

We are now in a position to apply our knowledge to any particular equation of the form

$$t^2 - p^2 D u^2 = 1.$$

For example let it be required to find the fundamental solution of the equation

$$t^2 - 6174\, u^2 = 1.$$

We first note that $6174 = 7^2 \times 126$. Therefore we put $p = 7$, $D = 126$, and we note that p divides D. From Legendre's or Degen's table we find that the fundamental solution of the equation

$$T^2 - 126\, U^2 = 1$$

is $T_1 = 449$, $U_1 = 40$. Since U_1 is not divisible by 7, we know by Theorem 3 that U_7 is the least U divisible by 7. We therefore calculate T_1 and U_1 as follows.

T_0	1,	U_0	0,
T_1	449,	U_1	40,
T_2	403201,	U_2	35920,
T_3	362074049,	U_3	32256120,
T_4	325142092801,	U_4	28965959840,
T_7	235451028081583642049,	U_7	20975644724880384280.

We notice that U_7 is actually divisible by 7 which constitutes a check on the work. Dividing by 7, we have for the required solution:

$$t_1 = 235451028081583642049,$$
$$u_1 = 2996520674982912040.$$

This equation can be solved in another way. Since $6174 = 3^2 \times 686$ we put $p = 3$ and $D = 686$. From the tables

$$T_1 = 10850138895,$$
$$U_1 = 414260228.$$

Now since 3 is a non-residue of 686, we have $(p - \epsilon)/2 = 2$. Therefore either U_1 or U_2 is divisible by 3. But we see that U_1 is not divisible by 3 and therefore U_2 must be. From (5) we have immediately

$$T_2 = 235451028081583642049,$$
$$U_2 = 8989562024948736120.$$

Dividing U_2 by 3, we arrive at the same solution as before.

By way of comparison with the usual method of solving the Pell equation, it is found that $\sqrt{6174}$ expands in a continued fraction having 40 terms to the period. Thus, although we cannot hope to reduce the size of the solution, we obtain the solution much more readily by using the short method. Of course we cannot always apply this method since the D must contain a square factor. The number of cases however in which the method does apply is larger than one might at first suspect. For instance there are 627 numbers between 1 and 1700 (not counting perfect squares) for which the above method could have been used in preparing the existing tables of the solutions of the Pell equation.

It is of interest to note that Legendre used the solution for D to solve the equation for $4D$. Although he gives no account of his calculations, we can deduce this fact from the list of errors in the first edition of his table. For instance there is an error in the value of T for $D = 214$. The idenfical error appears again in the solution for $D = 856$. In fact for every D less than 250 which has an error in the T the same error occurs in the solution for $4D$. More striking evidence is found in the solutions for 86 and 344. Here the same entry is given for both arguments, the division by 2 having been omitted in the case of 344. In all these cases the U_1 is divisible by two and other evidence seems to show that Legendre did not employ non-fundamental solutions or any other value for p than 2. For instance there is an error in $D = 343 = 7^2 \times 7$ whereas the solution for 7 is quite correct. The same is true of $801 = 3^2 \times 89$. In fact the present paper is the result of investigations which originated in the examination of the errors in this table.

THE OPERATIONS OF BOOLEAN ALGEBRAS.*

By Orrin Frink.

An operation on n variables in an algebra may be thought of as a rule of combination of n elements of the algebra or as a function of n variables. It is a well known fact that the most general Boolean operation on n variables contains at most 2^n independent parameters. Thus the most general rule of combination of two letters in a Boolean algebra can be expressed in the form

$$F(a,b) = Aab + Ba'b + Cab' + Da'b'.$$

I use here the notation $a+b$ for the logical sum of a and b, ab for the logical product of a and b, and a' for the negation of a. The capital letters stand for constants and the small letters for variables. The parameters A, B, C and D are called the discriminants of the operation.

Similarly the most general function of one variable can be written in the form $f(a) = Aa + Ba'$. A relation between several elements of a Boolean algebra is expressed by one or more equations or inequations between such functions of the elements related.

A common type of problem concerning the operations of these algebras is that of finding all operations which have a certain formal property. Another problem, and one which I here solve, is that of finding all the operations which have all formal properties in common with a given operation, and in particular to find all unique operations, that is, those that share all their formal properties with no one other operation.

Wiener, in a paper to which I shall later refer, makes the statement "It would seem that a system whose postulates deal with entities which occupy a unique position in it has in some sense received a more thoroughgoing analysis than one where this is not the case." I give here later a postulate system for Boolean algebras in terms of a fundamental operation which is unique in this sense, which, as we shall see is not true of systems based on the operations usually taken as fundamental, such as logical sum and product.

Many postulate systems have been given for Boolean algebras. Of these, Huntington's first set of ten postulates,† has in a sense become the standard. Sheffer has given a set of only five postulates, using the one

* Presented to the American Mathematical Society, May 2, 1925; manuscript received December 1925.

† Trans. Amer. Math. Soc., vol. 5 (1904), p. 288.

fundamental operation of "rejection", and Bernstein later reduced the number to four, using the same operation. The operation which Russell calls "incompatability" would do just as well. The systems of both Sheffer and Bernstein are based on Huntington's, however, in that they prove as theorems Huntington's postulates in order to establish the sufficiency of their own postulates. All these posulate systems have the defect which Wiener has shown, namely that they do not serve to characterize uniquely the fundamental operations which they concern; in fact there are as many operations with the formal properties of logical addition, for example, as there are elements of the algebra.

Wiener has shown that the most general operation having the formal properties of logical sum and hence satisfying Huntington's third set of postulates for the algebra of logic is $ab+Aab'+Aa'b$.* I will prove by a different method the equivalent proposition that this generalization of logical sum and the corresponding generalization of logical product $ab+A'a'b'+A'a'b$ are the most general operations satisfying the postulates of Huntington's first set of postulates mentioned above.

There occur in the postulates the two Boolean constants 0 and 1, which have the properties that $a+0=a$, $a+1=1$, $a0=0$, $a1=a$, where a is any element of the algebra. It might be expected that for the generalizations of logical sum and logical product, the Boolean constants demanded by the postulates and corresponding to 0 and 1 above might be two quite arbitrary and independent elements of the algebra. The generalized fundamental operations would then contain two independent parameters. Of course, since $1=0'$ and $1'=0$, the two constants, which we may call A and B, would have to be the negatives of each other. That means that the operation of negation would have to be generalized also. For example, the operation $f(a)=(AB+A'B')a+(AB'+A'B)a'$ has the formal properties of negation, for instance $f[f(a)]=a$, and it also changes A into B and B into A.

Nevertheless it turns out of course that A and B must be negatives in the ordinary sense, that the generalized operations can contain only one parameter, and that no generalization of negation is possible. I will now prove a few preliminary theorems.

Let

$$F(a,b)=Aab+Bab'+Ca'b+Da'b'. \tag{1}$$

Suppose F is such that there is an element I such that $F(a,I)=a$ for all values of a. Then we will say that F has the "right idemfacient element I." Applying this condition to (1) we have

* Trans. Amer. Math. Soc., vol. 8 (1917), p. 17.

$$AaI+BaI'+Ca'I+Da'I',$$

or

$$(AI+BI')a+(CI+DI')a' = a.$$

If this is to be true for all values of a we have

(2) $AI = I$; (3) $BI' = I'$; (4) $CI = 0$; (5) $DI' = 0$.

THEOREM I. *The most general Boolean operation having the right idemfacient element I is $Aab+Bab'+Ca'b+Da'b'$, where $AI=I$, $BI' = I'$, $CI = 0$, $DI' = 0$.*

A similar theorem for operations with left idemfacient elements can be had by interchanging the conditions on B and C. It is interesting to note that from conditions (3) and (4) we get $BC = C$ irrespective of the value of I, and from (2) and (5) we get $AD = D$. Hence if the operation has both a left and right idemfacient element, $B = C$ and the operation is commutative.

If the operation is such that there exists an element N such that $F(a, N) = N$ for all values of a, we shall say that $F(a, b)$ has the "right nilfacient element N". Applying now this condition to (1) we will have

$$AaN+BaN'+Ca'N+Da'N' = N,$$

or

$$(Aa+Ca')N+(Ba+Da')N' = N.$$

From this we get readily

(6) $AN = N$; (7) $BN' = 0$; (8) $CN = N$; (9) $DN' = 0$.

THEOREM II. *The most general Boolean operation having the right nilfacient element N is $Aab+Bab'+Ca'b+Da'b'$, where $AN = N$, $BN' = 0$, $CN = N$, $DN' = 0$.*

If now we want the condition that the same function $F(a, b)$ have both the right idemfacient element I and the right nilfacient element N, we must combine the conditions of Theorems I and II. Combining (3) $BI' = I'$ and (7) $BN' = 0$, we get $I'N' = 0$. Combining (4) $CI = 0$ and (8) $CN = N$, we get $IN = 0$. Hence

(10) $$I = N'.$$

Combining (2), (6), and (10) we have $A = 1$.
Combining (5), (9), and (10) we have $D = 0$.

From (2), (7), and (10) we get $B = N$.

From (3), (8), and (10) we get $C = N$.

THEOREM III. *If $F(a, b)$ has the right idemfacient element I and the right nilfacient element N, then $I = N'$ and F is of the form*

$$ab + Nab' + Na'b.$$

It is possible for an operation with a right idemfacient element and a left nilfacient element to have for these two elements any two of the algebra whatever.

Huntington's first set of postulates, which concern a class K of elements and two operations, $a+b$ and ab, between them are as follows:

I (a). *$a+b$ is in the class whenever a and b are in the class.*

I (b). *ab is in the class whenever a and b are in the class.*

II (a). *There is an element Λ such that $a+\Lambda = a$ for every element a.*

II (b). *There is an element V such that $aV = a$ for every element a.*

III (a). *$a+b = b+a$ whenever a, b, $a+b$, and $b+a$ are in the class.*

III (b). *$ab = ba$ whenever a, b, ab, and ba are in the class.*

IV (a). *$a+(bc) = (a+b)(a+c)$ whenever a, b, c, $a+b$, $a+c$, bc, $a+bc$, and $(a+b)(a+c)$ are in the class.*

IV (b). *$a(b+c) = ab+ac$ whenever a, b, c, ab, ac, $b+c$, $a(b+c)$, and $ab+ac$ are in the class.*

V. *If the elements Λ and V in* II (a) *and* II (b) *exist and are unique, then for every element a there is an element a' such that $a+a' = V$ and $aa' = \Lambda$.*

VI. *There are at least 2 elements x and y in the class such that $x \neq y$.*

On the basis of these postulates Huntington proves certain theorems, of which I wish to use the two following:

IX (a). *$a+V = V$ for every element a.*

IX (b). *$a\Lambda = \Lambda$ for every element a.*

Postulate II (a) and Theorem IX (a) amount to the statement that the operation of logical sum has a right idemfacient element and a right nilfacient element; hence, by our Theorem III it must be of the form $ab + Nab' + Na'b$. Similarly, we deduce from II (b) and IX (b) that an operation with the formal properties of logical product must be of the same form.

Conversely we must establish the fact that all operations of the form $ab + Aab' + Aa'b$ and $ab + A'ab' + A'a'b$ satisfy the postulates when substituted for $a+b$ and ab respectively. In showing this I will use the ordinary rules of Boolean algebra, which are either expressed in the postulates, or, like the associative law, follow from them. Except in the

case of postulates IV (a), IV (b), and V, the verification is immediate and obvious.

Making the substitution in postulate IV (a) we find that we must prove that

$$a(bc+A'bc'+A'b'c)+Aa(bc+A'bc'+A'b'c)'+Aa'(bc+A'bc'+A'b'c)$$
$$=(ab+Aab'+Aa'b)(ac+Aac'+Aa'c)+A'(ab+Aab'+Aa'b)$$
$$\times(ac+Aac'+Aa'c)'+A'(ab+Aab'+Aa'b)'(ac+Aac'+Aa'c).$$

Making use of the fact that $(bc+A'bc'+A'b'c)'$ is equal to $b'c'+Ab'c+Abc'$, the left member on multiplying out reduces to $abc+A'abc'+A'ab'c+Aab'c'+Aab'c+Aabc'+Aa'bc+0$. This can be shortened to

$$abc+ab'c+abc'+Aab'c'+Aa'bc. \tag{12}$$

Similarly the right member reduces to $abc+Aabc'+Aab'c+Aab'c'$ $Aa'bc+A'abc'+A'ab'c$, which is equal to (12). For all the other terms are zero, using the fact that $aa'=0$ and $a0=0$. The first distributive law is thus verified.

The second distributive law IV (*b*) can be obtained from the first by interchanging the operations of sum and product. Since our operations $ab+Aab'+aa'b$ and $ab+A'ab'+A'a'b$ interchange when we interchange A and A', to prove IV (*b*) we merely go through the proof of IV (*a*) interchanging A and A'.

As for Postulate V, the element demanded by it is still the ordinary negative a'. For substituting it for b in the above operations we have

$$aa'+Aaa+Aa'a'=A,$$

and

$$aa'+A'aa+A'a'a'=A'.$$

As we have seen, A and A' are the $\vee$ and $\wedge$ of our system.

As it is of some interest, I give overleaf a table of the corresponding generalizations of some of the familiar dyadic Boolean operations and relations, which we derive at once from those of logical sum and product.

There are alternative ways of expressing the important operation which is the generalization of logical sum and product, e. g.,

$$ab+Aab'+Aa'b,$$
$$ab+A(a+b),$$
$$A'ab+A(a+b),$$
$$Aab+A'ab+Aab'+Aa'b.$$

The first form is the standard expanded form, the second is the one used by Wiener, the third serves to exhibit Peirce's principle of duality, and the fourth shows the very important fact that the operation is symmetrical in the three letters a, b, and A.

Name of operation	Symbol	Generalization
Logical sum	$a+b$	$ab+Aab'+Aa'b.$
Logical product	ab	$ab+A'ab'+A'a'b.$
Equivalence	$ab+a'b'$	$Aab+A'ab'+A'a'b+Aa'b'.$
Non-equivalence	$ab'+a'b$	$A'ab+Aab'+Aa'b+A'a'b'.$
Implication	$a'+b$	$Aab+a'b+Aa'b'.$
Exclusion	ab' or $a-b$	$A'ab+ab'+A'a'b'.$
Rejection	$a'b'$	$A'ab'+A'a'b+a'b'.$
Incompatability	$a'+b'$	$Aab'+Aa'b+a'b'.$
Inclusion, or Subsumption	$ab'=0$ or $a<b$	$A'ab+ab'+A'a'b'=A'$, or $Aab'+A'a'b=0.$
Some a is b	$ab\neq 0$	$Aab+A'a'b'\neq 0.$
No a is b	$ab=0$	$Aab+A'a'b'=0.$

TABLE.

Let us call for the moment the generalizations of logical sum and product $F_s(a,b)$ and $F_p(a,b)$. Since they obey the associative law, we can speak unambiguously of the logical sum and product of three or more elements, and it would be interesting to see the forms of

$$F_s(a,b,c) = F_s[a,F_s(b,c)] = F_s[F_s(a,b),c].$$

We have

$$F_s(a,b,c) = (ab+Aab'+Aa'b)c+A(a'b'+A'a'b+A'ab')c$$
$$+A(ab+Aab'+Aa'b)c' = abc+Aab'c+Aa'bc+Aabc'+Aab'c'$$
$$+Aa'bc'+Aa'b'c.$$

A short way of writing this is

$$F_s(a,b,c) = A'abc+A(a'b'c')', \text{ or } A(a+b+c)+A'(abc).$$

By the principle of duality we have $F_p(a,b,c) = Aabc+A'(a+b+c)$. Similarly for any number of variables

$$F_s(a,b,c,d,\cdots) = A(a+b+c+d+\cdots)+A'(abcd\cdots);$$
$$F_p(a,b,c,d,\cdots) = A(abcd\cdots)+A'(a+b+c+d+\cdots.$$

Let us now return to the general problem mentioned before, namely that of finding all the operations which have the same formal properties

as a given operation, and all unique operations. Since all Boolean operations can be defined in terms of the fundamental operations of logical sum, logical product and negation, we could solve the first of these problems by substituting in the expression of the operation in these terms the generalized operations $ab+Aab'+Aa'b$ and $ab+A'ab'+A'a'b$ for logical sum and product respectively. Similarly, for the second problem, we impose the condition that the parameter A vanish from the result. I think it better, however, to introduce here a new viewpoint, that of transformation, which will also bring out more clearly the relation which logical sum and product bear to their generalizations.

If there exists among the operations $f(a)$ on one variable of the algebra one which has an inverse operation $f^{-1}(a)$ such that $f^{-1}[f(a)] = a$ for all values of a, then we may transform any other operation, say on two or three variables, by $f(a)$ and the formal properties of the operation will be retained. This is a general method, which can be used in other than Boolean algebras. By the transform of $F(a,b)$ by $f(x)$ we mean $f[F(f^{-1}\{a\}; f^{-1}\{b\})]$. For example, suppose $F(a,b)$ obeys the associative law. Then so will its transform, for if $F[a,F(b,c)] = F[F(a,b),c]$ then

$$f\{F[f^{-1}(a),f^{-1}f(F[f^{-1}(b),f^{-1}(c)])]\}$$
$$= f\{F[f^{-1}f(F[f^{-1}(a),f^{-1}(b)],f^{-1}(c))]\},$$

and both sides are equal to $f\{F[f^{-1}(a),f^{-1}(b),f^{-1}(c)]\}$ where F is triadic.

The question now arises of finding Boolean operations with inverses. Let $f(a) = Aa+Ba'$ be a function with an inverse; then we want to find x as a function of a such that $f(x) = a$. This gives us $Ax+Bx' = a$, or $(Aa'+A'a)x+(Ba'+B'a)x' = 0$. This equation can be satisfied only if $(Aa'+A'a)(Ba'+B'a) = 0$, which reduces to $a = AB+aAB'+aA'B$. It is evident from $Ax+Bx' = a$ that x must be of the form $aB'+a'B$ $+K(Aa+A'a')$. Substituting $AB+aAB'+aA'B$ for a in the parenthesis we get $x = aB'+a'B+K(AB+A'B')$.

THEOREM IV. *There exists an inverse x of a with respect to the operation $f(a) = Ax+Bx'$ such that $f(x) = a$, when and only when a is of the form $AB+aAB'+aA'B$; x is then equal to $aB'+a'B+K(AB+A'B')$, K being an arbitary constant. The inverse x is unique when and only when $f(x)$ is such that the inverse exists for all values of a. This occurs when and only when $A = B'$, and $f(x) = Ax+A'x'$. The function is then its own inverse function.*

The only available functions with inverses are then those which are their own inverses, namely those of the form $Kx+K'x'$, which may be read "x is equivalent to K".

We can apply our Theorem IV to the subject of inverses of dyadic operations. If $F(a,b) = Aab + Bab' + Ca'b + Da'b'$ we shall say that the element b has the right inverse element x with respect to a and F if $F(a,x) = b$. If x exists for all values of a and b, then x expressed as a function of a and b is called the right inverse operation of $F(a,b)$. Similarly for left inverses. Schröder in his *Algebra der Logik*, vol. 2, pp. 493–4, determines the inverse operations of dyadic operations with inverses. Writing $F(a,x) = b$ in the form $(Aa + Ca')x + (Ba + Da')x' = b$ and applying Theorem IV we have

THEOREM V. *If an x exists such that $F(a,x) = b$ for all values of a and b, then $F(a,b)$ must be of the form $Aab + A'ab' + Ca'b + C'a'b'$. The inverse x is then unique and is of the same form, that is, the function is equal to its inverse.*

Applying the corresponding theorem for left inverses, we have the result that the most general Boolean operation which has always both a left and right inverse is of the form

$$F(a,b) = Aab + A'ab' + A'a'b + Aa'b'.$$

Bernstein has shown that for this operation (which may be written $a \equiv b \equiv A$), the elements form an Abelian group. I pointed out that it is also the most general operation with respect to which the elements of a Boolean algebra form a group, Bernstein having by an oversight given a more general operation as having that property.

The operation in question is the generalization of equivalence given in the table on page 482. For, substituting $ab + Aab' + Aa'b$ for $a + b$ in the definition of equivalence in terms of logical sum and negation, namely $(a+b)' + (a'+b')'$, we have

$$\begin{aligned}(ab + A'ab' + A'a'b)\,(a'b' + A'ab' + A'a'b) + A(ab + A'ab' + A'a'b)\\ \cdot (ab + Aab' + Aa'b) + A(A'b' + Aab' + Aa'b)(a'b' + A'ab' + A'a'b)\\ = Aab + A'a'b + A'ab + Aa'b'.\end{aligned}$$

Wiener has pointed out that the generalization of logical sum is the operation made to correspond to logical sum when the elements of the algebra are transformed by a one to one transformation. Let us see into what operation an arbitrary operation is transformed by the function $f(x) = Kx + K'x'$.

We will write the operation in the completely expanded form:

$$F(a,b,c,\cdots) = A_1\,abc\cdots + B_1\,a'b'c'\cdots + A_2a'bc\cdots + B_2ab'c'\cdots + \cdots.$$

Two terms in the expansion such as $A_1\, abc\cdots$ and $B_1\, a'b'c'\cdots$ we will call conjugate if each variable in one term is the negative of a variable in the other. The transform of F is

$$f\{F[f(a), f(b), f(c), \cdots]\}$$

or

$$\begin{aligned} &K[A_1\,(Ka + K'a')\,(Kb + K'b')\,(Kc + K'c')\cdots \\ &\quad + B_1\,(Ka' + K'a)\,(Kb' + K'b)\,(Kc' + K'c)\cdots + \cdots] \\ &+ K'[A_1'\,(Ka + K'a')\,(Kb + K'b')\cdots + B_1'\,(Ka' + K'a)\,(Kb' + K'b)\cdots + \cdots] \\ &\quad = (KA_1 + K'B_1')\,abc\cdots + (KB_1 + K'A_1')\,a'b'c'\cdots + \cdots. \end{aligned}$$

THEOREM VI. *Transformation of $F\,(a, b, c, \cdots)$ by $f\,(x) = Kx + K'x'$ replaces the coefficient A_n of any term in the completely expanded form of F by $(KA_n + K'B_n')$, where B_n is coefficient of the conjugate term.*

By giving the arbitrary parameter K all values in the algebra we get all transforms of the operation. In saying that this set of all transforms of an operation is the set of all operations with the same formal properties we are taking two things for granted. One is that the transform actually has the same formal properties. I have given a proof of this in the case of the associative law. It might be hard to prove in general, but we could get around this difficulty by defining "formal properties" to be those unchanged by transformation.

The other assumption is that all operations with the same formal properties as the given operation can be obtained by transformation. That this is actually the case may be seen in some such way as the following. Express the given operation in terms of logical sum and negation. The manner of its dependence on logical sum and negation is one of the formal properties of the operation. The other operation, supposed to have the same formal properties, must be capable of being expressed in the same way in terms of an operation having the same formal properties as logical sum. It easily follows that the second expression is a transform of the first, and hence also the second operation is a transform of the first.

The question naturally arises, when is the transformed operation the same as the original one? It is evident that $A_n = (KA_n + K'B_n')$ for all values of K when and only when $A_n = B_n'$.

Hence we have the

THEOREM VII. *The only unique operations of the algebra are those in whose complete expansion the coefficient of any term is the negative of the coefficient of the conjugate term.*

Let us see which of the simpler operations are unique. Of those in one variable, only the transforming operations themselves, of the form $Ax + A'x'$, are invariant. In fact they form an Abelian group, and of course all elements in such a group are self-conjugate or invariant. The negation is included in the set, and is therefore unique.

The unique dyadic operations are of the form $Aab + Bab' + B'a'b + A'a'b'$. They do not seem to be so important as we might have expected. Those of the form $ab + Bab' + B'a'b$ are particularly interesting. Such an operation may be written $Ba + B'b$; in other words it is the general function of one variable (B), considered as a function of the two coefficients. These latter operations also have the peculiar property that any dyadic Boolean operation is distributive with respect to them. An attempt to eliminate the parameters A and B by giving them the special values 1 or 0 or making them equals or negatives reduces the operation to a trivial one.

However, among the triadic operations we get some very important new results. We can get rid of the coefficients in the general form by giving them the values 1 or 0. The only unique triadic operations without parameters and which are symmetric in the three variables are the following:

$$\text{(i)}\quad abc + a'bc + ab'c + abc',$$
$$\text{(ii)}\quad a'b'c' + ab'c' + a'bc' + a'b'c,$$
$$\text{(iii)}\quad abc + ab'c' + a'bc' + a'b'c,$$
$$\text{(iv)}\quad a'b'c' + a'bc + ab'c + abc'.$$

Here (i) and (ii) correspond to our generalizations of logical sum and rejection, which are negatives, and (iii) and (iv) to those of equivalence and non-equivalence. For instance, the operation which we called the generalization of logical sum was $ab + Aab' + Aa'b$. If we let $c = A$ this becomes (i) above, namely $abc + a'bc + ab'c + abc'$, when written in the completely expanded form. This is an accord with the interesting observation we made before that the operation is symmetric in a, b, and A.

There are six non-symmetric operations on three variables without parameters, which we can get by substituting a' for a, b' for b, or c' for c in (i) or (ii). These correspond to implication, logical product, and so forth. For example, the operation $ab'c + a'bc + a'b'c + a'b'c'$ corresponds to $a' + b'$ with c as parameter, or to $b'c$ with a as parameter. Replacing a variable by its negative in (iii) changes the operation into (iv), and vice versa. Of course, changing two variables thus, leaves (iii) and (iv) unchanged.

Operation (i), which I will call triadic sum, can be read "At least two of the three propositions a, b, and c are true," and its negative (ii), triadic rejection, "at least two are false." Or if an interpretation in terms of sets is desired, the triadic addition can be read "that which is common to at least two of the sets a, b, and c." I have written the operations above in the standard form, where there is no overlapping between terms, but perhaps simpler expressions would be

$$ab + ac + bc \text{ for triadic sum,}$$

and

$$a'b' + a'c' + b'c' \text{ for triadic rejection.}$$

I shall now give a postulate system for Boolean algebras which will concern a class K of elements and a triadic operation F relating them, to be thought of as triadic rejection.

1. $F(a, b, c)$ is in K whenever a, b, and c are in K.
2. $F(a, b, c) = F(b, a, c)$ whenever a, b, and c are in K.
3. $F(a, b, c) = F(b, c, a)$ whenever a, b, and c are in K.
4. $F(a, a, b) = F(a, a, c)$ for all values of a, b, and c.
   Definition I $F(a, a, b) = a'$.
5. $(a')' = a$ for all values of a.
   Definition II $[F(a, b, c)]' = S(a, b, c,)$.
6. $S(a, a', b) = b$ for all values of a and b.
7. $S(a, [S(b, c, d')], d) = S[S(a, b, d), S(a, c, d), d']$.
8. There are at least two elements x and y in K such that $x \neq y$.
9. $F(a, b, c) = S\{S[a', S(b', c', \vee), \wedge], S(b', c', \wedge), \vee\}$.

It is easy to see that $S(a, b, \vee)$ and $S(a, b, \wedge)$ satisfy Huntington's postulates given above if substituted for $a + b$ and ab, if $\vee = \wedge'$. Postulate 7 is the distributive law. I have no doubt that a set of less than nine postulates for triadic rejection could be found.

I shall not discuss the subject of unique operations in four or more variables, although it is no doubt a very interesting subject.

One of the important applications of Boolean algebra is to the theory of sets. The subsets of a given set have the properties of the elements of a Boolean algebra. All finite Boolean algebras may be interpreted in this way, and thus we see that the number of elements, if finite, must be of the form 2^n, where n is some integer.

An important idea in set theory is that of relativity, where certain properties of sets are defined with respect to some arbitrary set instead

of the fundamental space. The foregoing results applied to the theory of sets lead to a different relativity principle, in which the logical sum and logical product of two sets is relative to some arbitrary set. This relative logical sum of two sets with respect to set A is the set of elements common to the two sets plus those elements of A which are in only one of the two sets. A' then has the properties of the null set. This leads to a generalization of the relation of subset. In fact, a is a subset of b relative to A when and only when $AaB'+A'a'b=0$. This relation has the properties of the subset relation, such as transitivity and the existence of an all-inclusive element and an all-included element analogous to the universal and null sets.

Since A and a occur symmetrically in the definition of relative subset, we get the result that if a is a subset of b with respect to c, then c is a subset of b with respect to a. This suggests that the triadic relation $abc'+a'b'c=0$ is an important one. In fact, Wiener has shown that it is an invariant relation, which has the formal properties of a relation studied by Kempe which may be read "c is between a and b." It should be pointed out that there is a great difference between invariant operations and invariant relations. Wiener has given the necessary and sufficient condition that a relation be unique or invariant. This condition is that the coefficients of conjugate terms of the completely expanded form of the operation occuring in the equation defining the relation be equal. It will be remembered that for unique operations the conjugate coefficients must be negatives.

It is the existence of unit sets that causes the chief formal difference between the theory of sets and Boolean algebras. A unit set has the property that its logical product with any other set is either itself or the null set; in other words, for all values of B it is included in either B or B'. An infinite Boolean algebra need contain no such unit elements with these properties. An interesting example of such a Boolean algebra whose elements cannot be interpreted as the subsets of a set is the following. Suppose that the atomic theory is false and that matter is infinitely subdivisible. Then the set of all subaggregations of matter in the universe forms a Boolean algebra for the usual interpretation of logical sum, logical product and negation or complement. There is nothing here corresponding to unit set, as a point is not an aggregation of matter. It is understood that the null aggregation is included in the algebra.

For our generalization of subset, the unit elements are not unit sets. For instance, if (a) is a unit set, then the "unit element" with respect to A corresponding to (a) is $A'+(a)$ if (a) is in A, or $A'-(a)$ if (a) is in A'. I am using element here to mean set; it has no reference to the members

of the sets, whose existence is not referred to in the postulates. The distinction between the relation of inclusion which subsists between classes and the relation of membership in a class is a familiar one which is emphasised by Whitehead and Russell.

The applications to the symbolic logic of the *Principia Mathematica* of Whitehead and Russell seem less important. It is understood that the elements of this logic are propositions. If we consider their system up to the introduction of propositional functions, which will not concern us, the chief peculiarity of the logic is the „assertion." "$\vdash p$," p is asserted, corresponds to the statement $p = 1$, or $p' = 0$ in Boolean logic. In this way Whitehead and Russell avoid the use of the Boolean constants 1 and 0. There is, however, no method of denying an assertion, and there is nothing corresponding to the "inequation" $p \neq 0$ of the Boolean logic.

As a result, although there is an analogue in the system of the *Principia* of the Boolean universal statement $ab' = 0$ (all a is b), there is no analogue of the particular statement $ab \neq 0$ (some a is b, or some b is a). That is why Whitehead and Russell say that their elementary propositions do not contain any indefinite terms, and why they introduce propositional functions, where the indefinite terms are made explicit.

The generalization of logical sum, $p \vee q$, can be written $(p \cdot q) \vee (p \cdot k) \vee (q \cdot k)$. We have the result that if we substitute this expression for $p \vee q$ and also substitute for $\vdash p$ the proposition $\vdash (p \equiv k)$, leaving negation unchanged, the elementary postulates of Whitehead and Russell's logic are satisfied, k being an arbitrary proposition.

If a relation between n variables is not invariant, there is a corresponding relation in $n+1$ variables which is. This may be derived as follows. Suppose the relation to be expressed in this form:

$$A_1\, abc\cdots + B_1\, a'b'c'\cdots + \cdots + A_r abc'\cdots + B_r a'b'c\cdots + \cdots = 0.$$

The operation forming the left member of this equation may be generalized by Theorem VI, according to which the general term has its coefficient A_r replaced by $kA_r + k'B'_r$. If we set this generalized operation equal to k', which corresponds to 0 in the new system, and reduce to the standard form we get the new relation

$$A_1(kabc\cdots + k'a'b'c'\cdots) + B_1(k'abc + ka'b'c'\cdots) + \cdots = 0.$$

This relation has the properties of the old relation, and if we consider k a variable, it is invariant. By this method we can derive the between relation, $a'bc + ab'c' = 0$, from the inclusion relation, $a'b = 0$.

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A PROOF OF PETERSEN'S THEOREM

By Orrin Frink jr.

By a regular linear graph of the third degree we mean a network of lines (1–cells) and points (0–cells or vertices) such that there are exactly three 1–cells meeting at every vertex. The number of vertices is called the order of the graph. A regular graph is said to be primitive if it cannot be obtained by superposing two regular graphs of the same order but of lower degree. Sylvester gave an example of a primitive graph of the third degree, but it contained three leaves. Petersen's theorem states that every primitive third degree graph has three or more leaves. By a leaf we mean a portion of the graph which is connected to the rest of the graph only by a single 1–cell, and no proper part of which has this property.

Petersen's proof* has been simplified by Brahana† and Errera.‡ The present proof is intended to be still simpler and to involve simpler notions. The method used is to consider first graphs without leaves and show that these are factorable (that is, not primitive). The proof is by induction, no shrinking or counting processes being used. A connected regular graph of the third degree without leaves will be called a *simple* graph. It is easy to see that a connected third degree graph is simple when and only when every 1–cell is on a closed circuit.

Theorem I. *Given a simple graph of order greater than two, we can always obtain from it a simple graph of lower order by removing any* 1*–cell and properly joining the four incident* 1*–cells in pairs.*

Thus in fig. 1 delete x and join 1 to 3 and 2 to 4 obtaining fig. 2. The result will be a simple graph unless the part of the graph containing 1–3 is connected to that containing 2–4 by a single 1-cell or

Fig. 1.

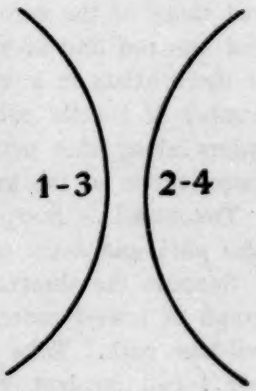

Fig. 2.

* Acta Math., vol. 15 (1891), pp. 193–220.

† Annals of Math., vol. 19 (1917), pp. 59–63.

‡ Dissertation, Brussels, 1921.

not at all. If this is the case there must be in the original graph a path from 1 to 3 and a path from 2 to 4. For example, if there were no path from 1 to 3 then either 1 or 3 would be the stem of a leaf, or an "isthmus". Then I say that if we delete x and join 1 to 4 instead and 2 to 3 we obtain a simple graph. To prove this it is sufficient to show that all the closed circuits that were disturbed by the deletion, that is, those through 1-cells 1, 2, 3 and 4 have been replaced by new ones. The circuits through 1 and 3 have been lengthened, the 1-cell x having been replaced by the path we proved to exist from 2 to 4. The same is true of the circuits through 2 and 4. As for circuits through 1 and 2, if the first deletion disconnected the graph these must also contain 3 and 4 and hence have not been disturbed. If the first deletion resulted in an isthmus however, a circuit through 1 and 2 need not contain 3 and 4 if it contains the isthmus. In this case there are two new circuits, joining 1-3 and 2-4 to the isthmus, one of which will do the work of the original circuit through 1 and 2. The same argument applies in the case of circuits through 3 and 4. The circuits through 1 and 4 and through 2 and 3 have not been disturbed.

It is unfortunate that such a trivial and obvious theorem should require such a long proof. It is to be noted that the argument holds even if a pair of the 1-cells 1, 2, 3 and 4 are identical or incident at another vertex. If a third degree graph is not primitive, it is said to be factorable, and the first and second degree graphs into which it can be decomposed are called its factors. If the 1-cells of the first degree factor are colored red and those of the second degree factor blue, there will be two blue lines and one red line at every vertex. The graph is then said to be *colored.* If there exists in a colored graph a closed path consisting of an even number of 1-cells colored alternately red and blue we can interchange colors along this path and obtain a new coloration and hence a new factorization of the graph.

THEOREM II. *Every 1-cell of a colored simple graph is on a closed red-blue path and hence can have its color changed.*

Suppose the theorem were false, then there would exist a colored simple graph of lowest order containing a blue 1-cell, say x, not on any closed red-blue path. Take a blue 1-cell adjacent to x, say y, and delete the red 1-cell incident with its other end joining loose ends according to Theorem I to obtain a simple graph of lower order. In the new graph which is also colored, there must exist a closed red-blue path through x. But this same path must exist in the original graph, because it is not disturbed by restoring the deleted 1-cell. It is possible that a closed red-blue path containing y might be disturbed by this restoration, but x

and y, being adjacent blue 1-cells, cannot be on the same red-blue path. Thus we obtain a contradiction. For a graph of order two Theorem I would not apply, but the conclusion is then obvious.

THEOREM III. *Every simple graph is colorable.*

If the theorem were false there would exist a non-colorable simple graph of lowest order. Lower its order according to Theorem I and color it. Three cases arise. Case I. If both new 1-cells are colored blue, restore the deleted 1-cell colored red. Case II. If one is colored blue and the other red, restore the deleted 1-cell colored blue. In either case we have a coloration of the original graph contrary to assumption. Case III. If both new 1-cells are colored red, change the color of one of them according to Theorem II. If this changes the color of the other also we have Case I, if not, Case II. For a graph of order two the theorem is obvious. This proves the theorem.

THEOREM IV (Petersen's Theorem). *A regular graph of the third degree with fewer than three leaves is colorable.*

We have shown that a simple graph is colorable. A graph with one leaf is impossible. To color one with two leaves, join the leaves by a new 1-cell. The resulting graph is simple and may be colored. If the new 1-cell is colored red, delete it. If it is colored blue, change its color according to Theorem II and then delete it. This restores the original graph, which is now colored.

GEOMETRY OF A SET OF n FUNCTIONS.*

By Louis Ingold.

1. Introduction. In the paper Functional Differential Geometry† the author has considered a geometry based in a certain way on a function $f(x; u^1, u^2, \cdots, u^n)$ of an independent variable x and the parameters u^λ $(\lambda = 1, 2, \cdots, n)$. In this treatment the function f was considered to be analogous to a vector from an arbitrary origin to points of the space under consideration. The variables u^λ were regarded as the coördinates of this space.

In the actual development the function f itself played almost no part. The functions f_λ $(= \partial f/\partial u^\lambda, \lambda = 1, 2, \cdots, n)$ really served as the basis of the geometry. These functions were interpreted as the tangent vectors to the parameter curves of the space, with the curvilinear coördinates u^λ. The geometry may be regarded as a geometry of a set of vectors of a very special type in which the fundamental vectors or functions, (the f_λ) are all partial derivatives of the same function f with respect to the coördinates.

The present paper contains a generalization of these ideas.

1. The fundamental functions (denoted by θ_i) selected for the basis of the geometry are not necessarily the partial derivatives of one and the same function with respect to the coördinates.

2. The number of the coördinates u^λ is not necessarily the same as the number of the fundamental functions θ_i.

3. The space of the fundamental vectors and the space of the parameters are (except in special applications) regarded as independent of each other.

The generalization 2. is especially desirable as it permits the study, from the same point of view, of a given vector space and of the enlarged vector spaces obtained by adjoining other vectors to the original fundamental system; for example we may adjoin the derivatives of the original vectors θ_i with respect to the coördinates.

Various interpretations of the theory are possible but in the following pages the geometric language employed is based chiefly on the interpretation given below.

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* Presented to the American Mathematical Society April 10, 1925. This paper also includes the main portions of "Extensions of the formulae of Gauss and Codazzi" presented to the Society February 28, 1925.

† Trans. Amer. Math. Soc., vol. 13 (1912), pp. 319-341.

Each function $\theta_i(x; u^1, \cdots, u^m)$ is interpreted as a vector field associated with the points of a manifold with the coördinates $u^1, \cdots u^m$. This manifold is referred to as the parameter space S_m. To emphasize this analogy we continually refer to functions of x and the parameters u^λ indifferently as *functions* or *vectors*.

The n functions θ_i taken together represent a set of n vectors associated with the points of the parameter space S_m. From this point of view the theory may be regarded as a generalization of the theory of the *moving trihedral* which has been so successfully employed by French geometers.

It need not be supposed that the vectors represented by the functions θ_i are tangent to the parameter curves of the manifold S_m; in fact this would not be possible unless the functions θ_i satisfied certain conditions.

The parameter space may be dispensed with altogether, that is, we may suppose that the vectors represented by the functions θ_i are rotated about a fixed origin as the parameters u^λ vary. The parameters are then merely *parameters of displacement.*

The vectors θ_i and all linear combinations of them whose coefficients are functions of the coördinates u^λ only, (i. e. independent of x) will be said to belong to the vector space V_n. The paper is chiefly concerned with the space V_n.

The relations among the vectors θ_i are expressed by means of certain functions of the coördinates obtained by a process analogous to scalar multiplication. The same process is used to study the relations between the vectors θ_i and the vectors $\theta_{i\lambda} = \partial\,\theta_i/\partial\,u^\lambda$ which in general are not supposed to be linearly expressible in terms of the θ_i.

The modifications that arise when a set* of functions $\varphi_i = a_i^j\,\theta_j (i = 1, 2, \cdots, n)$ are used as basis, are studied and necessary conditions are obtained that the space V_n be a linear space.

The earlier portion of the paper is based on a former paper, Displacements in a Function Space, the detailed results of which are contained in the abstract in the Bulletin of the American Mathematical Society, vol. 18 (1911–1912), pp. 272–273.

The paper considers not only the space V_n of the vectors θ_i but also the larger spaces obtained by adjoining the first and higher partial derivatives of the θ_i with respect to the coördinates. The necessary conditions that any of these enlarged spaces be linear are expressed by

* We adopt the convenient custom of omitting the sign of summation Σ when the index of summation occurs twice in the sample term of the sum; once as a subscript and once as a superscript. Latin indices, referring to the vector system θ_i are supposed to run from 1 to n and Greek indices, referring to the coördinates, u^λ, run from 1 to m.

equations which may be regarded as generalizations of the formulas of Gauss and Codazzi.

Conditions altogether similar to those just mentioned are obtained as the necessary and sufficient conditions that a linear vector space V_n exists whose fundamental functions θ_i satisfy a certain given system of linear partial differential equations. This result is a very general form of the fundamental theorem of differential geometry.

A number of special cases are introduced in order to illustrate the general theory.

2. **The fundamental vectors.** We consider a set of n functions $\theta_1, \cdots, \theta_n$ of the variable x and the parameters $u^1, \cdots, u^m$. These functions and their derivatives with respect to the parameters u are supposed to be defined for the variable x on the interval $0 \leqq x \leqq 1$, and for a given region of the parameter space.

The integral from 0 to 1 with respect to x of the product of two functions φ, ψ, will be taken to represent the scalar product of the corresponding pair of vectors. This integral will be written simply as a product:

$$\int_0^1 \varphi \psi \, dx = \varphi \psi. \tag{1}$$

For fixed values of the coördinates the θ's can be regarded as a system of n independent vectors in a euclidean n-space at the point u. The scalar products of the vectors θ_i in pairs form a set of quantities that we shall call the first fundamental quantities. We write,

$$\theta_i \theta_j = g_{ij} \tag{2}$$

and assume that $g = |g_{ij}|$ does not vanish at points of the parameter manifold under consideration.

In terms of these products the magnitudes of the vectors θ_i and the angle φ_{ij} between the pairs of vectors θ_i, θ_j can be defined in the usual manner by the formulas

$$\text{magnitude of } \theta_i = \sqrt{g_{ii}}, \qquad \cos \varphi_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}}. \tag{3}$$

The fundamental quantities are used, not to define distance between points in the parameter space, but to define magnitude and relative direction in the vector space V_n.

3. **The derivatives $\theta_{i,\lambda}$: normals to V_n.** It is evident that the derivatives $\partial\theta_i/\partial u^\lambda$ will be linearly expressible in terms of the θ_i only in very special cases. We are able, however, by means of the process of

scalar multiplication (defined by integration as in § 2) to define the components of these quantities which lie in V_n and also to define the normal components. By a normal to V_n we mean any function $\psi(x; u^1, u^2, \cdots, u^m)$ such that

$$\int_a^1 \theta^i \psi \, dx = 0.$$

We write

$$\theta_{i,\lambda} = \frac{\partial \theta_i}{\partial u^\lambda} = \Gamma_{i\lambda}{}^k \theta_k + N_{i\lambda}, \tag{4}$$

where the coefficients $\Gamma_{i\lambda}{}^k$ are functions of the coördinates u and the $N_{i\lambda}$ are functions of x and the parameters u^λ which represent normal vectors to the space V_n.

If we form the products (i. e. the integrated products) of both sides of (4) by θ_r we obtain

$$\theta_r \frac{\partial \theta_i}{\partial u^\lambda} = \Gamma_{i\lambda}{}^k g_{kr}. \tag{5}$$

We denote the left hand side $\Gamma_{ir\lambda}$.* The solutions of these equations for the quantities $\Gamma_{i\lambda}{}^k$ are given by

$$\Gamma_{i\lambda}{}^k = g^{kr} \Gamma_{ir\lambda}, \tag{6}$$

where gg^{kr} is the cofactor of g_{kr} in the determinant $|g_{kr}|$.

If the number of variables u^α is equal to n and if the θ_i are the derivatives of some function θ with respect to the u^i, the quantities $\Gamma_{ir\lambda}$ are expressible in terms of the derivatives of the quantities g_{ij}, but in general this is not possible. One set of relations, however, is easily obtained; from $\theta_i \theta_j = g_{ij}$ follows by differentiation,

$$\theta_i \frac{\partial \theta_j}{\partial u^\lambda} + \theta_j \frac{\partial \theta_i}{\partial u^\lambda} = \frac{\partial g_{ij}}{\partial u^\lambda}, \qquad \Gamma_{ji\lambda} + \Gamma_{ij\lambda} = \frac{\partial g_{ij}}{\partial u^\lambda}. \tag{7}$$

When the quantities $\Gamma_{ij\lambda}$ are known, the coefficients of the θ_k on the right in equation (4) are known and these determine the projections of the vectors $\theta_{i,\lambda}$ in the space V_n. These quantities do not determine the normals $N_{i\lambda}$.

The normals depending on the fundamental functions θ_i and on their first derivatives will be called the first normals to the space V_n. There are, in general, mn independent first normals. In case the functions θ_i are the derivatives with respect to the parameters of some function θ there

* These quantities were called rotations of the vector system θ_i in the paper "Displacements in a Function Space". See abstract, loc. cit.

are in general $n(n+1)/2$ independent first normals, but this number will be further reduced if there exist linear relations between the derivatives of the functions θ_i.

The quantities g_{ij} have no necessary connection with the metric of the parameter manifold. If $m = n$ this metric may be introduced by means of the differential form

$$ds^2 = g_{ij}\, du^i\, du^j. \tag{8}$$

The quantities $\Gamma_{r\lambda}^{\ i}$ may then be used as the basis of the definition of covariant differentiation so that if q^i are the components of any contravariant vector

$$q_\lambda^i = \frac{\partial q^i}{\partial u^\lambda} + \Gamma_{r\lambda}^{\ i}\, q^r. \tag{9}$$

4. Transformation of the fundamental functions. There are certain features of the geometry of a set of functions θ_i which will not be changed by a change to the new basis φ_i given by the formulas

$$\varphi_i = a_i^j\, \theta_j, \tag{10}$$

where the coefficients a are functions of the coördinates u^λ. For example, any first normal to the original space will also be a first normal to the space determined by the new fundamental functions.

We may look upon equations (10) as defining a displacement (not necessarily rigid) of the vector system θ_i in the space V_n. The connections between the formulas corresponding to the two sets of fundamental vectors will now be obtained.

Quantities corresponding to the quantities g_{ij}, $\Gamma_{ij\lambda}$, etc. are defined as follows:

$$\varphi_i \varphi_j = h_{ij}, \quad |h_{ij}| = h, \quad \varphi_j \frac{\partial \varphi_i}{\partial u^\lambda} = \overline{\Gamma}_{ij\lambda}, \quad h^{jm}\, \overline{\Gamma}_{ij\lambda} = \overline{\Gamma}_{i\lambda}^{\ m}, \tag{11}$$

where hh^{jm} is the cofactor of h_{jm} in the determinant $|h_{jm}|$. In addition we shall make use of the quantities p_{ik} defined by

$$\varphi_i \theta_k = p_{ik}. \tag{12}$$

When the quantities p_{ik} are known the transformation (10) is known and conversely.

The following formulas are readily obtained,

$$\begin{aligned} \varphi_i \theta_k &= p_{ik} = a_i^j g_{jk} = A_k^j h_{ij}, \\ a_k^j &= p_{kr} g^{rj}, \qquad A_k^j = p_{rk} h^{rj}, \end{aligned} \tag{13}$$

where $|a_s^r|A_k^j$ is the cofactor of a_j^k in the determinant $|a_s^r|$. When the transformation a_i^j is the identical transformation we evidently have

$$a_i^j = \delta_i^j, \qquad p_{ik} = g_{ik} = h_{ik}. \tag{14}$$

For the new quantities $\bar{\Gamma}_{ij\lambda}$ we have by definition

$$\begin{aligned}\bar{\Gamma}_{ij\lambda} &= \varphi_j \frac{\partial \varphi_i}{\partial u^\lambda} = a_j^r \theta_r \frac{\partial (a_i^s \theta_s)}{\partial u^\lambda} \\ &= a_j^r \theta_r \theta_s \frac{\partial a_i^s}{\partial u^\lambda} + a_j^r a_i^s \theta_r \frac{\partial \theta_s}{\partial u^\lambda} \\ &= a_j^r g_{rs} \frac{\partial a_i^s}{\partial u^\lambda} + a_j^r a_i^s \Gamma_{sr\lambda}.\end{aligned} \tag{15}$$

Similarly for the new quantities $\bar{\Gamma}_{i\lambda}{}^m$ we have by formula (6)

$$\begin{aligned}\bar{\Gamma}_{i\lambda}{}^m &= h^{jm} \bar{\Gamma}_{ij\lambda} \\ &= h^{jm}\left[a_j^r g_{rs} \frac{\partial a_i^s}{\partial u^\lambda} + a_j^r a_i^s \Gamma_{sr\lambda}\right],\end{aligned}$$

but

$$a_j^r = p_{jt} g^{tr}, \qquad h^{jm} a_j^r = h^{jm} p_{jt} g^{tr} = A_t^m g^{tr}.$$

Consequently

$$\begin{aligned}\bar{\Gamma}_{i\lambda}{}^m &= A_t^m \left[g^{tr} g_{rs} \frac{\partial a_i^r}{du^\lambda} + g^{tr} a_i^s \Gamma_{sr\lambda}\right] \\ &= A_t^m \left[\delta_r^t \frac{\partial a_i^r}{du^\lambda} + a_i^s \Gamma_{s\lambda}{}^t\right] \\ &= A_t^m \left[\frac{\partial a_i^t}{\partial u^\lambda} + a_i^s \Gamma_{s\lambda}{}^t\right],\end{aligned} \tag{16}$$

which is the formula desired.

5. **Relation to affine geometry.** If $m = n$, the quantities Γ_{ik}^j have similar properties to the coefficients of affine connection employed by Weyl and by Eisenhart and Veblen* in their treatment of affine geometry. In a recent paper† the author has used formula (16) to define a transformation from one affine geometry to another. The Γ_{ik}^j were assumed to be symmetric

*H. Weyl, Raum, Zeit und Materie; Eisenhart and Veblen, The Riemann Geometry and its generalisations, Proc. Nat. Acad. Sci., vol. 8 (1922), pp. 19–23.

†Associated types of linear connection, Proc. Nat. Acad. Sci., vol. 11 (1925), pp. 252–256.

in i and λ, but most of the results are valid without this assumption. The result of § 4 suggests one possible interpretation of that transformation and indicates a generalization to cases in which $m \neq n$.

In the paper just referred to it is shown that the resultant of two transformations of the type (16) associated with two matrices $\| a_i^j \|$, $\| b_i^j \|$ is again a transformation of the same type associated with the resultant of these two matrices. The same result evidently holds in the case $m \neq n$ considered in the present paper.

After transposing, multiplying through by a_m^r and summing on the index m, formula (16) becomes

$$\frac{\partial a_i^r}{\partial u^\lambda} = a_m^r \bar{\Gamma}_{i\lambda}^{\ m} - a_i^s \Gamma_{s\lambda}^{\ r}. \tag{17}$$

If we suppose the quantities $\Gamma_{s\lambda}^{\ r}$ and $\bar{\Gamma}_{i\lambda}^{\ m}$ to be known, equations (17) may be regarded as differential equations in the a's. The integrability conditions may be written

$$a_s^t \left[\frac{\partial \bar{\Gamma}_{m\lambda}^{\ s}}{\partial u^\mu} - \frac{\partial \bar{\Gamma}_{m\mu}^{\ s}}{\partial u^\lambda} + \bar{\Gamma}_{m\lambda}^{\ r} \bar{\Gamma}_{r\mu}^{\ s} - \bar{\Gamma}_{m\mu}^{\ r} \bar{\Gamma}_{r\lambda}^{\ s} \right]$$

$$= a_m^r \left[\frac{\partial \Gamma_{r\lambda}^{\ t}}{\partial u^\mu} - \frac{\partial \Gamma_{r\mu}^{\ t}}{\partial u^\lambda} + \Gamma_{r\lambda}^{\ s} \Gamma_{s\mu}^{\ t} - \Gamma_{r\mu}^{\ s} \Gamma_{s\lambda}^{\ t} \right]$$

or also

$$a_s^t \bar{R}_{m\lambda\mu}^{\ \ s} = a_m^r R_{r\lambda\mu}^{\ \ t}. \tag{18}$$

The set of quantities $R_{r\lambda\mu}^{\ \ s}$ play the same role in the present paper as the components of the curvature tensor in the case $m = n$. We shall call them the curvature components of the vector system θ_i. The usual processes used in tensor analysis may also be employed here.

Equations (18) may be solved either for the $\bar{R}_{m\lambda\mu}^{\ \ s}$ or for the $R_{m\lambda\mu}^{\ \ s}$. The solutions are,

$$\bar{R}_{m\lambda\mu}^{\ \ s} = A_t^s a_m^r R_{r\lambda\mu}^{\ \ t}, \qquad R_{m\lambda\mu}^{\ \ t} = A_m^i a_s^t \bar{R}_{i\lambda\mu}^{\ \ s} \tag{19}$$

where the Greek indices take on the values from 1 to m while the Latin indices take on the values from 1 to n.

6. **Invariant relations.** From equations (18) a number of invariant relations may be obtained connecting the curvature components $R_{i\lambda\mu}^{\ \ j}$ and $\bar{R}_{i\lambda\mu}^{\ \ j}$. For given values of μ and λ these may be regarded as a system of n^2 homogeneous equations in the n^2 quantities a_i^j. It follows that for values of the a_i^j which are not all zero the determinant of the coefficients must

vanish.* The equations thus obtained connecting the curvature components have the same form for all possible transformations (10). The relations are therefore invariant with respect to these transformations. There are $m(m-1)/2$ of these invariant relations.

We also obtain a set of invariants from equations (19) by placing $s = m$ and taking the sum for the index m. The equations become

$$\bar{R}_{m\lambda\mu}{}^{m} = R_{m\lambda\mu}{}^{m}$$

showing that the sets of quantities $R_{m\lambda\mu}{}^{m}$ are invariant.

Relations equivalent to equations (18) may be obtained by using the quantities p_{ik} instead of the a_i^j. It will be recalled that either set may be used to define the transformation (10). From the definition of the p_{ik}

$$\frac{\partial p_{ik}}{\partial u^\lambda} = \varphi_i \frac{\partial \theta_k}{\partial u^\lambda} + \theta_k \frac{\partial \varphi_i}{\partial u^\lambda} = a_i^j \theta_j \frac{\partial \theta_k}{\partial u^\lambda} + A_k^j \varphi_j \frac{\partial \varphi_i}{\partial u^\lambda}$$
$$= p_{ir} g^{rj} \Gamma_{kj\lambda} + p_{rk} h^{rj} \bar{\Gamma}_{ij\lambda} = p_{ir} \Gamma_{k\lambda}{}^{r} + p_{rk} \bar{\Gamma}_{i\lambda}{}^{r}.$$

The integrability conditions of this system of equations in p_{ik} are

$$p_{ir} R_{k\lambda\mu}{}^{r} + p_{rk} \bar{R}_{i\mu\lambda}{}^{r} = 0. \tag{20}$$

It is interesting to note that, when the transformation a_i^j is the identical transformation, the quantities $p_{ik} = g_{ik} = h_{ik}$ and equations (20) reduce to

$$g_{ir} R_{k\lambda\mu}{}^{r} + g_{rk} R_{i\lambda\mu}{}^{r} = 0. \tag{21}$$

7. Conditions for a linear space. If the functions θ_i can be expressed in the form

$$\theta_i = A_i^j e_j(x) \qquad (|A_i^j| \neq 0) \tag{22}$$

where the functions $e_j(x)$ are independent of the parameters u and the coefficients A_i^j are independent of x, we shall call the space V_n a linear space.

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* For example, in the case $n = 2$ we have the relation

$$\begin{vmatrix} \bar{R}_{1\lambda\mu}{}^{1} - R_{1\lambda\mu}{}^{1} & \bar{R}_{2\lambda\mu}{}^{1} & \bar{R}_{1\lambda\mu}{}^{2} & 0 \\ -R_{1\lambda\mu}{}^{2} & \bar{R}_{1\lambda\mu}{}^{1} - R_{2\lambda\mu}{}^{2} & 0 & \bar{R}_{1\lambda\mu}{}^{2} \\ \bar{R}_{2\lambda\mu}{}^{1} & 0 & \bar{R}_{2\lambda\mu}{}^{2} - R_{1\lambda\mu}{}^{1} & -R_{2\lambda\mu}{}^{1} \\ 0 & \bar{R}_{2\lambda\mu}{}^{1} & -R_{1\lambda\mu}{}^{2} & \bar{R}_{2\lambda\mu}{}^{2} - R_{2\lambda\mu}{}^{2} \end{vmatrix} = 0.$$

This relation between the R's and the $\bar{R}$'s is independent of the transformation.

Necessary conditions that V_n be a linear space are easily obtained from equations (19); for in a linear space we take the functions e_j as the new fundamental functions φ_j of § 4. The quantities $\bar{\Gamma}_{i\lambda}{}^{j}$ are then all equal to zero. It follows that the curvature components $\bar{R}_{k\lambda\mu}{}^{r}$ all vanish and therefore by equations (19) the curvature components $R_{k\lambda\mu}{}^{r}$ of the original system θ_i must also vanish. Hence,

In order that the space V_n be linear it is necessary that the curvature components $R_{k\lambda\mu}{}^{r}$ should all vanish.

8. **Normal extensions of V_n.** If the functions $\theta_{i,\lambda}$ are not expressible linearly in terms of the θ_i, at least all can be expressed in terms of the θ_i and a certain set θ_{i_1,λ_1}, θ_{i_2,λ_2}, $\cdots$, θ_{i_k,λ_k}. We adjoin these additional functions to the set θ_i and denote them by θ_{n+1}, θ_{n+2}, $\cdots$, θ_{n+k}. We also suppose the set θ_1, $\cdots$, θ_{n+k} to be linearly independent.

The enlarged space V_{n+k} based on the linearly independent set of functions θ_1, $\cdots$, θ_{n+k} we shall call the *first normal extension of V_n*.

The first normal extension of V_{n+k} will be called, *the second normal extension of V_n*, and so on.

It is clear that if a vector space V_n be linear, it coincides with its first normal extension. We also clearly have the following corollary to the theorem of § 7.

In order that V_{n+k}, the first normal extension of V_n, be a linear space it is necessary that the curvature components $R_{i\lambda\mu}{}^{j}$, determined by the set of functions θ_1, $\cdots$, θ_{n+k}, should all vanish.

9. **A special case.** Before proceeding further we consider the special case in which $m = 2$, $n = 2$; we denote the parameters by u and v. It will be convenient (and it is always possible) to select the original system of functions so that $\theta_1\theta_1 = 1$, $\theta_2\theta_2 = 1$, $\theta_1\theta_2 = 0$. We denote $\theta_2\dfrac{\partial\theta_1}{\partial u} = \Gamma_{121}$ by p_1 and $\theta_2\dfrac{\partial\theta_1}{\partial v} = \Gamma_{122}$ by p_2.

It is clear from the way in which the θ_i are chosen that

$$\Gamma_{111} = \Gamma_{112} = \Gamma_{221} = \Gamma_{222} = 0. \qquad \Gamma_{121} = -\Gamma_{211} = p_1,$$
$$\Gamma_{122} = -\Gamma_{222} = p_2.$$

Without loss of generality* we may suppose that the φ_i of equations (10) satisfy the equations $\varphi_1\varphi_1 = \varphi_2\varphi_2 = 1$. The product $\varphi_1\varphi_2$ will be denoted by l; l is the cosine of the angle between φ_1 and φ_2.

* If transformations of this type are followed by transformations which alter the lengths of the fundamental vectors the formulas obtained below will contain the ratio of these lengths but this ratio eliminates along with the a's so that the invariant relation $R^2 = P^2$ holds for the general class of transformations.

The following notations will be used for the $\bar{\Gamma}_{ij\lambda}$;

$$\bar{\Gamma}_{121} = q_1, \quad \bar{\Gamma}_{122} = q_2, \qquad \bar{\Gamma}_{211} = q_1', \quad \bar{\Gamma}_{212} = q_2'.$$

From $\varphi_1 \varphi_2 = l$ we have

$$q_1' + q_1 = \frac{\partial l}{\partial u}, \qquad q_2' + q_2 = \frac{\partial l}{\partial v},$$

and by means of these equations q_1' and q_2' may be eliminated.

By an easy computation equations (18) reduce to

$$Pa_1^2 = \frac{R}{\sqrt{1-l^2}}(a_2^1 - l\,a_1^1), \qquad Pa_1^1 = \frac{R}{\sqrt{1-l^2}}(la_1^2 - a_2^2),$$

$$Pa_2^1 = \frac{R}{\sqrt{1-l^2}}(a_1^2 - l\,a_2^2), \qquad Pa_2^2 = \frac{R}{\sqrt{1-l^2}}(la_2^1 - a_1^1),$$

where

$$P = \frac{\partial p_1}{\partial v} - \frac{\partial p_2}{\partial u} \text{ and } R = \frac{\partial}{\partial v}\frac{q_1}{\sqrt{1-l^2}} - \frac{\partial}{\partial u}\frac{q_2}{\sqrt{1-l^2}}.$$

Elimination of the a's leads to the invariant relation

$$R^2 = P^2.$$

Since P contains only the quantities p_1 and p_2, belonging to the system θ_i, and R contains only the quantities q_1, q_2 and l, belonging to the system φ_i, it follows that R is an invariant.

If the space V_2 is linear, $R = 0$ but the following example shows that the converse is not true in general.

Let $\psi(x;v)$ be any function satisfying the condition

$$\frac{\partial \psi}{\partial v}\frac{\partial \psi}{\partial v} = 1$$

and take

$$\theta_1 = \frac{\partial \psi}{\partial v}, \qquad \theta_2 = \frac{r}{\sqrt{u^2+r^2}}\left[\frac{\partial \psi}{\partial v} + u\frac{\partial^2 \psi}{\partial v^2}\right],$$

where

$$\frac{1}{r^2} = \frac{\partial^2 \psi}{\partial v^2}\frac{\partial^2 \psi}{\partial v^2}.$$

It will be found that the invariant R vanishes but the function ψ can be chosen so that V_2 is not linear.

We mention in passing that if θ_1 and θ_2 are tangent vectors to the parameter curves of a two dimensional surface, the invariant R equals the Gaussian curvature multiplied by $\sqrt{EG(1-l^2)}$.

10. **Spaces with a single first normal.** It is clear that we may always select a normed orthogonal set of functions as the basis functions θ_i. This choice will not affect the vectors which belong to the space V_n or any of its normal extensions.

It will be convenient to use such a system to study the first normal extension of a given space. We shall first consider a special case; namely, the case in which V_{n+k} contains only one additional function.

We assume, then, that the derivatives of the θ_i are all expressible in terms of the θ_i and one additional function θ_{n+1} orthogonal to each of the θ_i. We may also assume that θ_{n+1} is normed; i. e., $\theta_{n+1}\,\theta_{n+1} = 1$.

Since we have assumed the θ_i to be normed and mutually orthogonal, we have

$$g_{ij} = g^{ij} = \begin{cases} 1, & (i = j), \\ 0, & (i \neq j), \end{cases}$$

where now i and j run from 1 to $n+1$.

Equations (4) may now be written

$$\frac{\partial\,\theta_i}{\partial\,u^\lambda} = L_{i\lambda}\,\theta_{n+1} + \Gamma_{i\lambda}{}^k\,\theta_k. \tag{24}$$

Equations (6) reduce to $\Gamma_{i\lambda}{}^k = \Gamma_{ik\lambda}$, and by equations (7)

$$\Gamma_{ik\lambda} = -\Gamma_{ki\lambda}, \qquad \Gamma_{i\lambda}{}^k = -\Gamma_{k\lambda}{}^i.$$

It is easily seen that the $\Gamma_{ir\lambda}$ and the $\Gamma_{i\lambda}{}^r$ have the same values for the new set as for the original set when i and r are different from $n+1$, but for the new set we have the additional quantities

$$\Gamma_{i\,n+1\,\lambda},\quad \Gamma_{n+1\,i\lambda},\quad \Gamma_{n+1\,n+1\,\lambda},\quad \Gamma_{i\lambda}{}^{n+1},\quad \Gamma_{n+1\,\lambda}{}^i,\quad \Gamma_{n+1\,\lambda}{}^{n+1}.$$

If we multiply both sides of (24) by θ_{n+1} we obtain $\Gamma_{i\,n+1\,\lambda} = L_{i\lambda}$ and therefore $\Gamma_{n+1\,i\lambda} = -L_{i\lambda}$, $\Gamma_{i\lambda}{}^{n+1} = L_{i\lambda}$, $\Gamma_{n+1\,\lambda}{}^i = -L_{i\lambda}$. Finally, since $\theta_{n+1}\,\theta_{n+1} = 1$, we have $\Gamma_{n+1\,n+1\,\lambda} = \Gamma_{n+1\,\lambda}{}^{n+1} = 0$.

With these results we are able to compare the curvature components of the system $\theta_1, \cdots, \theta_n$ with those of the system $\theta_1, \cdots, \theta_n, \theta_{n+1}$. We write the new curvature components in parentheses, $(R_{i\lambda\mu}{}^s)$.

From the definition of these quantities we have

$$(25)\qquad (R_{i\lambda\mu}{}^{s}) = R_{i\lambda\mu}{}^{s} + L_{i\mu}L_{s\lambda} - L_{i\lambda}L_{s\mu}, \qquad (i \neq n+1, s \neq n+1),$$

$$(26)\qquad (R_{n+1\lambda\mu}{}^{s}) = -\frac{\partial L_{s\lambda}}{\partial u^{\mu}} + \frac{\partial L_{s\mu}}{\partial u^{\lambda}} + L_{r\lambda}\Gamma_{s\mu}{}^{r} - L_{r\mu}\Gamma_{s\lambda}{}^{r},$$

$$(27)\qquad (R_{m\lambda\mu}{}^{n+1}) = \frac{\partial L_{m\lambda}}{\partial u^{\mu}} - \frac{\partial L_{m\mu}}{\partial u^{\lambda}} + L_{r\mu}\Gamma_{m\lambda}{}^{r} - L_{r\lambda}\Gamma_{m\mu}{}^{r}.$$

If V_{n+1} is a linear space, these must all vanish. The equations resulting in this way from (26) and (27) are the same and may be regarded as a form of the Codazzi relations, while (25) gives a form of the Gauss relation.*

11. **Curvature components of subspaces.**† The formulas of the previous article may be generalized in the following manner. We may divide the set of functions $\theta_1, \cdots, \theta_n$ into two sets which may be denoted by $\theta_1, \cdots, \theta_p$ and $\theta_{\bar{1}}, \cdots, \theta_{\bar{q}}$. The fundamental functions of the space V_n are thus obtained by combining the fundamental functions of two spaces V_p and V_q where $p+q=n$.

The curvature components of the entire space V_n will be indicated by R's in parentheses. These components are divided into the following four classes

$$(R_{i\lambda\mu}{}^{s}),\ (R_{\bar{i}\lambda\mu}{}^{s}),\ (R_{i\lambda\mu}{}^{\bar{s}}),\ (R_{\bar{i}\lambda\mu}{}^{\bar{s}}),$$

where the unmarked indices refer to the p dimensional space V_p determined by the functions $\theta_1, \cdots, \theta_p$ and the indices with the stroke over them refer to the q dimensional space V_q determined by the functions $\theta_{\bar{1}}, \cdots, \theta_{\bar{q}}$.

We use the notation $\Gamma_{i\lambda}{}^{s}$ and $R_{i\lambda\mu}{}^{s}$ to denote quantities belonging to the space V_p while $\Gamma_{\bar{i}\lambda}{}^{\bar{s}}$ and $R_{\bar{i}\lambda\mu}{}^{\bar{s}}$ denote quantities belonging to V_q. It is evident that the quantities $\Gamma_{i\lambda}{}^{s}$ and $\Gamma_{\bar{i}\lambda}{}^{\bar{s}}$ belong also to the space V_n. The remaining Γ's belonging to V_n will be denoted by $\Gamma_{\bar{i}\lambda}{}^{s}$ and $\Gamma_{i\lambda}{}^{\bar{s}}$.

We have immediately from the definition of the (R)'s the following formulas

$$(25')\qquad (R_{i\lambda\mu}{}^{s}) = R_{i\lambda\mu}{}^{s} + \Gamma_{i\lambda}{}^{\bar{t}}\Gamma_{\bar{t}\mu}{}^{s} - \Gamma_{i\mu}{}^{\bar{t}}\Gamma_{\bar{t}\lambda}{}^{s},$$

* See the paper Functional Differential Geometry, loc. cit., p. 341, equations (22) and (23). It is to be observed that the present discussion is more general than the former since it is not assumed that $\partial\theta_i/\partial u^j = \partial\theta_j/\partial u^i$. The assumption that V_{n+1} is linear was not made in the former paper and is not really necessary here since by equating the two values of $\partial^2\theta_i/\partial u^j\partial u^k$ the argument could proceed as in the paper just cited.

† Added June 5, 1926.

$$(26') \quad (R_{\bar{\imath}\lambda\mu}{}^{\bar{s}}) = \frac{\partial \Gamma_{\bar{\imath}\lambda}{}^{\bar{s}}}{\partial u^{\mu}} - \frac{\partial \Gamma_{\bar{\imath}\mu}{}^{\bar{s}}}{\partial u^{\lambda}} + \Gamma_{\bar{\imath}\lambda}{}^{t}\,\Gamma_{t\mu}{}^{\bar{s}} - \Gamma_{\bar{\imath}\mu}{}^{t}\,\Gamma_{t\lambda}{}^{\bar{s}} + \Gamma_{\bar{\imath}\lambda}{}^{\bar{t}}\,\Gamma_{\bar{t}\mu}{}^{\bar{s}} - \Gamma_{\bar{\imath}\mu}{}^{\bar{t}}\,\Gamma_{\bar{t}\lambda}{}^{\bar{s}},$$

$$(27') \quad (R_{i\lambda\mu}{}^{\bar{s}}) = \frac{\partial \Gamma_{i\lambda}{}^{\bar{s}}}{\partial u^{\mu}} - \frac{\partial \Gamma_{i\mu}{}^{\bar{s}}}{\partial u^{\lambda}} + \Gamma_{i\lambda}{}^{t}\,\Gamma_{t\mu}{}^{\bar{s}} - \Gamma_{i\mu}{}^{t}\,\Gamma_{t\lambda}{}^{\bar{s}} + \Gamma_{i\lambda}{}^{\bar{t}}\,\Gamma_{\bar{t}\mu}{}^{\bar{s}} - \Gamma_{i\mu}{}^{\bar{t}}\,\Gamma_{\bar{t}\lambda}{}^{\bar{s}},$$

$$(25'') \quad (R_{\bar{\imath}\lambda\mu}{}^{\bar{s}}) = \bar{R}_{\bar{\imath}\lambda\mu}{}^{\bar{s}} + \Gamma_{\bar{\imath}\lambda}{}^{t}\,\Gamma_{t\mu}{}^{\bar{s}} - \Gamma_{\bar{\imath}\mu}{}^{t}\,\Gamma_{t\lambda}{}^{\bar{s}}.$$

If the space V_p is regarded as the original space and V_n is the normal extension these formulas are generalizations of the equations of Gauss and Codazzi. The spaces V_r and V_q may, however, be regarded as any two spaces whose fundamental functions are independent.

In the case of Riemannian spaces immersed in higher euclidean space the second and higher fundamental quantities may usually be looked upon as belonging to an extended system of Γ's for the containing space.

12. **Vector differential equations.** If the space V_n is linear, equations (4) may be written

$$(28) \qquad \frac{\partial \theta_i}{\partial u^{\lambda}} = \Gamma_{i\lambda}{}^{k}\,\theta_k;$$

for the derivatives $\partial \theta_i / \partial u^{\lambda}$ by equations (22) may be computed in terms of the θ_i.

By substituting the values of θ_i from (22) in (28) it is found that the coefficients A_i^j must satisfy the equations

$$(29) \qquad \frac{\partial A_i^j}{\partial u^{\lambda}} = \Gamma_{i\lambda}{}^{k}\,A_k^j;$$

that is, the system of coefficients A_i^j for each value of the index j satisfies the same differential equations as the set of vectors θ_i.

On the other hand, if the differential equations (28) for the vectors θ_i are given we may at once write down equations (29). The necessary and sufficient conditions that these equations have a system of solutions A_i^j, for a fixed value of j, reducing to preassigned values $(A_i^j)^0$ for given values $(u^{\lambda})^0$ of the parameters is that the curvature components $R_{m\lambda\mu}{}^{s}$ all vanish.

It is possible to find n different systems of solutions A_i^j, one system of quantities for each value of j, and these may be selected in such a way that the determinant $|A_i^j|$ does not vanish; for example they may be taken to be the systems of solutions that reduce to

$$\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 \end{matrix}$$

when the parameters take on the values $(u^\lambda)^0$.

The system A_i^j just found may be used with any linearly independent set of functions $e_j(x)$ to set up the functions θ_i of equations (22). The functions so constructed will satisfy equations (28). It follows, therefore, that

The necessary and sufficient condition that there exists a linear space V_n with fundamental functions θ_i satisfying the differential equations (28) *is that the curvature components $R_{m\lambda\mu}{}^s$ all vanish.*

The quantities g_{ij} belonging to a system of solutions of (28) will depend on the particular system of solutions chosen, but for any given set θ_i they will satisfy equations (6) and (7) of § 3. Equations (7) may be reduced by means of (28) to the form

$$\Gamma_{i\lambda}{}^k g_{kj} + \Gamma_{j\lambda}{}^k g_{ki} = \frac{\partial g_{ij}}{\partial u^\lambda}.$$

The complete integrability of this system of equations for the quantities g_{ij} follows from the integrability of equations (28).

It is clear that the result of this section depends in no way upon the source of the coefficients Γ in equations (28) and is therefore applicable to any space when the coefficients of affine connection are given.

The theorem serves to unify the various special branches of differential geometry, and in particular, makes evident the intimate relation between the integrability conditions in projective differential geometry and the usual conditions of Gauss and Codazzi.

13. **Differential geometry.** As has been stated, the usual differential geometry of n dimensions is a special case of the more general theory which precedes. The principal special assumption for this case is that there exists in the vector space V_n a set of vectors f_i which are the partial derivatives, with respect to the parameters, of some function or vector f. This, of course, does not preclude the use of other sets φ_i given by the transformation (10) but the set f_i, when it exists, is obviously the most convenient for many purposes.

The spaces may be studied by means of differential equations but for the function f the equations analogous to (28) become second order differential equations.

The differential equations of projective differential geometry will serve as an illustration. These may be written*

$$(30)\qquad \begin{aligned} \frac{\partial^2 f}{\partial u^2} &= a\frac{\partial^2 f}{\partial u\,\partial v} + b\frac{\partial f}{\partial u} + c\frac{\partial f}{\partial v}, \\ \frac{\partial^2 f}{\partial v^2} &= a'\frac{\partial^2 f}{\partial u\,\partial v} + b'\frac{\partial f}{\partial u} + c'\frac{\partial f}{\partial v}. \end{aligned}$$

If $a\,a'-1 \neq 0$, the higher derivatives of f are all expressible in terms of $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, and $\frac{\partial^2 f}{\partial u\,\partial v}$ and these may be taken as the θ_1, θ_2, θ_3 of § 12. The complete system of equations (28) is the following:

$$(31)\qquad \begin{aligned} \frac{\partial\theta_1}{\partial u} &= a\theta_3 + b\theta_1 + c\theta_2, & \frac{\partial\theta_1}{\partial v} &= \theta_3, \\ \frac{\partial\theta_2}{du} &= \theta_3, & \frac{\partial\theta_2}{\partial v} &= a'\theta_3 + b'\theta_1 + c'\theta_2, \\ \frac{\partial\theta_3}{du} &= A\theta_3 + B\theta_1 + C\theta_2, & \frac{\partial\theta_3}{dv} &= A'\theta_3 + B'\theta_1 + C'\theta_2, \end{aligned}$$

where A, B, C, A', B', C', can be computed in terms of a, b, c, a', b', c', The usual conditions of integrability are simply the conditions that the first normal extension, V_3 be linear. Six of the nine conditions merely give the values of A, B, C, etc. in terms of a, b, c, etc. The complete list is given below.

$$(32)\qquad \begin{aligned} R_{1\,12}{}^1 &= \frac{\partial b}{\partial v} + cb' + aB' - B = 0, \\ R_{1\,12}{}^2 &= \frac{\partial c}{dv} + cc' + aC' - C = 0, \\ R_{1\,12}{}^3 &= \frac{\partial a}{\partial v} + ca' + b + aA' - A = 0, \\ R_{2\,12}{}^1 &= -\frac{\partial b'}{\partial u} - bb' + B' - a'B = 0, \\ R_{2\,12}{}^2 &= -\frac{\partial c'}{\partial u} - b'c + C' - a'C = 0, \\ R_{2\,12}{}^3 &= -\frac{\partial a'}{\partial u} - b'a - c' + A' - a'A = 0, \end{aligned}$$

* Wilczynski, Projective differential geometry of curved surfaces, Trans. Amer. Math. Soc., vol. 8 (1907), pp. 233–260. These are Wilczynski's equations (3) with $d = d' = 0$.

$$R_{312}{}^{1} = \frac{\partial B}{\partial v} - \frac{\partial B'}{du} + Cb' - B'b + AB' - A'B = 0,$$

$$R_{312}{}^{2} = \frac{\partial C}{\partial v} - \frac{\partial C'}{\partial u} + Cc' - B'c + AC' - A'C = 0,$$

$$R_{312}{}^{3} = \frac{\partial A}{\partial v} - \frac{\partial A'}{\partial u} + Ca' - B'a + B - C' = 0.$$

The last three conditions correspond to Wilczynski's integrability conditions but differ from them somewhat because we have used $\frac{\partial^2 f}{\partial u\, \partial v}$ instead of f for the third independent vector θ_3.

14. **Special solutions.** In studying a definite solution of the equations of § 13 in a metric space it is advantageous to employ the unit normal N, together with $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, for the fundamental system of vectors. With this system we use the three index symbols $\Gamma_{ik}{}^{r}$* and the second fundamental quantities L_{ij}. We may write

$$\frac{\partial^2 f}{\partial u^2} = L_{11}N + \Gamma_{11}{}^{r} f_r, \quad \frac{\partial^2 f}{\partial u\, \partial v} = L_{12}N + \Gamma_{12}{}^{r} f_r, \tag{33}$$

$$\frac{\partial^2 f}{\partial v^2} = L_{22}N + \Gamma_{22}{}^{r} f_r.$$

Comparing these with equations (30) we find

$$\begin{aligned} [\Gamma_{11}{}^{r} - a\Gamma_{12}{}^{r}] f_r + [L_{11} - aL_{12}]N &= bf_1 + cf_2, \\ [\Gamma_{22}{}^{r} - a'\Gamma_{12}{}^{r}] f_r + [L_{22} - a'L_{22}]N &= b'f_1 + c'f_2. \end{aligned} \tag{34}$$

These equations enable us to express the coefficients in equations (30) in terms of the fundamental quantities of the surface represented by any particular solution.

$$\begin{aligned} a &= \frac{L_{11}}{L_{12}}, \quad b = \Gamma_{11}{}^{1} - \frac{L_{11}}{L_{12}}\Gamma_{12}{}^{1}, \quad c = \Gamma_{11}{}^{2} - \frac{L_{11}}{L_{12}}\Gamma_{12}{}^{2}, \\ a' &= \frac{L_{22}}{L_{12}}, \quad b' = \Gamma_{22}{}^{1} - \frac{L_{22}}{L_{12}}\Gamma_{12}{}^{1}, \quad c' = \Gamma_{22}{}^{2} - \frac{L_{22}}{L_{12}}\Gamma_{12}{}^{2}. \end{aligned} \tag{35}$$

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* These are the Γ's for the system of two vectors $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$. The Γ's for the system f_1, f_2, N can be expressed in terms of these Γ's and the L_{ij}.

We conclude, therefore, that it is posssible to express the projective invariants of the surfaces determined by equations (30) in terms of the usual first and second fundamental quantities of those surfaces.

By using the equations for a and a' the expression for the Gaussian curvature $\frac{L_{11}L_{22}-L_{12}^2}{EG-F^2}$ reduces to $\frac{aa'-1}{EG-F^2}L_{12}^2$ from which it follows that, when $aa' = 1$, the surfaces determined by (30) are developables. It should not be inferred, however, that they are necessarily contained in a linear space of three dimensions.*

The method just used may be extended to other special problems in an obvious way; thus, to deal with a two dimensional surface in a linear space of four dimensions, we may distinguish the cases in which there are

1) two independent first normals;

2) one first normal and one second normal.

Only developable surfaces occur in the second of these classes.

In the case of two independent first normals only one differential equation of the second order is needed but this must be supplemented by a sufficient number of third order equations to make possible the expression of all third derivatives in terms of those of lower order.

Under suitable conditions on the coefficients the complete system analogous to equations (31) may be written down, using two of the second derivatives of f as θ_3 and θ_4. The condition that θ_1, θ_2, θ_3 and θ_4, form a linear space are the integrability conditions.

* In the paper referred to Wilczynski was dealing only with the three dimensional case.

IRREGULAR FIELDS.

By E. T. Bell.

I. Introduction.

1. In numerous investigations concerning sequences of functions, particularly of certain polynomials and their generalizations, a modification of an infinite abstract field F, which we shall call an *irregular field* IF, appears as the underlying algebraic structure. An IF differs from an F only in the exclusion of division in IF by infinite number of distinct elements, called *irregular*. Both F (with $m = 1$) and IF (with $m = \infty$) are varieties of the system Φ (or Φ_m) defined in § 2. The postulates for Φ, as will be seen presently, are a modification of the "second set" of Dickson[1],* p. 200, for a field. For concreteness in § 2 the relation $\mathcal{A}$ may be visualized as *equality*, and S, P as *sum*, *product*. We emphasize however that $\mathcal{A}$, S, P in the sequel are to receive *interpretations* differing from those current in instances of F. The consistency of the postulates is proved in the course of the paper by exhibiting systems which satisfy them. We recall that Vandiver[2] has discussed finite algebras in which divison by $n > 1$ elements is impermissible. His investigations are in an entirely different direction from those of this paper.

The point of view of the present paper is purely abstract. It is an attempt to place upon a rigorous basis the use in algebra and analysis of the powerful umbral calculus originated by Blissard[3] and employed by many later writers. It also aims to offer several concrete instances of the solution of a set of postulates and the interconnections of the resulting theories, all abstractly identical but differing in their interpretations, each one of which is an *instance* of the *general theory* consisting of the set of postulates and its implications based on them alone. It should be mentioned however that the various algebras developed here have immediate and fruitful application to a remarkable new class of polynomials in two complex variables which include as very special cases the Bernoullian and Eulerian polynomials and numbers and the anharmonic polynomial generalizations of these discussed in a former paper[4]. There is a short account of the new polynomials in another paper[5], also, with indications of their characteristic properties, in Section VII of the present. I hope to return in more detail to this application later, as the theory of the Bernoullian and allied polynomials falls out of the other on replacing either of the

*References are at the end of the paper.

two complex variables by 0 or 1. For the present an abstract treatment seemed of sufficient interest to merit isolated exposition.

2. A Φ is a system consisting of a relation Λ, a set Σ of elements $\alpha, \beta, \cdots, \gamma, \cdots$, and two operations S, P which may be performed upon any two distinct or identical elements α, β of Σ to produce uniquely determined elements $S(\alpha, \beta)$, $P(\alpha, \beta)$ of Σ such that the postulates $\Pi_0 - \Pi_5$ for Λ, S, P are satisfied. Elements of Σ will be called elements of Φ.

(Π_0) Λ is reflexive, symmetrical and transitive with respect to elements of Φ, so that if α, β, γ are any three elements of Φ,

$$\alpha \Lambda \alpha,$$
$$\alpha \Lambda \beta :\equiv: \beta \Lambda \alpha,$$
$$\alpha \Lambda \alpha \cdot \beta \Lambda \gamma :\supset: \alpha \Lambda \gamma,$$

the sign $:\equiv:$ in the second being that of formal equivalence (as in mathematical logic). Further, if $\alpha \Lambda \beta$, then either of α, β may replace the other in any Λ relation concerning either.

(Π_1) If α and β are any two elements of Φ, then $S(\alpha, \beta)$ and $P(\alpha, \beta)$ are uniquely determined elements of Φ, and

$$S(\beta, \alpha) \Lambda S(\alpha, \beta), \qquad P(\beta, \alpha) \Lambda P(\alpha, \beta).$$

(Π_2) If α, β, γ are any three elements of Φ, then

$$S(S(\alpha, \beta), \gamma) \Lambda S(\alpha, S(\beta, \gamma)),$$
$$P(P(\alpha, \beta), \gamma) \Lambda P(\alpha, P(\beta, \gamma)),$$
$$P(\alpha, S(\beta, \gamma)) \Lambda S(P(\alpha, \beta), P(\alpha, \gamma)).$$

(Π_3) There exist in Φ two distinct elements, denoted by ω and η, such that, if α is any element of Φ, then

$$S(\alpha, \omega) \Lambda \alpha, \qquad P(\alpha, \eta) \Lambda \alpha,$$

(whence $S(\omega, \alpha) \Lambda \alpha$, $P(\eta, \alpha) \Lambda \alpha$, by Π_1).

(Π_4) Whatever be the element α of Φ, there exists in Φ an element ξ such that $S(\alpha, \xi) \Lambda \omega$ (whence $S(\xi, \alpha) \Lambda \omega$, by Π_1).

(Π_5) Whatever be the element α of Φ different from each of the m distinct elements $\omega_0 \equiv \omega, \omega_1, \cdots, \omega_{m-1}$ of Φ, there exists in Φ an element ζ such that $P(\alpha, \zeta) \Lambda \eta$ (whence $P(\zeta, \alpha) \Lambda \eta$, by Π_1).

3. The elements $\omega_j (j = 0, 1, \cdots, m-1)$ in Π_5 are called *irregular*: all other elements of Φ are *regular*; $S(\alpha, \beta)$, $P(\alpha, \beta)$ in Π_1 are called the

sum, product respectively of α and β, and ω, η in Π_3 are the *zero, unity* of Φ; the relation $\mathcal{A}$ in Π_0 is called *equality.* A particular Σ, $\mathcal{A}$, S, P satisfying $\Pi_0 - \Pi_5$ is called an *instance* of Φ, which is *regular* or *irregular* according as $m = 1$ or $m > 1$ in Π_5.

Taking $m = 1$ and comparing with Dickson[1] pp. 200–201, we see that $\Pi_1 - \Pi_5$ then become identical with his postulates for a field when for $\mathcal{A}$ we write $=$, and for ω, η we put his 0, 1.

To indicate that Φ contains precisely m irregular elements we shall write $\Phi_m \equiv \Phi$. Thus F is an instance of Φ_1. When $m = \infty$ we write IF for Φ_m.

Adopting for a field the notation of Dickson[1], we shall designate the elements of F by $a, b, \cdots, c, \cdots$, the zero and unity in F by 0, 1 and the product of a, b by ab (without a dot between $a, b \cdots$; the dot notation is reserved for another type of product).

Elements of F will be called *ordinaries.* The field of complex numbers $a + bi$, $i = \sqrt{-1}$, a, b real numbers, will be designated by CF. It is to be observed in all that follows that F is not its instance CF unless so stated. Theorems proved for the various IF's later constructed imply precisely similar results when for F is put its instance CF.

4. From his postulates Dickson proves five fundamental theorems for a field. If in his proofs we replace the equality sign by $\mathcal{A}$, $\alpha + \beta$, $\alpha\beta$ by $S(\alpha, \beta)$, $P(\alpha, \beta)$ respectively, and write ω, η for his 0, 1, we see at once that $T_1 - T_5$ below hold for Φ_m on taking account in T_4, T_5 of a few slight changes occasioned by $m \geqq 1$ in Π_5.

(T_1) ω, η are unique in Φ_m.

(T_2) If α, β, γ are elements of Φ_m such that $S(\alpha, \beta) \mathcal{A} S(\alpha, \gamma)$, then $\beta \mathcal{A} \gamma$. Hence the element ξ in Π_4 is uniquely determined by α; it is called the *negative* of α and is designated in Φ_1 (but not in Φ_m, $m > 1$) by $-\alpha$.

(T_3) If α and β are any elements of Φ_m, there exists one and (by T_2) only one element ξ of Φ_m for which $S(\alpha, \xi) \mathcal{A} \beta$.

The resulting element ξ will be called the result of *subtracting* α from β; in Φ_1 it is written $\alpha - \beta$.

(T_4) If α, β, γ are elements of Φ_m such that $P(\alpha, \beta) \mathcal{A} P(\alpha, \gamma)$ and α is regular, then $\beta \mathcal{A} \gamma$.

In particular, if $P(\alpha, \beta) \mathcal{A} \eta$ and $P(\alpha, \gamma) \mathcal{A} \eta$, then $\beta \mathcal{A} \gamma$. Hence the element ζ in Π_5 is uniquely determined by α; it is called the *reciprocal* (or *inverse*) of α; in Φ_1 it is designated by $1/\alpha$ or α^{-1}.

(T_5) If α and β are elements of Φ_m and α is regular, there exists one and (by T_4) only one element θ of Φ_m such that $P(\alpha, \theta) \mathcal{A} \beta$.

The resulting element θ is called the *quotient* of β by α, or the result of *dividing* β by α; in Φ_1 it is designated by β/α.

As a precaution for reading succeeding sections we remark that the Greek letters later receive a *specific* meaning consistent with the foregoing *general* significance.

5. The simultaneous discussion of F and IF affords a very simple illustration of the multiple interpretation of a postulate system, and it gives two instances of the *solution* of such a system, provided we can show that at least one IF ($\equiv \Phi_\infty$) exists. Several will be obtained in the course of the paper. The abstract set $\Pi_0 - \Pi_5$ together with its implications is the *general theory* of which the several solutions are *instances*. The propositions of any one instance of the general theory can be placed in simple isomorphism with those of any other so that the propotitions of any isomorphic pair are formally equivalent. Hence it is necessary to elaborate only one solution in order to write down the propositions in any other—analogously to the principle of duality in geometry.

Starting from any F we shall construct an IF and apply to it the preceding remarks. In particular we shall proceed from the IF just mentioned to the construction of further IF's, all of which are *distinct*. Two Φ's are distinct if either contains at least one element or one operation not in the other, the implied difference between operations being in their interpretations.

When there can be no confusion we shall replace $\mathcal{A}$ by $=$.

II. The umbral irregular field *UIF*.

6. We recall that the elements of F are $a, b, \cdots, c, \cdots$; the zero and unity in F are written 0, 1, and the four fundamental operations in F are indicated as in common algebra; elements of F are called ordinaries. In what follows $a, b, \cdots, c, \cdots, \alpha_n, \beta_n, \cdots, \gamma_n, \cdots, \xi_n, \cdots (n = 0, 1, \cdots)$ are ordinaries.

The *umbra* of the sequence $\alpha_n (n = 0, 1, \cdots)$ of ordinaries is denoted by α; it is a symbol for the entire class of elements of F composing the sequence. Umbrae are defined by their laws of combination as stated in § 7. For $n \geqq 0$ an integer, the umbral power α^n is idendical with the element α_n of F; α itself is not an element of F. Small Greek letters without suffixes or exponents will be used to denote umbrae; an umbra with a suffix is an ordinary; an umbra with both an exponent and a suffix is meaningless. Umbrae are defined only for sequences of elements in the given F.

7. The umbra of the sequence $\alpha_n \beta_n (n = 0, 1 \cdots)$ is written $\alpha\beta$. This is the definition of the symbol $\alpha\beta$. It does not follow that $\alpha\beta$ is to be interpreted as a product. In particular, if $\alpha_n = a^n (n = 0, 1, \cdots)$, $\alpha\beta$ is the umbra of $a^n \beta_n (n = 0, 1, \cdots)$, since a^n is an element of F.

From the definition $\alpha\beta$ and $\beta\alpha$ are the same.

If α, β are any umbrae, and $n \geqq 0$ is an integer,

$$(\alpha+\beta)^n \equiv \sum_{r=0}^{n} \binom{n}{r} \alpha_{n-r}\beta_r,$$

which gives the definition of $(\alpha+\beta)^n$, is an element of F, and likewise for

$$(\alpha+\beta+\cdots+\gamma)^n \equiv \sum \frac{n!}{p!\ q!\cdots t!}\alpha_p\beta_q\cdots\gamma_t,$$

where $\alpha, \beta, \cdots, \gamma$ are r in number and $\sum$ refers to all sets of r integers $p, q, \cdots, t$ each $\geqq 0$ whose sum is n. If in the last $\alpha, \beta, \cdots, \gamma$ are replaced by $a\alpha, b\beta, \cdots, c\gamma$ respectively, we have

$$(a\alpha+b\beta+\cdots+c\gamma)^n = \sum \frac{n!}{p!\ q!\cdots t!} a^p b^q\cdots c^t \alpha_p\beta_q\cdots\gamma_t.$$

In particular, for $a, b, \cdots, c$ all different from the zero in F,

$$(a\alpha+b\beta+\cdots+c\gamma)^0 = \alpha_0\beta_0\cdots\gamma_0.$$

For convenience $\alpha+(-1)\beta$ is written $\alpha-\beta$. We shall write the umbra of the sequence

$$(a\alpha+b\beta+\cdots+c\gamma)^n \qquad (n = 0, 1, \cdots)$$

as $a\alpha+b\beta+\cdots+c\gamma$. Hence, by what precedes, we have defined $\pm a\alpha \pm b\beta \pm \cdots \pm c\gamma$, and it follows that $a\alpha+b\beta+\cdots+c\gamma$ is invariant under all permutations of $a\alpha, b\beta, \cdots, c\gamma$, and that in any term, say $a\alpha$, the Greek and Latin letter may be interchanged.

We remark that $\alpha+\beta$ is not to be interpreted as the sum of α and β; it is so far merely the symbol of the umbra assigned to the sequence of ordinaries

$$\sum_{r=0}^{n} \binom{n}{r} \alpha_{n-r}\beta_r \qquad (n = 0, 1, \cdots).$$

As a convenience $a\alpha \pm a\beta$ may be written $a(\alpha \pm \beta)$. The symbols $\alpha+\beta \equiv (\alpha+\beta)$ and $(\alpha+\beta)^1$ must not be confused; the former is the umbra of the sequence whose nth term is $(\alpha+\beta)^n$, the latter denotes the ordinary $\alpha_1\beta_0+\alpha_0\beta_1$.

The umbra of the sequence of ordinaries

$$a\alpha_n+b\beta_n\cdots+c\gamma_n \qquad (n = 0, 1, \cdots)$$

will be written $a\cdot\alpha+b\cdot\beta+\cdots+c\cdot\gamma$. Hence the last is invariant under all permutations of $a\cdot\alpha, b\cdot\beta, \cdots, c\cdot\gamma$, and in any term the Greek and

Latin letters may be interchanged. For $n \geqq 0$ an integer we shall write, by definition,

$$(a\cdot\alpha+b\cdot\beta+\cdots+c\cdot\gamma)^n \equiv a\alpha_n+b\beta_n+\cdots+c\gamma_n,$$

so that $(a\cdot\alpha+b\cdot\beta+\cdots+c\cdot\gamma)^n$ $(n=0,1,\cdots)$ is the sequence whose umbra is $a\cdot\alpha+b\cdot\beta+\cdots+c\cdot\gamma$, or, what is the same, this is the umbra of $a\alpha_n+b\beta_n+\cdots+c\gamma_n$ $(n=0,1,\cdots)$. In these definitions the plus sign and the dot are to be regarded as constituting a single symbol of operation. The meaning of $a\cdot\alpha-b\cdot\beta$ is $a\cdot\alpha+(-b)\cdot\beta$, and $a\cdot\alpha$ is the umbra of $a\alpha_n$ $(n=0,1,\cdots)$. Hence $a\cdot(\alpha+\beta)$ is the umbra of $a(\alpha+\beta)^n$ $(n=0,1,\cdots)$.

There can be no confusion in reading combinations of umbrae with and without the dot if it be noted in any case what sequence the complete umbra represents. Thus, the umbra of $a\alpha_n+b\beta_n$ $(n=0,1,\cdots)$ is $a\cdot\alpha+b\cdot\beta$, which cannot be confused with $a\alpha+b\cdot\beta$, which is the umbra of

$$(a\alpha+b\cdot\beta)^n \equiv \sum_{r=0}^{n}\binom{n}{r}(a\alpha)^r\,(b\cdot\beta)^{n-r} \equiv b\sum\binom{n}{r}a^r\alpha_r\beta_{n-r}\ (n=0,1,\cdots).$$

The umbra of $\alpha_n+\beta_n$ $(n=0,1,\cdots)$ is $1\cdot\alpha+1\cdot\beta$; that of $(\alpha+\beta)^n$ $(n=0,1,\cdots)$ is $\alpha+\beta$, etc.

8. We shall now solve Π_0—Π_5 of § 2 in terms of umbrae $\alpha,\beta,\cdots,\gamma\cdots$ of § 7. That is, we shall assign to A, S, P, such meanings that the set of postulates is satisfied.

For Σ (§ 2) we take the set of all umbrae of sequences of ordinaries (elements of F).

Referring now to Π_0 we define two umbrae α, β to be *equal*, $\alpha\,A\,\beta$, when and only when $\alpha_n=\beta_n$ $(n=0,1,\cdots)$. Clearly this definition satisfies all the requirements on A stated in Π_0. For convenience we shall write henceforth $\alpha\,A\,\beta$ as $\alpha=\beta$.

In particular, as frequently useful instances of equality, we have for $n=0,1,\cdots$,

$$\alpha = \beta+\gamma :\equiv: \alpha_n = (\beta+\gamma)^n \equiv \sum_{r=0}^{n}\binom{n}{r}\beta_{n-r}\gamma_r,$$

$$\alpha = 1\cdot\beta+1\cdot\gamma :\equiv: \alpha_n = \beta_n+\gamma_n,$$

$$\alpha = \delta+1\cdot\beta+1\cdot\gamma :\equiv: \alpha_n = \sum_{r=0}^{n}\binom{n}{r}\delta_{n-r}(\beta_r+\gamma_r),$$

where, as before, $:\equiv:$ indicates formal equivalence.

In Π_1 we now take

$$S(\alpha,\beta)\equiv 1\cdot\alpha+1\cdot\beta, \qquad P(\alpha,\beta)\equiv\alpha+\beta.$$

Evidently these satisfy the commutativity demanded by Π_1.

It must be shown that this choice of P, S satisfies Π_2, and hence we must show that

$$1\cdot(\alpha+\beta)+1\cdot\gamma = 1\cdot\alpha+1\cdot(\beta+\gamma),$$
$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma),$$
$$\alpha+1\cdot(\beta+\gamma) = 1\cdot(\alpha+\beta)+1\cdot(\alpha+\gamma).$$

But, as in § 7, end, these are obvious. For example, the last states that for $n = 0, 1, \cdots$,

$$\sum_{r=0}^{n}\binom{n}{r}\alpha_{n-r}(\beta_r+\gamma_r) = (\alpha+\beta)^n+(\alpha+\gamma)^n.$$

To satisfy Π_3 we take for ω the umbra of the sequence each of whose terms is the zero in F, and for η the umbra of the sequence whose first term is the unity in F and each of whose remaining terms is the zero in F. That these are the required zero, unity is evident from

$$1\cdot\alpha+1\cdot\omega = \alpha, \qquad \alpha+\eta = \alpha,$$

which are obvious as above for Π_2.

To see that ξ exists to satisfy Π_4 we must have $1\cdot\alpha+1\cdot\xi = \omega$, of which the solution is $\omega-1\cdot\alpha$, which is equivalent to either $1\cdot\omega-1\cdot\alpha$ or $-1\cdot\alpha$. As before this $\xi = -1\cdot\alpha$ evidently satisfies Π_4.

To find the element ζ demanded by Π_5, and at the same time to exhibit the irregular elements, we must solve $\alpha+\zeta = \eta$, and hence we must solve

$$\sum_{r=0}^{n}\binom{n}{r}\alpha_{n-r}\zeta_r = \eta_n \qquad (n = 0, 1, \cdots).$$

Provided $\alpha_0 \neq 0$, this set of linear equations for $\zeta_r (r = 0, 1, \cdots)$ may be solved for $\zeta_0, \zeta_1, \zeta_2, \cdots$ successively. The umbra of the resulting unique sequence ζ_n $(n = 0, 1, \cdots)$ will be written $\frac{\eta}{\alpha}$ or η/α. Hence the ζ required by Π_5 is η/α, and an umbra β is irregular if and only if $\beta_0 = 0$ (the zero in F). In particular ω is irregular.

If desired an interesting explicit formula for ζ_n, involving the partitions of n, can easily be found, but as it is only occasionally useful in applications we may dispense with it here.

9. Recapitulating the results of § 8, we have shown that for A, S, P, ω, η as there defined A, S, P and the set of all umbrae of sequences in F constitute in IF, which we shall call the *umbral irregular field UIF*, in which the irregular elements are the umbrae of sequences of elements

in F each having its initial term the zero of F, and in which the zero and unity are ω, η respectively.

From § 3 it follows that T_1—T_5 hold in *UIF*..

The unique result of subtracting α from β in *UIF* given by T_3 will be written $1\cdot\alpha - 1\cdot\beta$.

The unique result of dividing β by the regular element α in *UIF* as given by T_5 will be written β/α or $\frac{\beta}{\alpha}$.

The product in *UIF* of $a\cdot\alpha$ and $b\cdot\beta$ may be written $(a\cdot\alpha)+(b\cdot\beta)$, indicating that each of $a\cdot\alpha$, $b\cdot\beta$ is a single umbra, and distinguishing this product from the sum $a\cdot\alpha+b\cdot\beta$. A separate notation is however unnecessary, since $(a\cdot\alpha)+(b\cdot\beta) = ab\cdot(\alpha+\beta)$.

III. The irregular field *AIF* of generalized Appell polynomials.

10. In this section we obtain a further solution *AIF* of Π_0—Π_5 in § 2, distinct from *UIF*. The elements of *AIF* are some but not all of those of *UIF*; they are selected for their usefulness in applications. Before proceeding to *AIF* we shall require some preliminaries concerning derivatives in F.

11. Let $f \equiv f(u, v, \cdots, w, a, b \cdots, c)$ be either a polynomial or an entire function in the complex variables $u, v, \cdots, w$, and let $a, b, \cdots, c$ be constant complex numbers of which the coefficients in f are algebraic functions, so that f is an element of *CF*. The extension of all that follows to any f is immediate and need not be elaborated. We shall write $f(\xi, \theta, \cdots, \zeta, \alpha, \beta, \cdots, \gamma)$, where the Greek letters as always are elements of *UIF*, for the result of replacing $u, v, \cdots, w, a, b, \cdots, c$, in f by $\xi, \theta, \cdots \zeta, \alpha, \beta, \cdots, \gamma$ respectively.

If ξ may represent any one of several elements of *UIF* it is called a *variable element of UIF*, briefly a *variable*; if α may represent only the single element α of *UIF* it is called a *constant*, precisely as in *CF*. We shall use $\xi, \theta, \cdots, \zeta$ to denote variables, $\alpha, \beta, \cdots, \gamma$ constants. The meaning of *arbitrary constant* in *UIF* is abstractly identical with the like in F.

Since

$$f' \equiv f'(u, v, \cdots, w, a, b, \cdots, c) \equiv \frac{\partial^{r+s+\cdots+t} f}{\partial u^r \partial v^s \cdots \partial w^t}$$

is of the same species (polynomial or entire function) as f, it follows from the definition of $f(\xi, \theta, \cdots \zeta, \alpha, \beta, \cdots, \gamma)$ that $f'(\xi, \theta, \cdots, \zeta, \alpha, \beta, \cdots, \gamma)$ is uniquely known. As the definition of the *umbral* $(r+s+\cdots+t)$*th partial derivative* on the left of the following we shall write

$$\frac{\partial^{r+s+\cdots+t}}{\partial \xi^r \partial \theta^s \cdots \partial \zeta^t} f(\xi, \theta, \cdots, \zeta, \alpha, \beta \cdots, \gamma) \equiv f'(\xi, \theta, \cdots, \zeta, \alpha, \beta, \cdots, \gamma).$$

Note that $f(\xi, \theta, \cdots, \zeta, \alpha, \beta, \cdots, \gamma)$ is linear in the elements of each of the sequences $\xi_n, \cdots, \gamma_n (n = 0, 1, \cdots)$.

12. Evidently a solution of

$$(1)\quad \frac{\partial}{\partial \xi} A_n = n A_{n-1}\ (n = 0, 1, \cdots), \text{ or } \frac{\partial}{\partial \xi} A^n = n A^{n-1} (n = 0, 1, \cdots),$$

is given by

$$(2)\qquad A_n \equiv A_n(\xi, \alpha) \equiv (\xi + \alpha)^n = \sum_{r=0}^{n} \binom{n}{r} \xi_{n-r} \alpha_r \qquad (n = 0, 1, \cdots),$$

where, by § 11, ξ denotes a variable and α a constant (arbitrary) in *UIF*. Note that (1) is a set of differential equations in the usual sense of that term only when F is its instance *CF*. From any set of solutions $A_n(\xi, \alpha), A_n(\xi, \beta), \cdots$ of (1) we shall now obtain an infinity more, and we shall show that the set of umbrae of all solutions is an *IF*. (Suffixes of umbrae are raised, thus ξ^n, until after the performance of all indicated operations; e. g., $\frac{\partial}{\partial \xi} \xi^n = n \xi^{n-1} = n \xi_{n-1}$.)

Since by (2) $A_n(\xi, \alpha,)$ is an element of F for $n = 0, 1, \cdots$, it follows that the umbra $A(\xi, \alpha)$ of $A_n(\xi, \alpha)\ (n = 0, 1, \cdots)$ is an element of *UIF*. Hence the sum $1 \cdot A(\xi, \alpha) + 1 \cdot A(\xi, \beta)$, the product $A(\xi, \alpha) + A(\xi, \beta)$, the difference $1 \cdot A(\xi, \alpha) - 1 \cdot A(\xi, \beta)$ and the quotient, provided the divisor be regular, $A(\xi, \alpha)/A(\xi, \beta)$, of $A(\xi, \alpha)$ and $A(\xi, \beta)$, in *UIF*, are elements of *UIF*. Obviously the above sum and difference are umbrae of further solutions of (1), while the like for the product and quotient is false. We call α the *exponent* of $A(\xi, \alpha)$.

13. We shall next devise an *IF*, the *AIF*, of which each element is the umbra of a solution of (1). The fundamental operations in *AIF* are indicated by notations differing slightly from those in F and *UIF*, so that all three systems may be employed simultaneously. This is merely a convenience to avoid possible confusion in interpreting results, as the meanings of the four fundamental operations in *F*, *UIF*, *AIF* are distinct.

Since α in § 12 (2) is an arbitrary constant element of *UIF*, (1) holds when α is replaced by any one of $1 \cdot \alpha + 1 \cdot \beta$, $1 \cdot \alpha - 1 \cdot \beta$, $a \cdot \alpha + b \cdot \beta + \cdots + c \cdot \gamma$, $\alpha + \beta + \cdots + \gamma$, α/β, provided in the last that β be regular.

We now define the uniquely determined element $A(\xi, \gamma)$ of *UIF* in which γ is any one of the sum $1 \cdot \alpha + 1 \cdot \beta$, the difference $1 \cdot \alpha - 1 \cdot \beta$, the product $\alpha + \beta$, the quotient α/β (provided that β in α/β be regular), to be that one of the *sum* $1A(\xi, \alpha) + 1A(\xi, \beta)$, the *difference* $1A(\xi, \alpha) - 1A(\xi, \beta)$, the *product* $A(\xi, \alpha) \cdot A(\xi, \beta)$, the *quotient* $A(\xi, \alpha)/A(\xi, \beta)$ of the elements $A(\xi, \alpha)$, $A(\xi, \beta)$ in *AIF*, in which the *zero* is $A(\xi, \omega)$ and the *unity* is

$A(\xi, \eta)$. Further, defining the fundamental operations in AIF as indicated on the left of the following, we shall write

$$\begin{aligned} 1\,A(\xi,\alpha)+1\,A(\xi,\beta) &= A(\xi, 1\cdot\alpha+1\cdot\beta),\\ 1\,A(\xi,\alpha)-1\,A(\xi,\beta) &= A(\xi, 1\cdot\alpha-1\cdot\beta),\\ A(\xi,\alpha)\cdot A(\xi,\beta) &= A(\xi,\alpha+\beta),\\ A(\xi,\alpha)/A(\xi,\beta) &= A(\xi,\alpha/\beta), \end{aligned}$$

giving the four fundamental operations in AIF, with the generalization of the first pair,

$$aA(\xi,\alpha)+bA(\xi,\beta)+\cdots+cA(\xi,\gamma) = A(\xi, a\cdot\alpha+b\cdot\beta+\cdots+c\cdot\gamma).$$

The irregular elements of AIF are the $A(\xi, \iota)$, where ι runs through all irregular elements of UIF.

The foregoing definitions assume that with respect to the addition, multiplication, zero and unity stated the elements $A(\xi, \alpha)$, where α is any element of UIF, do in fact form an IF in which the subtraction and division are as stated, and hence that they satisfy $T_1 - T_5$ of § 3. That this is so is obvious from the (1, 1) correspondence which had been established with UIF, or it may, if preferred, be established in detail, although this (as a matter of logic) is superfluous.

We can summarize as follows. If $\delta = \varphi(\alpha, \beta, \cdots, \gamma)$ is any function in UIF, then the element $A(\xi, \delta)$ of AIF whose exponent (§ 12, end) is δ is the same function in AIF of the elements $A(\xi, \alpha)$, $A(\xi, \beta)$, $\cdots$, $A(\xi, \gamma)$ whose respective exponents are $\alpha, \beta, \cdots, \gamma$; that is,

$$\delta = \varphi(\alpha, \beta, \cdots, \gamma) :\supset: A(\xi, \delta) = \varphi(A(\xi, \alpha), \cdots, A(\xi, \beta), \cdots, A(\xi, \gamma)).$$

The operations by which δ is constructed from $\alpha, \beta, \cdots, \gamma$ to produce $\varphi(\alpha, \beta, \cdots, \gamma)$ belong to UIF; those by which $A(\xi, \delta)$ is obtained from $A(\xi, \alpha)$, $A(\xi, \beta)$, $\cdots$, $A(\xi, \gamma)$ belong to AIF. For example,

$$\begin{aligned} \delta = \alpha+\beta :\supset: A(\xi,\delta) &= A(\xi,\alpha)\cdot A(\xi,\beta),\\ \delta = 1\cdot\alpha+1\cdot\beta :\supset: A(\xi,\delta) &= 1\,A(\xi,\alpha)+1\,A(\xi,\beta). \end{aligned}$$

We might therefore indicate both sets of operations by the same notation and then obtain theorems in UIF or in AIF by interpreting the letters denoting elements in the formulas as belonging to UIF or to AIF. This illustrates § 5.

14. Each element of AIF is the umbra of a solution of § 12 (1). Hence from any given solutions of (1) we derive an infinity more by operating

in AIF. Thus the square, cube, $\cdots$ $A(\xi, \alpha+\alpha)$, $A(\xi, \alpha+\alpha+\alpha)$, $\cdots$ of $A(\xi, \alpha)$ give further solutions. In practice it is convenient to write $\alpha+\alpha = \alpha^{(2)}$, $\alpha+\alpha+\alpha = \alpha^{(3)}$, etc.

When F is its instance CF, we may designate the AIF in § 13 by $AICF$. In $AICF$ the equations (1) are a set of differential equations in the complex variable ξ, and α_r $(r = 0, 1, \cdots)$ are complex constants; the $A_n(\xi, \alpha)$ in this case are the ordinary Appell[6] polynomials. For the development of the connections between Appell polynomials AIF is useful in addition to $AICF$, since in any F the α in $A(\xi, \alpha)$ is an umbra.

15. We shall now derive a few simple properties of the elements of AIF. Consider the product $A(\xi, \alpha) \cdot A(\xi, \beta)$ in AIF. Since $A(\xi, \alpha) = \xi+\alpha$ we have

$$A(\xi, \alpha) \cdot A(\xi, \beta) = A(\xi, \alpha+\beta) = \xi+\alpha+\beta = (\xi+\beta)+\alpha,$$

and therefore

$$A(\xi, \alpha) \cdot A(\xi, \beta) = A(A(\xi, \beta), \alpha) = A(A(\xi, \alpha), \beta).$$

Hence the nth element of the sequence in F whose umbra is the product in AIF of $A(\xi, \alpha)$, $A(\xi, \beta)$ may be obtained from the nth element of the sequence in F whose umbra is $A(\xi, \alpha)$ by replacing ξ_r by $A_r(\xi, \alpha)$ $(r = 0, 1, \cdots, n)$, and similarly with α, β interchanged throughout.

In the same way, including as special cases addition and subtraction in AIF, the nth element of the sequence whose umbra is

$$aA(\xi, \alpha)+bA(\xi, \beta)+\cdots+cA(\xi, \gamma) \text{ is } a(\xi+\alpha)^n+b(\xi+\beta)^n+\cdots+c(\xi+\gamma)^n.$$

From division in UIF we obtain, analogously to the known expansions of polynomials or of entire functions (provided the resulting series for the latter be convergent) in CF in terms of Hermite or of other Appell polynomials in CF, the expansion of the element

$$\varphi(\xi) = a_0\xi_0+a_1\xi_1+\cdots+a_n\xi_n+\cdots$$

of F in terms of $A_n(\xi, \alpha)$ $(n = 0, 1, \cdots)$, where α is an arbitrary constant element of UIF. Writing for the moment $\gamma \equiv A(\xi, \alpha) = \xi+\alpha$, we have, in UIF, $\xi = \gamma/\alpha = \gamma+\eta/\alpha$, η/α being the reciprocal in UIF of α. Hence, writing $\eta/\alpha = \alpha'$ we have $\xi_n = (\gamma+\alpha')^n$ $(n = 0, 1, \cdots)$, and therefore

$$\xi_n = \sum_{r=0}^{n} \binom{n}{r} \alpha'_{n-r} A_r(\xi, \alpha) \qquad (n = 0, 1, \cdots);$$

whence by substitution in $\varphi(\xi)$ we obtain the required expansion.

16. If f is as in § 11, we have in F the following analogue of Taylor's theorem in CF, θ being variable, λ constant in UIF,

$$f(\theta+\lambda) = \lambda^0 f(\theta)+\frac{\lambda^1}{1!}\frac{\partial f(\theta)}{\partial \theta}+\frac{\lambda^2}{2!}\frac{\partial^2 f(\theta)}{\partial \theta^2}+\cdots,$$

which terminates with $\frac{\lambda^n}{n!}\frac{\partial^n f(\theta)}{\partial \theta^n}$ if $f(u)$ is a polynomial in CF of degree n in the complex variable u. Since $f(\theta)$ is of the form (cf. § 11) $\alpha_0\theta_0+\alpha_1\theta_1+\alpha_2\theta_2+\cdots$, where the α_j are elements of f it suffices to prove the Taylor theorem for $f(\theta) = \theta^n$, so that here $f(\theta+\lambda) = (\theta+\lambda)^n$. But for this f the result is obvious. After completion of the indicated operations exponents of umbrae are lowered as usual.

17. Let $\alpha, \beta, \cdots, \gamma, \delta$ be connected by the relation in UIF, $a\cdot\alpha+b\cdot\beta+\cdots+c\cdot\gamma = d\cdot\delta$.
Multiplying in UIF both sides of this equation be ξ we have (cf. § 8)

$$a\cdot(\xi+\alpha)+b\cdot(\xi+\beta)+\cdots+c\cdot(\xi+\gamma) = d\cdot(\xi+\delta),$$

or what is the same

$$a\cdot A(\xi.\alpha)+b\cdot A(\xi,\beta)+\cdots+c\cdot A(\xi,\gamma) = d\cdot A(\xi,\delta).$$

From the last it follows, with f as before and ζ as usual a variable in UIF, that

$$af(\zeta+A(\xi,\alpha))+bf(\zeta+A(\xi,\beta))+\cdots+cf(\zeta+A(\xi,\gamma)) = df(\zeta+A(\xi,\delta)),$$

since the coefficients of ζ^n in the Taylor expansions of both members are equal by the given relation between $\alpha, \beta, \cdots, \gamma, \delta$.

A special case of this very general theorem is the prototype in F of all recurrence theorems in CF concerning Appell polynomials (including therefore the Bernoullian, Eulerian, Hermitian, also the "modular" generalizations of the first pair discussed in a former paper[5]).

A special case of this theorem has numerous applications. Suppose for the particular value λ of ξ that $A_j(\lambda,\delta) = 0$ if $j \neq r, s, \cdots, t$. Then

$$af(\zeta+A(\lambda,\alpha))+bf(\zeta+A(\lambda,\beta))+\cdots+cf(\zeta+A(\lambda,\gamma))$$
$$= d\left[A_r(\lambda,\delta)\frac{\partial^r f(\zeta)}{\partial \zeta^r}+A_s(\lambda,\delta)\frac{\partial^s f(\zeta)}{\partial \zeta^s}+\cdots+A_t(\lambda,\delta)\frac{\partial^t f(\zeta)}{\partial \zeta^t}\right].$$

By giving to ζ particular values we obtain theorems concerning the $A(\xi,\alpha), \cdots$ or the $A(\lambda,\alpha), \cdots$. For example,

$$af(A(\xi,\alpha))+bf(A(\xi,\beta))+\cdots+cf(A(\xi,\gamma)) = df(A(\xi,\delta))$$

is implied by $a \cdot \alpha + b \cdot \beta + \cdots + c \cdot \gamma = d \cdot \delta$. The last is the prototype of the recurrence relations for the *numbers* of Bernoulli and others.

IV. The irregular exponential field *EIF*.

18. From *AIF* of section III we derive next an *IF*, the *EIF*. We recall (cf. § 7) that if t is an element of F and θ an element of *UIF*, $t\theta$ is the umbra of $t^n\theta_n\,(n = 0, 1, \cdots)$; $t^n\theta_n$ is an element of F.

The *umbral exponential function* $\exp \theta t$ is

$$\exp\ \theta t \equiv \sum_{n=0}^{\infty} \frac{\theta_n t^n}{n!} \equiv \sum_{n=0}^{\infty} \frac{\theta^n t^n}{n!};$$

θ is called the *exponent*, t the *parameter* of $\exp \theta t$.

If as in § 13, $\varphi(\alpha, \beta, \cdots, \gamma)$ is any function in *UIF*, the operations in *EIF* are given by the following comprehensive definition

$$\varphi(\exp \alpha t,\ \exp \beta t,\ \cdots,\ \exp \gamma t) = \exp \varphi(\alpha, \beta, \cdots, \gamma)\, t,$$

where the definition of equality, $\exp \zeta t = \exp \xi t$, in *EIF* is

$$\exp \zeta t = \exp \xi t :\equiv\colon \zeta = \xi,$$

and hence $\zeta_n = \xi_n\ (n = 0, 1, \cdots)$ by the definition of equality in *UIF*. Note that if umbral exponentials occur only in relations of equality it is unnecessary to discuss their convergence.

The elements of *EIF* are the $\exp \theta t$, where θ runs through all elements of *UIF*.

Since $\exp \theta t$ is the symbol of an ordinary (not that of an umbra), we may indicate operations in *EIF* by the same notations as in *IF*. Here it is interesting to notice we have formed a closed chain,

$$IF \to UIF \to EIF \to IF$$

the arrows signifying that any member of the chain is generated from its predecessor.

In particular then, attending to the above definition concerning φ, we have in *EIF*,

$$a \exp \alpha t + b \exp \beta t + \cdots + c \exp \gamma t = \exp(a \cdot \alpha + b \cdot \beta + \cdots + c \cdot \gamma)\, t,$$

including as special cases addition and subtraction in *EIF*;

$$\exp \alpha t \exp \beta t = \exp(\alpha + \beta)\, t,$$

for multiplication in *EIF*, and for division,

$$\exp \alpha t / \exp \beta t = \exp (\alpha/\beta) t;$$

while zero and unity in *EIF* are $\exp \omega t$ and $\exp \eta t$ respectively.

That *EIF* as defined is indeed in *IF*, and hence a solution of $\Pi_0 - \Pi_5$ of § 2, is evident as in previous solutions.

19. The connection between *UIF*, *AIF*, *EIF* is as follows. The exponents $\alpha, \beta, \cdots, \gamma$ of the elements $A(\xi, \alpha)$, $A(\xi, \beta)$, $\cdots$, $A(\xi, \gamma)$ in *AIF* belong to *UIF*; each element $A(\xi, \lambda)$ of *AIF* is the product in *UIF* of two elements ξ, λ of *UIF*; while in *EIF* the element whose exponent is $A(\xi, \lambda)$ is the product (in *EIF*) of the elements whose exponents are ξ, λ respectively,

$$\exp A(\xi, \lambda) t = \exp \xi t \exp \lambda t.$$

If, for an obvious reason, we call $\exp \lambda t$ the *constant divisor*, or simply the *divisor* of $A(\xi, \lambda)$, then *EIF* is the field of divisors of elements of *AIF*. For φ as before, the divisor of $A(\xi, \varphi(\alpha, \beta, \cdots, \gamma))$ is $\varphi(\exp \alpha t, \exp \beta t, \cdots, \exp \gamma t)$. In particular the divisor of $A(\xi, a \cdot \alpha + b \cdot \beta + \cdots + c \cdot \gamma)$ is $a \exp \alpha t + b \exp \beta t + \cdots + c \exp \gamma t$; that of $A(\xi, \alpha/\beta)$ is $\exp \alpha t / \exp \beta t$, and $\exp \omega t$, $\exp \eta t$ are respectively the zero and the unit divisor. On this basis the elements of *AIF* are placed, through their divisors, in (1, 1) correspondence with those of *EIF*. The introduction of *EIF* is made in order to reach the following developments.

V. Application of *EIF* to *AIF*.

20. We now extend to *IF* the notions of differential equations and difference equations in *CF*. What follows applies to any Φ_m (§ 2); when $m = 1$ we have *F*, and when further *F* is its instance *CF*, we recover from the following discussion the corresponding theory for the ordinary Appell[6] polynomials.

Equality in *EIF* (§ 18) has been defined in a manner independent of the convergence of the elements. The derivatives next introduced occur only in relations of equality. Hence we need not discuss convergence. When *F* is *CF* all series are convergent except at ∞, so that all equalities between series in the case of *CF* are valid in the usual sense.

The derivative indicated on the left of the following is defined by

$$\frac{\partial^r}{\partial t^r} \exp \alpha t = \sum_{n=0}^{\infty} n(n-1) \cdots (n-r+1) \alpha_n \frac{t^{n-r}}{n!};$$

hence, precisely as in *CF*,

$$\frac{\partial^r}{\partial t^r} \exp \alpha t = \alpha^r \exp \alpha t,$$

where here, however, as always in the umbral calculus, the raised suffix r in α^r indicates that the multiplication by α^r is to be performed before the degradation of exponents to suffixes. The like is to be observed where an umbra with an exponent (raised suffix) occurs, particularly in the formulas of this section, since otherwise incorrect results are obtained. This derivative being an element of F, so also is its product by the polynomial $f_r(t)$ in F.

21. If β is a given element of *UIF*, we shall say, in accordance with the definition of equality in *EIF*, that $\exp \alpha t$ is a solution of the following *linear differential equation in F of order r*,

$$\sum_{j=0}^{r} f_{r-j}(t) \frac{\partial^{r-j} y}{\partial t^{r-j}} = \exp \beta t \tag{1}$$

if, when y is replaced by $\exp \alpha t$, the coefficients of $t^n (n = 0, 1, \cdots)$ in (1) are severally equal. It does not follow of course that for β and the polynomials f preassigned, a solution $y = \exp \alpha t$ exists. We are interested only in those cases where a solution does exist. The means for writing down the necessary and sufficient conditions that this be so, in terms of r, the coefficients of the f's, and $\beta_n (n = 0, 1, \cdots)$, will be evident from the following discussion.

Let $y = \exp \alpha t$ be a solution of (1). Then we have

$$\sum_{j=0}^{r} f_{r-j}(t)\, \alpha^{r-j} \exp \alpha t = \exp \beta t. \tag{2}$$

If in (2) the f's are replaced by their explicit forms a result of the type

$$\sum_{j=0}^{s} a_j t^j \alpha^{q_j} \exp \alpha t = \exp \beta t. \tag{3}$$

is obtained, where s, q_j are integers $\geqq 0$, and a_j is an element of F; some of the a's may be zero. Equating coefficients of t^n in (3) we have

$$\sum_{j=0}^{s} \frac{1}{(n-j)!} a_j \alpha_{n-j+q_j} = \frac{\beta_n}{n!} \qquad (n = 0, 1, \cdots) \tag{4}$$

a *linear difference equation in F* for the determination of $\alpha_n (n = 0, 1, \cdots)$. If in (4) $n < s$, those terms for which $j > n$ are to be omitted from the left.

Conversely, if (4) is solvable for $\alpha_n (n = 0, 1, \cdots)$, then (1) has the solution $y = \exp \alpha t$, or, what is the same, (4) and

$$\sum_{j=0}^{r} f_{r-j}(t) \frac{\partial^{r-j}}{\partial t^{r-j}} \exp \alpha t = \exp \beta t \tag{5}$$

are formally equivalent. The coefficients a_j and the integers q_j in (4) are given at once by (5). When $\beta = \omega$ (the zero in *UIF*), the right of (1) or of (5) is 0 (the zero in *F*).

22. Assuming that (4) is solvable, and hence that $y = \exp \alpha t$ is a solution of (1) we shall now show that the $A_n(\xi, \alpha)$ $(n = 0, 1, \cdots)$ satisfy a linear difference equation in F and that $A_n(\xi, \alpha)$ satisfies a linear differential equation in F; moreover it will be shown in the course of the proof how these equations can readily be constructed.

For this we shall require the useful theorem

$$\alpha^r A^n = A^n (A - \xi)^r, \qquad A \equiv A(\xi, \alpha), \tag{6}$$

where r, n are integers $\geqq 0$. In full (6) is

$$\sum_{j=0}^{n} \binom{n}{j} \alpha_{n+r-j} \xi_j = \sum_{j=0}^{r} (-1)^r \binom{r}{j} \xi^j A^{n+r-j}(\xi, \alpha), \tag{7}$$

the raised suffixes on the right indicating as usual that the multiplication by ξ^j is to be performed on

$$A^{n+r-j}(\xi, \alpha) \equiv \sum_{s=0}^{n+r-j} \binom{n+r-j}{s} \xi^{n+r-j-s} \alpha^s$$

before exponents of ξ, α are degraded to suffixes. The theorem is evident from the identity

$$\binom{n}{s} = \sum_{j=0}^{s} (-1)^j \binom{r}{j} \binom{n+r-j}{s-j}$$

between binomial coefficients, valid for any integers n, s, r each $\geqq 0$ such that $n \geqq s$.

Multiply (3) throughout by $\exp \xi t$. Then

$$\sum_{j=0}^{s} a_j t^j \alpha^{q_j} \exp A(\xi, \alpha) t = \exp A(\xi, \beta) t. \tag{8}$$

Each coefficient of a power of t in (8) is of the form $a \alpha^r A^n$. Apply (6) or (7) to each, and equate coefficients of like powers of t. Denote by $Q^j(\xi)$ a polynomial in ξ's with raised suffixes. Then (8) gives, on reduction by (7) as indicated, a result of the form

$$\sum_{j=0}^{m} Q^j(\xi) A^{n+j}(\xi, \alpha) = A^n(\xi, \beta) \qquad (n = 0, 1, \cdots), \tag{9}$$

where m is a constant integer which obviously is determined by the order r of (5) and the maximum degree in t of the $f_{r-j}(t)$ in (5); some of the $Q^j(\xi)$, $j<m$, may be zero. Using

$$A^{n-r}(\xi,\alpha) = \frac{(n-r)!}{n!}\frac{\partial^r}{\partial\xi^r}A^n(\xi,\alpha) \qquad (0\leqq r\leqq n;\ n=0,1,\cdots)$$

on (9) we get

$$(10)\quad \sum_{j=0}^{m}(n+m-j)!\ Q^{m-j}(\xi)\frac{\partial^j}{\partial\xi^j}A^{n+m}(\xi,\alpha) = A^n(\xi,\beta) \qquad (n=0,1,\cdots),$$

as the *linear differential equation in* F satisfied by $A^{n+m}(\xi,\alpha)$.

Hence a relation of either type (4) or (5) implies equations of the forms (9), (10).

23. To illustrate (9), (10) we apply §§ 21, 22 to the simple example of the *Hermite polynomials in* F. Let $y = \exp\alpha t$ be a solution of

$$\frac{\partial y}{\partial t}+2ty = \exp\omega t,$$

where ω is the zero in UIF. The right hand member may then be replaced by 0. Hence

$$\alpha\ \exp\alpha t+2t\ \exp\alpha t = 0;$$

whence, corresponding to § 21 (4), we have the following difference equations in F,

$$(4.1)\quad \alpha_{n+2}+2(n+1)\alpha_n = 0, \qquad \alpha_{2n+1}=\alpha_1=0 \qquad (n=0,1,\cdots),$$

and on multiplying by $\exp\xi t$,

$$\alpha\ \exp A(\xi,\alpha)t+2t\ \exp A(\xi,\alpha)t = 0,$$

so that, as in § 22 (9),

$$\alpha A^{n+1}(\xi,\alpha)+2(n+1)A^n(\xi,\alpha) = 0.$$

We now apply § 22 (6) with $r=1$,

$$\alpha A^{n+1} = A^{n+1}(A-\xi)^1 = \xi^0 A^{n+2}-\xi^1 A^{n+1},$$

getting finally, as in (9), (10), for $n=0,1,\cdots$,

$$\xi^0 A^{n+2}(\xi,\alpha)-\xi^1 A^{n+1}(\xi,\alpha)+2(n+1)A^n(\xi,\alpha) = 0,$$

$$\left[2\frac{\partial^2}{\partial\xi^2}-\xi^1\frac{\partial}{\partial\xi}+(n+2)\right]A^{n+2}(\xi,\alpha) = 0.$$

both of which can be verified directly from (4.1). When F is CF the above becomes the algebra for the ordinary Hermite polynomials according to one of their usual definitions.

VI. Circular functions in F.

24. Regarding complex numbers as scalars we can take F over CF as in Dickson[1] pp. 9–10 for an algebra over a field. The postulates for scalar multiplication in what follows are abstractly identical with the like for an algebra; $i = \sqrt{-1}$. We shall require in particular the postulates

$$(i\alpha)^n = (\alpha i)^n = i^n \alpha_n, \quad (i\alpha)^n + (i\beta)^n = i^n(\alpha_n + \beta_n) \qquad (n = 0, 1, \cdots),$$

where α, β are any elements of UIF.

The *umbral cosine and sine* are defined by

$$2\cos \alpha t = \exp(i\alpha)t + \exp(-i\alpha)t = \sum_{n=0}^{\infty} \alpha_{2n} \frac{(-1)^n t^{2n}}{(2n)!},$$

$$2i \sin \alpha t = \exp(i\alpha)t - \exp(-i\alpha)t = \sum_{n=0}^{\infty} \alpha_{2n+1} \frac{(-1)^n t^{2n+1}}{(2n+1)!}.$$

Indicating as in EIF multiplication by juxtaposition, thus $\cos \alpha t \sin \beta t$, we have

$$2\cos \alpha t \cos \beta t = \cos(\alpha+\beta)t + \cos(\alpha-\beta)t,$$

with three others of a similar nature, precisely as in CF, all of which may be proved directly from the definitions. We have also $\cos \omega t = \exp \eta t$, $\sin \omega t = \exp \omega t$.

25. The theory of these umbral circular functions in F is abstractly identical with that of the circular functions in CF or in the field of real numbers, except as regards periodicity, which is lacking in F. This theory includes that of the φ, ψ, χ functions and their algebra discussed in detail in a former paper[7]. It is sufficient to remark here that if the arguments of the φ, ψ, χ are umbrae, one of which is ξ, then each of φ_n, ψ_n, χ_n is a solution of § 12 (1), and hence φ, ψ, χ become elements of AIF. Hence the theory of the present paper is the abstract structure of that of the φ, ψ, χ functions and their numerous applications.

VII. The field of elliptic polynomials.

26. To illustrate some of the preceding sections we shall briefly outline their application to a new class of polynomials of considerable interest.

These are the *elliptic polynomials* $E_n(u, z)$ $(n = 0, 1, \cdots)$ in the complex variables u, z. As the theory of the $E_n(u, z)$ is relatively as much richer than is that of any of its degenerate cases as is the theory of the elliptic functions correspondingly richer than that of the circular or the hyperbolic functions, we can indicate here only the barest outline of the new theory. First let us point out the manner in which the theory for the $E_n(u, z)$ $(n = 0, 1, \cdots)$ includes its degenerate cases. The proofs of the few properties of the E's presently stated follow immediately, as noted in § 29, from the symbolic generating identities for the $E_n(u, z)$ in § 28. *Throughout this entire section we are operating in CF.*

27. We shall say that a sequence of polynomials $A_n = A_n(z_1, z_2, \cdots, z_r)$ $(n = 0, 1, \cdots)$ in the complex variables z_i (j $= 1, \cdots, r$) is *Appell in* z_j if

$$\frac{\partial A_n}{\partial z_j} = n A_{n-1} \qquad (n = 0, 1, \cdots);$$

A_n is the polynomial of *rank* n. It is readily seen that if $r = 1$, $z_1 = u$, then $A_n(u) = A_n(u, \alpha)$ in the notation of § 12; viz $A_n(u) = (u+\alpha)^n$, the α_j $(j = 0, \cdots, n)$ being here arbitrary constant elements of CF.

Since α is arbitrary to the extent stated, we may consider in particular the α_r in $A_n(u, \alpha)$ to be polynomials in the complex variable $z \neq u$. Hence we have, in this case,

$$A_n(u, \alpha(z)) \equiv [u+\alpha(z)]^n, \quad \frac{\delta}{\partial u} A_n(u, \alpha(z)) = n A_{n-1}(u, \alpha(z)),$$

so that this A_n is Appell in u.

Further conditions may now be imposed on the $A_n(u, \alpha(z))$, as for example that they shall satisfy with respect to z, for $n = 0, 1, \cdots$, a certain partial differential equation. The algebra of such a situation being an immediate instance of that developed in preceding sections, there is no point in an intirely arbitrary assignment of the z-differential equation, as the resulting polynomials would be of no use. From all the possibilities we choose that which leads to the $E_n(u, z)$, which are of interest because they include as limiting cases the *numbers* of Bernoulli, Euler, Genocchi and Lucas, the classic *polynomials* (in one complex variable) of the same names, based on the respective numbers, and the *anharmonic polynomial generalizations* of these numbers which were discussed in a previous paper.[4] The $E_n(u, z)$ themselves are the appropriate elements for the determination of syzygies between numbers of representations of integers in quadratic forms in any nnmber of indeterminates, and it was in this connection that they first arose. They are further of interest because their theory

is abstractly identical with the algebra of elliptic functions, including the theory of transformation of the latter. A third claim to attention is the following: the recurrences, differential equations, etc., satisfied by the $E_n(u, z)$ degenerate immediately to the like for the limiting cases of the $E_n(u, z)$ above mentioned, so that the theories of the Bernoulli and allied numbers, polynomials and their anharmonic generalizations are recovered at once on replacing, according to the case, either or both of u, z by constants. This will be seen presently. The theory of these elliptic polynomials is more readily developed than is that of any of its degenerate forms, because in the perfectly general case (z, u complex variables) the underlying symmetry, which is that of the linear group abstractly identical with the cross ratio group, is not masked as it is in the limiting cases.

We briefly enumerate a few characteristic properties of these remarkable polynomials

(I) The $E^n(u,z)$ are Appell in u but not in z.

(II) The $E^n(u,z)$ fall into twelve mutually exclusive sequences.

(III) The twelve sequences in (II) fall into six pairs, called *reciprocals*, having the following properties. If for the moment we denote any pair of reciprocal sequence of $E_n(u,z)$'s by $P_n(u,z)$, $Q_n(u,z)$ $(n = 0, 1, \cdots)$, then, by (I),

$$\frac{\partial}{\partial u}P_n(u,z) = nP_{n-1}(u,z), \quad \frac{\partial}{\partial u}Q_n(u,z) = nQ_{n-1}(u,z) \quad (n = 0, 1, \cdots);$$

while with respect to z, P, Q satisfy

$$\frac{\partial}{\partial z}\sum_{r=0}^{[n/2]}\binom{n}{2r}P_{n-2r}(u,z)Q_{2r}(u,z) = 0 \quad (n = 0, 1, \cdots).$$

More symmetrically, with umbral notation, we may write

$$P^n \equiv P_n(u,z), \qquad Q^n \equiv Q_n(u,z),$$
$$\frac{\partial P^n}{\partial u} = nP^{n-1}, \qquad \frac{\partial Q^n}{\partial u} = nQ^{n-1},$$

$$\frac{\partial}{\partial z}[(P+Q)^n+(P-Q)^n] = 0,$$

$$\frac{\partial}{\partial u}[(P+Q)^n+(P-Q)^n] = n[(P+Q)^{n-1}+(P-Q)^{n-1}], \quad (n=0,1,\cdots),$$

the last of which shows that $(P+Q)^n+(P-Q)^n$ $(n = 0, 1, \cdots)$ is a sequence of polynomials Appell in u which, by the second, are independent of z. Hence a characteristic property of a pair of reciprocal sequences

of elliptic polynomials is that from them may be constructed a sequence of polynomials Appell in one argument and independent of the other.

(IV) The z-partial differential equation in (III) is an *algebraic* relation between the z-derivatives of $P_n, P_{n-2}, P_{n-4}, \cdots, Q_n, Q_{n-2}, Q_{n-4}, \cdots$, and these elliptic polynomials, linear in the polynomials and in their derivatives.

There is a corresponding algebraic relation between polynomials selected from any pair of sequences of elliptic polynomials, not necessarily reciprocals. When the sequences are not reciprocals the differential equation involves also a bilinear function of the polynomials alone, in analogy with the derivatives of elliptic functions. Note that *pairs* and their derivatives are *algebraically* connected.

(V) An elliptic polynomial $E_n(u, z)$ satisfies no equation of the type

$$\frac{\partial}{\partial z} E_n(u, z) = f(E) \qquad (n = 0, 1, \cdots),$$

where $f(E)$ is an algebraic function of polynomials $E_j(u, z)$ of any ranks $j = n_1, n_2, \cdots$. Since the $E_n(u, z)$ are Appell in u, it thus appears that their propertiers with respect to u, z are radically different.

(VI) Omitted here.

(VII) In a former paper[3] the polynomials $\beta_n(u)$, $\eta_n(u)$, $\gamma_n(u)$, $\varrho_n(u)$ $(n = 0, 1, \cdots)$ based respectively on the numbers B_n, E_n, G_n, R_n $(n = 0, 1, \cdots)$ of Bernoulli, Euler, Genocchi and Lucas, with the even suffix notation of the last, were named correspondingly. For $u = 2v$ or $u = 2v \pm 1$, these functions are equivalent to the Bernoullian and Eulerian functions of other writers. All are Appell in u.

For $z = 0$ the $E_n(u, z)$ degenerate as follows: each of six of the $E_n(u, 0)$ of rank n becomes a certain one of $\beta_n(u)$, $\eta_n(u)$, $\gamma_n(u)$, $\varrho_n(u)$; two become $-n[(u+1)^n - (u-1)^n]$; two degenerate to $(u+1)^n + (u-1)^n$, and one to $2u^n$.

(VIII) We previously defined and investigated the anharmonic polynomials[4] in the single complex variable z. It was shown in the paper just cited that the anharmonic polynomials of rank n degenerate for $z = 0$ or $z = 1$ either to numerical constants or to the numbers B_n, E_n, G_n, R_n. Parallel to the $E_n(u, z)$, the anharmonic polynomials fall into twelve mutually exclusive sequences which may be arranged into reciprocal pairs in a sense explained in the paper cited.

For $u = 0$ the twelve elliptic polynomials $E_n(u, z)$ of rank n degenerate to the twelve anharmonic polynomials of the same rank.

For $u = z = 0$ the $E_n(u, z)$ degenerate as follows: two to $-4n$; two to 2; two to 0; two to $2E_n$; two to $2R_n$; one to $2^{n+1} B_n$ and one to $2^{n+1} G_n$.

(IX) To any relation between the elliptic functions

$$\begin{array}{lll} \text{sn}, & \text{cn}, & \text{dn}, \\ \text{ns}, & \text{nc}, & \text{nd}, \\ \text{sc}, & \text{sd}, & \text{cd}, \\ \text{cs}, & \text{ds}, & \text{dc}, \end{array}$$

(in a familiar notation much used by Glaisher and others), there is a unique correspondent in terms of elliptic polynomials, which is such that either relation in a pair implies the other.

28. We proceed to show how the properties stated in § 27 follow by means of the preceding sections from the following definitions of the $E_n(u, z)$. Revising slightly the notation of the anharmonic polynomials, $\eta s_n(z), \cdots, \beta p_n(z)$ $(n = 0, 1, \cdots)$ we now write, as symbolic equations of definition,

$$x\,\text{sn}\,(x, k) = \cos \eta s(z)x, \tag{1.1}$$

$$2x\,\text{sc}\,(x, k) = \cos 2\alpha s(z)x, \tag{1.2}$$

$$x\,\text{sd}\,(x, k) = \cos \beta s(z)x; \tag{1.3}$$

$$\text{cn}\,(x, k) = \cos \eta c(z)x, \tag{2.1}$$

$$\text{dc}\,(x, k) = \cos \alpha c(z)x, \tag{2.2}$$

$$\text{nd}\,(x, k) = \cos \beta c\,(z)x, \tag{2.3}$$

$$\text{dn}\,(x, k) = \cos \gamma c\,(z)x, \tag{2.4}$$

$$\text{nc}\,(x, k) = \cos \delta c\,(z)x, \tag{2.5}$$

$$\text{cd}\,(x, k) = \cos \varepsilon c\,(z)x; \tag{2.6}$$

$$x\,\text{ns}\,(x, k) = 2\cos \eta p(z)x, \tag{3.1}$$

$$x\,\text{cs}\,(x, k) = \cos 2\alpha p(z)x, \tag{3.2}$$

$$x\,\text{ds}\,(x, k) = 2\cos \beta p(z)x, \tag{3.3}$$

in which x is a parameter, $\text{sn}\,(x, k)$ indicates an elliptic function with argument x and modulus k^2; $z = k^2$ when, as above, z occurs in a *generator* (umbral sine or cosine), $z =$ an arbitrary complex variable in any context not involving generators; and $\eta s, \cdots, \beta p$ are double-letter symbols of umbrae analogous to the two letter notation sn, cn, $\cdots$ for the elliptic functions. The Greek letters $\eta, \alpha, \beta, \gamma, \delta, \varepsilon$, recall the specific substitution of the cross ratio group appertaining to the elliptic function in the definition concerned; η is the identity of the group. That is, as explained in the previous paper[4], by a certain substitution $\eta, \alpha, \beta, \gamma, \delta, \varepsilon$ operating on z the set of twelve elliptic functions is obtained, up to constant multipliers,

from the three, sn (x, k), cn (x, k), dn (x, k). It will be noticed that the definitions fall into sets of 3, 6, 3; in a given set the Latin letter s, c or p remains the same. The sets of 3 are cyclic; the set of 6 is anharmonic, in the sense of the previous paper. The notation $\eta s, \cdots, \beta p$ therefore summarizes the mutual relations of the polynomials $\eta s_n(z), \cdots, \beta p_n(z)$ $(n = 0, 1, \cdots)$ as z is transformed by the group $(\eta, \alpha, \beta, \gamma, \delta, \varepsilon)$.

The definition (2. 1), for example, states that the coefficient of $x^{2n}/(2n)!$ in the Maclaurin expansion of x sn (x, k) is $\mu c_{2n}(z)$, and similarly for the others.

Putting $z = 0$ in these we have

$$(1.11)\qquad x\,\mathrm{sn}(x,0) = \cos\ \eta s(0)\,x = x\sin x,$$
$$(1.21)\qquad 2x\,\mathrm{sc}(x,0) = \cos 2\alpha s(0)\,x = \cos 2Gx,$$
$$(1.31)\qquad x\,\mathrm{sd}(x,0) = \cos\ \beta s(0)\,x = x\sin x;$$

$$(2.11)\qquad \mathrm{cn}(x,0) = \cos\ \eta c(0)\,x = \cos x,$$
$$(2.21)\qquad \mathrm{dc}(x,0) = \cos\ \alpha c(0)\,x = \cos Ex,$$
$$(2,31)\qquad \mathrm{nd}(x,0) = \cos\ \beta c(0)\,x = 1,$$
$$(2.41)\qquad \mathrm{dn}(x,0) = \cos\ \gamma c(0)\,x = 1,$$
$$(2.51)\qquad \mathrm{nc}(x,0) = \cos\ \delta c(0)\,x = \cos Ex,$$
$$(2.61)\qquad \mathrm{cd}(x,0) = \cos\ \varepsilon c(0)\,x = \cos x;$$

$$(3.11)\qquad x\,\mathrm{ns}(x,0) = 2\cos\eta p(0)\,x = 2\cos Rx,$$
$$(3.12)\qquad x\,\mathrm{cs}(x,0) = \cos 2\alpha p(0)\,x = \cos 2\,Bx,$$
$$(3.13)\qquad x\,\mathrm{ds}(x,0) = 2\cos\beta p(0)\,x = 2\cos Rx,$$

where B, E, G, R are as in 27. These give the degenerations of the anharmonic polynomials.

Generalizing the Bernoulli and allied polynomials, we now define the $E_n(u, z)$ of § 27 by means of their generators as follows.

$$(4.1)\quad 2x\,\mathrm{sn}(x,k)\,\mathrm{soc}\,ux = 2\cos\ \eta s(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\eta\,\sigma(u,z)x,$$
$$(4.2)\quad 4x\,\mathrm{sc}(x,k)\,\mathrm{soc}\,ux = 2\cos 2\,\alpha s(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\alpha\,\sigma(u,z)x,$$
$$(4.3)\quad 2x\,\mathrm{sd}(x,k)\,\mathrm{soc}\,ux = 2\cos\ \beta s(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\beta\,\sigma(u,z)x;$$

$$(5.1)\quad 2\,\mathrm{cn}(x,k)\,\mathrm{soc}\,ux = 2\cos\eta\,c(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\eta\varkappa(u,z)x,$$
$$(5.2)\quad 2\,\mathrm{dc}(x,k)\,\mathrm{soc}\,ux = 2\cos\alpha\,c(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\alpha\varkappa(u,z)x,$$
$$(5.3)\quad 2\,\mathrm{nd}(x,k)\,\mathrm{soc}\,ux = 2\cos\beta\,c(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\beta\varkappa(u,z)x,$$
$$(5.4)\quad 2\,\mathrm{dn}(x,k)\,\mathrm{soc}\,ux = 2\cos\gamma\,c(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\gamma\varkappa(u,z)x,$$
$$(5.5)\quad 2\,\mathrm{nc}(x,k)\,\mathrm{soc}\,ux = 2\cos\delta\,c(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\delta\varkappa(u,z)x,$$
$$(5.6)\quad 2\,\mathrm{cd}(x,k)\,\mathrm{soc}\,ux = 2\cos\varepsilon\,c(z)x\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\varepsilon\varkappa(u,z)x;$$

$$(6.1)\quad x\,\mathrm{ns}(x,k)\,\mathrm{soc}\,ux = 2\cos\ \eta p(z)\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\eta\pi(u,z)x,$$
$$(6.2)\quad 2x\,\mathrm{cs}(x,k)\,\mathrm{soc}\,ux = 2\cos 2\,\alpha p(z)\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\alpha\pi(u,z)x,$$
$$(6.3)\quad x\,\mathrm{ds}(x,k)\,\mathrm{soc}\,ux = 2\cos\ \beta p(z)\,\mathrm{soc}\,ux \equiv \mathrm{soc}\,\beta\pi(u,z)x,$$

in all of which note that σ, $\varkappa$ or π corresponds to s, c or p respectively. The elliptic polynomials are the $\eta\sigma_n(u,z), \cdots, \beta\pi_n(u,z)$ $(n = 0, 1, \cdots)$ thus defined. As in a former paper[7] "soc" is an abbreviation for 'sin or cos'; the same one of sin or cos is understood for soc in any given equation. Thus each of (4.1)–(6.3) consists of two definitions; the sin gives the elliptic polynomials of odd ranks, the cos those of even.

If, as in the previous paper[7], we define the φ, ψ functions of any umbrae α, β by

$$\varphi_n(\alpha,\beta) = (\alpha+\beta)^n+(\alpha-\beta)^n, \qquad \psi_n(\alpha,\beta) = (\alpha+\beta)^n-(\alpha-\beta)^n,$$

for $n = 0, 1, \cdots$, and note the following

$$\begin{aligned} 2\cos\alpha x\cos\beta x &= \cos\varphi(\alpha,\beta)x,\\ 2\cos\alpha x\sin\beta x &= \sin\psi(\alpha,\beta)x,\\ 2\sin\alpha x\cos\beta x &= \sin\varphi(\alpha,\beta)x,\\ 2\sin\alpha x\sin\beta x &= -\cos\psi(\alpha,\beta)x, \end{aligned}$$

(cf. section VI), we have from (4.1)–(6.3) the following explicit forms of the elliptic polynomials and their degenerates the Bernoullian, Eulerian, etc; cf. § 27:

$$(4.11)\quad \eta\sigma_n(u,z) = \varphi_n(u,\eta s(z)), \qquad \eta\sigma_n(u,0) = -n\psi_{n-1}(u,1),$$
$$(4.21)\quad \alpha\sigma_n(u,z) = \varphi_n(u,2\alpha s(z)), \qquad \alpha\sigma_n(u,0) = \gamma_n(u),$$
$$(4.31)\quad \beta\sigma_n(u,z) = \varphi_n(u,\beta s(z)), \qquad \beta\sigma_n(u,0) = -n\psi_{n-1}(u,1);$$

$$(5.11)\quad \eta\varkappa_n(u,z) = \varphi_n(u,\eta c(z)), \qquad \eta\varkappa_n(u,0) = \varphi_n(u,1),$$
$$(5.21)\quad \alpha\varkappa_n(u,z) = \varphi_n(u,\alpha c(z)), \qquad \alpha\varkappa_n(u,0) = \eta_n(u),$$
$$(5.31)\quad \beta\varkappa_n(u,z) = \varphi_n(u,\beta c(z)), \qquad \beta\varkappa_n(u,0) = 2u^n,$$
$$(5.41)\quad \gamma\varkappa_n(u,z) = \varphi_n(u,\gamma c(z)), \qquad \gamma\varkappa_n(u,0) = 2u^n,$$
$$(5.51)\quad \delta\varkappa_n(u,z) = \varphi_n(u,\delta c(z)), \qquad \delta\varkappa_n(u,0) = \eta_n(u),$$
$$(5.61)\quad \varepsilon\varkappa_n(u,z) = \varphi_n(u,\varepsilon c(z)), \qquad \varepsilon\varkappa_n(u,0) = \varphi_n(u,1);$$

$$(6.11)\quad \eta\pi_n(u,z) = \varphi_n(u,\eta p(z)), \qquad \eta\pi_n(u,0) = \varrho_n(u),$$
$$(6.21)\quad \alpha\pi_n(u,z) = \varphi_n(u,2\alpha p(z)), \qquad \alpha\pi_n(u,0) = \beta_n(u),$$
$$(6.31)\quad \beta\pi_n(u,z) = \varphi_n(u,\beta p(z)), \qquad \beta\pi_n(u,0) = \varrho_n(u).$$

From these we see at once, as follows, that the theory of the elliptic polynomials includes those of its degenerate cases as mentioned in § 27. For we have

$$(4.12)\quad \eta\sigma_{2n}(0,z) = 2\eta s_{2n}(z), \qquad \eta\sigma_{2n}(0,0) = -4n,$$
$$(4.22)\quad \alpha\sigma_{2n}(0,z) = 2^{2n+1}\alpha s_{2n}(z), \qquad \alpha\sigma_{2n}(0,0) = 2^{2n+1}G_{2n},$$
$$(4.32)\quad \beta\sigma_{2n}(0,z) = 2\beta s_{2n}(z), \qquad \beta\sigma_{2n}(0,0) = -4n;$$

$$(5.12)\quad \eta\varkappa_{2n}(0,z) = 2\eta c_{2n}(z), \qquad \eta\varkappa_{2n}(0,0) = 2,$$
$$(5.22)\quad \alpha\varkappa_{2n}(0,z) = 2\alpha c_{2n}(z), \qquad \alpha\varkappa_{2n}(0,0) = 2E_{2n},$$
$$(5.32)\quad \beta\varkappa_{2n}(0,z) = 2\beta c_{2n}(z), \qquad \beta\varkappa_{0}(0,0) = 2, \qquad \beta\varkappa_{2n+2}(0,0) = 0,$$
$$(5.42)\quad \gamma\varkappa_{2n}(0,z) = 2\gamma c_{2n}(z), \qquad \gamma\varkappa_{0}(0,0) = 2, \qquad \gamma\varkappa_{2n+2}(0,0) = 0,$$
$$(5.52)\quad \delta\varkappa_{2n}(0,z) = 2\delta c_{2n}(z), \qquad \delta\varkappa_{2n}(0,0) = 2E_{2n},$$
$$(5.62)\quad \varepsilon\varkappa_{2n}(0,z) = 2\varepsilon c_{2n}(z), \qquad \varepsilon\varkappa_{2n}(0,0) = 2;$$

$$(6.12)\quad \eta\pi_{2n}(0,z) = 2\eta p_{2n}(z), \qquad \eta\pi_{2n}(0,0) = 2R_{2n},$$
$$(6.22)\quad \alpha\pi_{2n}(0,z) = 2^{2n+1}\alpha p_{2n}(z), \qquad \alpha\pi_{2n}(0,0) = 2^{2n+1}B_{2n},$$
$$(6.31)\quad \beta\pi_{2n}(0,z) = 2\beta p_{2n}(z), \qquad \beta\pi_{2n}(0,0) = 2R_{2n},$$

together with

$$\alpha s_1(z) = 1, \qquad \alpha_{2n+3}(z) = \alpha p_{2p+3}(z) = 0, \qquad \alpha p_1(z) = -\frac{1}{2},$$

all other functions or numbers of odd ranks being zero identically.

29. Let us now check in somewhat more detail the statements in § 27; (I) is obvious, as also is (II). For (III), the separation into reciprocals proceeds from the following: the elliptic function $\mathrm{sn}(x,k), \cdots, \mathrm{ds}(x,k)$ in the definition (4.1), $\cdots$, (6.3) may be paired with precisely one other in the set of 12 definitions which is its reciprocal, thus for example in (5.1), (5.5) we have $\mathrm{cn}(x,k)$ and $\mathrm{nc}(x,k)$ whose product is unity. Multiply together, member by member, such pairs of generating equations and reduce; thus, for example,

$$8\,\mathrm{soc}^2 ux = 2\,\mathrm{soc}\,\eta\varkappa(u,z)x\,\mathrm{soc}\,\delta\varkappa(u,z)x,$$

whence we have

$$4(1-\cos 2ux) = -\cos\psi(\eta\varkappa(u,z),\ \delta\varkappa(u,z))x,$$
$$4(1+\cos 2ux) = \cos\varphi(\eta\varkappa(u,z),\ \delta\varkappa(u,z))x,$$

whence, on equating coefficients of like powers of x we have results of the type stated in (III).

The algebraic relations mentioned in (IV) are obtained in precisely the same way by multiplications of any two generating identities. The derivative relations follow from the like for the x-derivatives of generators.

To see (V) we note that if the contrary were the case $\int \mathrm{dn}^2 u\, du$ would be an elliptic function, which it is not.

The degenerations in (VII) are obvious from the lists in § 28, and likewise for (VIII).

The isomorphism with elliptic functions noted in (IX) is sufficiently evident. It may be systematically developed by setting up a (1,1) correspondence between the elliptic polynomials and their respective elliptic function generators in the same way that in § 19 we established the "divisor" correspondence for the $A(\xi, \alpha)$. The result gives a new field, that of the elliptic polynomials.

30. In the field just described we obtain relations between elliptic pnlynomials by translating relations between elliptic functions, viâ the generators of the polynomials. Since, considered as functions of u, the umbrae of elliptic polynomials are elements of $AICF$, the entire theory of this paper applies to them without further elaboration. In particular we call attention to the results of applying §§ 15, 16, 17, 20–23, and the theory of the φ, ψ, χ developed in the previous paper[7]. To reach from these developments the degenerate theories, it is sufficient to replace in the formulas u, z by the special values indicated in § 28.

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ON THE APPROXIMATE SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS.

By Nicolas Kryloff.

In their papers M. C. Fubini* and M. L. Lichtenstein† were occupied with the following integro-differential system:

$$(1)\qquad \frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right]+q(x)u(x)+\int_0^1 K(x,y)u(y)\,dy = L(u) = f(x),$$
$$u(0) = u(1) = 0,$$

studying it from the point of view of the existence of the solution. Among the different methods that can be actually applied in obtaining an approximate solution of (1) the most convenient will be, as it seems to us, the method based on minimizing the integral

$$(2)\qquad \int_0^1 [L(u)-f(x)]^2\,dx$$

by means of terminating series of the form

$$u_m(x) = \sum_{i=1}^{m} a_i^{(m)}\,\psi_i(x),$$

where $[\psi_i(x)]$ is the system of easily calculable functions‡ verifying the boundary conditions (1) which are used in mathematical physics for the development of arbitrary functions. This method of least squares, recently developed by me from the point of view of the convergence of the solutions in a series of articles,§ leads us for the determination of the unknown coefficients $a_i^{(m)}$ to the following system of linear equations:

$$\int_0^1 [L(u_m)-f]\,L(\psi_i)\,dx = 0,\ (i \leqq m),$$

obtained in the usual way and in this case evidently soluble.

* Fubini, Alcuni nuovi problemi di calcolo di variazioni etc., Ann. di Mat., ser. 3, vol. 20, pp. 217–244.

† Lichtenstein, Über eine Integro-Differentialgleichung etc., Schwarz-Festschr. (1914), pp. 274–285.

‡ For example, the sine function.

§ Comptes Rendus, vol. 181, p. 86; Bull. Acad. des Sci. d'Ukraine, vol. 1, fasc. 3, 4.

Because our aim is to elaborate a method of approximate solution of (1), we can suppose that this solution exists according to the papers mentioned above; therefore we have

$$\int_0^1 [L(u)-f]\,L(\psi_i)\,dx = 0$$

and hence

$$\int_0^1 [L(u-u_m)]\,h(\psi_i)\,dx = 0, \qquad (i \leqq m). \tag{3}$$

Multiplying now the equations of the system (3) respectively by $A_i^{(m)}-a_i^{(m)}$ and summing we easily obtain

$$\int_0^1 L(u-u_m)\,L(U_m-u_m)\,dx = 0, \tag{4}$$

where

$$U_m(x) = \sum_{i=1}^{m} A_i^{(m)}\,\psi_i(x)$$

is the sum of order m obtained by the application of one of the well known methods of summation (for instance, that of Prof. L. Fejér or that of Prof. D. Jackson) to the development of the integral of (1). This being stated we observe that

$$U_m-u_m = u-u_m+U_m-u;$$

therefore from (4) we obtain

$$\int_0^1 [L(u-u_m)]^2\,dx = \int_0^1 L(u-u_m)\,L(U_m-u)\,dx$$

and by Schwarz's inequality we have easily

$$\int_0^1 [L(u-u_m)]^2\,dx < \varepsilon_m, \tag{5}$$

where the order of ε_m can be fixed according to the restrictions imposed on the coefficients of the integro-differential equations (1); if, for example, $p(x)$, $q(x)$, $K(x,y)$, $f(x)$ satisfy a Lipschitz condition of order one, the order of smallness of ε_m will be $1/m$ in virtue of the well known theorems* concerning the approximation of functions of a real variable.

* Cf. Vallée-Poussin, Leçons sur l'approximation des fonctions, Paris 1919.

By integrating by parts we have evidently

$$(6)\qquad \int_0^1 L(u-u_m)[u_m-u]\,dx = \int_0^1 \left\{p(x)\left[\frac{d(u-u_m)}{dx}\right]^2 - q(x)[u-u_m]^2\right\} dx - \int_0^1\int_0^1 K(x,y)[u(x)-u_m(x)][u(y)-u_m(y)]\,dx\,dy;$$

therefore, if $p(x) \geqq p_1$, $q(x) \leqq 0$ and, according to the terminology of the theory of integral equations, the kernel $K(x, y)$ is definitely negative, we obtain from (6)

$$\int_0^1 p(x)\left[\frac{d(u-u_m)}{dx}\right]^2 dx < \sqrt{\varepsilon_m}\,|u-u_m|,$$

and hence

$$(7)\qquad |u-u_m| < \frac{\sqrt{\varepsilon_m}}{p_1},$$

which proves, not only the convergence of the algorithm based on minimizing (2), but also makes it possible to estimate the error of the mth approximation. Utilizing now inequalities (5) and (7) we can easily obtain by means of (6) the orders of

$$\int_0^1 [r'_m(x)]^2\,dx \quad \text{and} \quad \int_0^1 [r''_m(x)]^2\,dx,$$

where

$$r'_m(x) = \frac{du(x)}{dx} - \frac{du_m(x)}{dx}, \qquad r''_m(x) = \frac{d^2u(x)}{dx^2} - \frac{d^2u_m(x)}{dx^2}.$$

Then from the well known identity

$$[r'_m(\varepsilon)]^2 = [r'_m(x)]^2 - 2\int_\varepsilon^x r'_m(x)\,r''_m(x)\,dx,$$

where x, ε are any two points of the interval (0,1), we get immediately from Schwarz's inequality

$$[r'_m(\varepsilon)]^2 < \int_0^1 [r'_m(x)]^2\,dx + 2\sqrt{\int_0^1 [r'_m(x)]^2\,dx \int_0^1 [r''_m(x)]^2\,dx},$$

and this makes it possible for us to estimate the order of smallness of $|du/dx - du_m/dx|$ and at the same time to state that the derivative of the sequence u_m converges to the derivative of the integral of (1).

The following theorem is therefore established: *If* $p(x) \geqq p_1 > 0$, $q(x) \leqq 0$ *and* $K(x, y)$ *is definitely negative, then for the approximate integration of the integro-differential system* (1) *there can be applied the method based on the minimizing of the integral* (2), *and this method of least squares gives us, not only a convergent algorithm, but also permits us to estimate the orders of smallness of the errors committed at the mth approximation in the evaluation of the integral and its derivative.*

This method of least squares, very important from a practical point of view, is not without some interest, as it seems to us, from the theoretical side also, because it gives the approximate solution (together with the order of the error committed) of some problems where, till now, only existence theorems were known. It is for this reason that I take pleasure in communicating it to American readers. More detailed explanations of different generalisations and applications of this method can be found in my paper, the first chapter of which is now on the point of publication in the *Annales de la Faculté des Sciences de Toulouse.*

KIEFF, UKRAINE.

March 10, 1926.

A NEW PROOF OF PARSEVAL'S IDENTITY FOR TRIGONOMETRIC FUNCTIONS.

By J. Tamarkin.

The number of proofs of Parseval's identity

$$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx,$$

is large. Nevertheless the following proof is apparently new and has the advantage of simplicity.

1. Let (a, b) be a given interval and

$$u_1(x), u_2(x), \cdots u_n(x), \cdots \tag{1}$$

a given set of orthogonalized normalized functions on this interval. The sense in which we employ these terms implies that $u_1(x), \cdots u_n(x), \cdots$ belong to the class (L_2) of functions which are integrable in the sense of Lebesgue together with their squares and satisfy the equations

$$\int_a^b u_i(x)\,u_j(x)\,dx = \begin{cases} 0 \text{ if } j \neq i, \\ 1 \text{ if } j = i. \end{cases} \tag{2}$$

Let $f(x)$ be an arbitrary function of the class (L_2). We denote by f_i the Fouriers's coefficients of $f(x)$ with respect to the set (1), i. e.

$$f_i = \int_a^b f(x)\,u_i(x)\,dx \qquad (i = 1, 2, \cdots). \tag{3}$$

It is well known that the quantity

$$\varepsilon_n(f) = \int_a^b \left\{f(x) - \sum_{i=1}^{n} f_i\,u_i(x)\right\}^2 dx = \int_a^b f^2\,dx - \sum_{i=1}^{n} f_i^2 \geqq 0$$

represents the minimum error, in the sense of least squares, for approximation to the function $f(x)$ by linear aggregates of the form

$$\sum_{i=1}^{n} a_i\,u_i(x).$$

This fact yields immediately Bessel's inequality:

$$\sum_{i=1}^{n} f_i^2 \leqq \int_a^b [f(x)]^2\,dx, \tag{4}$$

which proves that the series

$$\sum_{i=1}^{\infty} f_i^2$$

is convergent for any functron $f(x)$ in (L_2), and that

$$f_n \to 0 \text{ as } n \to \infty.$$

If $\varphi(x)$ is any other function in (L_2) the inequality

$$\left| \sum_{i=n+1}^{n+p} f_i \varphi_i \right| \leqq \sqrt{\sum_{i=n+1}^{n+p} f_i^2 \sum_{i=n+1}^{n+p} \varphi_i^2} \leqq \sqrt{\sum_{i=n+1}^{n+p} f_i^2} \sqrt{\int_a^b \varphi^2\,dx} \tag{5}$$

shows that the series

$$\sum_{i=1}^{\infty} f_i \varphi_i \tag{6}$$

is absolutely convergent. If $\varphi(x)$ depends on certain parameters and is uniformly bounded, the same inequality (5) shows that the series (6) is uniformly convergent with respect to these parameters. We shall discuss some necessary and sufficient conditions under which Parseval's identity

$$\int_a^b f^2\,dx = \sum_{i=1}^{\infty} f_i^2 \tag{7}$$

holds for any $f(x)$ in (L_2). It is obvious that the equation (7) which we shall call the equation of completeness for the set (1), is entirely equivalent to the fact that the minimum quadratic error

$$\varepsilon_n(f) \to 0 \quad \text{as} \quad n \to \infty.$$

If (7) holds true, we say that the set (1) is complete with respect to the function $f(x)$.

2. Suppose that

$$\xi_1 < x < \xi_2$$

is any subinterval of the interval (a, b). The "elementary step function" $\varphi_{\xi_1, \xi_2}(x)$ is defined by the relations

$$\varphi_{\xi_1, \xi_2}(x) = \begin{cases} 1 \text{ if } x \text{ is within } (\xi_1, \xi_2), \\ 0 \text{ if } x \text{ is without } (\xi_1, \xi_2). \end{cases} \tag{8}$$

The values assigned to $\varphi_{\xi_1,\xi_2}(x)$ at the points $x=\xi_1$ or $x=\xi_2$ are of no importance.

THEOREM 1. *Necessary and sufficient condition that the set* (1) *be complete with respect to any function in* (L_2) *is that the set* (1) *be complete with respect to any elementary step function.*

The condition is obviously necessary. It remains only to prove its sufficiency. We shall construct this proof in a series of successive steps.

Step 1: *If the set* (1) *is complete with respect to the functions*

$$\varphi^{(1)}(x), \cdots, \varphi^{(r)}(x), \tag{9}$$

it is also complete with respect to any linear combination of functions (9) *with constant coefficients.*

It is obvious that if (1) is complete with respect to $\varphi(x)$, the set (1) is also complete with respect to $c\varphi(x)$, where c is an arbitrary constant.

Now suppose that the set (1) is complete with respect to the functions $\varphi(x)$ and $\psi(x)$, that is

$$\int_a^b \varphi^2\,dx = \sum_{i=1}^{\infty} \varphi_i^2; \qquad \int_a^b \psi^2\,dx = \sum_{i=1}^{\infty} \psi_i^2. \tag{10}$$

We set

$$\Phi(x) = \varphi(x)+\psi(x); \qquad \Psi(x) = \varphi(x)-\psi(x).$$

Then we have on the one hand:

$$\int_a^b \Phi^2\,dx+\int_a^b \Psi^2\,dx = 2\int_a^b \varphi^2\,dx+2\int_a^b \psi^2\,dx.$$

On the other hand

$$\sum_{i=1}^{\infty} \Phi_i^2+\sum_{i=1}^{\infty} \Psi_i^2 = \sum_{i=1}^{\infty}(\varphi_i+\psi_i)^2+\sum_{i=1}^{\infty}(\varphi_i-\psi_i)^2 = 2\sum_{i=1}^{\infty}\varphi_i^2+2\sum_{i=1}^{\infty}\psi_i^2.$$

Hence, in virtue of (10):

$$\sum_{i=1}^{\infty} \Phi_i^2+\sum_{i=1}^{\infty} \Psi_i^2 = \int_a^b \Phi^2\,dx+\int_a^b \Psi^2\,dx.$$

By Bessel's inequality each term of the left hand member cannot exceed the corresponding term of the right hand member, whence it follows that

$$\int_a^b \Phi^2\,dx = \sum_{i=1}^{\infty} \Phi_i^2; \qquad \int_a^b \Psi^2\,dx = \sum_{i=1}^{\infty} \Psi_i^2.$$

An easy application of the method of complete induction will achieve the proof of our statement.

Step 2. We designate as a general step function any single-valued function whose graph in (a, b) consists of finite number of rectilinear segments parallel to the x-axis, the values of the function at the points of discontinuity being of no importance. It is evident that every general step function can be represented as a linear combination of the elementary step functions. Hence:

If the (1) *set is complete with respect to every elementary step function, the set* (1) *is also complete with respect to every general step function.*

Step 3. *Any function in* (L_0) *can be approximated by means of a general step function so that the quadratic error of the approximation is less than any preassigned positive constant, i. e.* $\varepsilon > 0$ *being given, it is always possible to find a step function* $\varphi(x)$ *such that*

$$\delta(f, \varphi) = \int_a^b \{f(x) - \varphi(x)\}^2 \, dx < \varepsilon. \tag{11}$$

This fact, which is trivial if the integral is in the sense of Riemann, is one of the fundamental properties of the integral of Lebesgue.*

Step. 4. *If* $f(x)$ *and* $\varphi(x)$ *are any functions of the class* (L_2), *then*

$$\varepsilon_n(f) \leqq \sqrt{\int_a^b f^2 \, dx} \left\{ \sqrt{\varepsilon_n(\varphi)} + \sqrt{\delta(f, \varphi)} \right\}. \dagger \tag{12}$$

This may be established as follows: let

$$f(x) = \sum_{i=1}^{n} f_i \, u_i(x) + r^{(n)}(x), \tag{13}$$

$$\varphi(x) = \sum_{i=1}^{n} \varphi_i \, u_i(x) + \varrho^{(n)}(x). \tag{14}$$

Then we have

$$r_j^{(n)} = \int_a^b \left\{ f(x) - \sum_{i=1}^{n} f_i \, u_i(x) \right\} u_j(x) \, dx = f_j - f_j = 0 \qquad (j = 1, 2, \cdots, n),$$

and also

$$\varrho_j^{(n)} = 0 \qquad (j = 1, 2, \cdots, n).$$

Multiplying (13) by $r^{(n)}(x)$ and integrating we obtain

$$\varepsilon_n(f) = \int_a^b \left\{ f(x) - \sum_{i=1}^{n} f_i \, u_i(x) \right\}^2 dx = \int_a^b r^{(n)}(x)^2 \, dx = \int_a^b f(x) \, r^{(n)}(x) \, dx,$$

* E. Hobson, The Theory of Functions of a Real Variable, 2nd. ed., vol I (Cambr. 1921), articles 430, 434.

† This inequality and the following proof have been given by W. Stekloff; see, for instance, Sur la théorie de fermeture des systèmes de fonctions orthogonales, Mém. de l'Acad. Impér. des Sciences de St. Pétersbourg, vol. 30, no. 4 (1911).

and in the same way

$$\varepsilon_n(\varphi) = \int_a^b \varrho^{(n)}(x)^2\,dx = \int_a^b \varphi(x)\,\varrho^{(n)}(x)\,dx.$$

Finally, multiplying (13) by $\varphi(x)$, (14) by $f(x)$, integrating and subtracting we have

$$\int_a^b \varphi(x)\,r^{(n)}(x)\,dx = \int_a^b f(x)\,\varrho^{(n)}(x)\,dx.$$

Now we can write

$$\begin{aligned}\varepsilon_n(f) &= \int_a^b f(x)\,r^{(n)}(x)\,dx = \int_a^b \varphi(x)\,r^{(n)}(x)\,dx + \int_a^b \{f(x)-\varphi(x)\}\,r^{(n)}(x)\,dx \\ &= \int_a^b f(x)\,\varrho^{(n)}(x)\,dx + \int_a^b \{f(x)-\varphi(x)\}\,r^{(n)}(x)\,dx.\end{aligned} \tag{15}$$

A simple application of the inequality of Bouniakovsky-Schwarz shows:

$$\begin{aligned}\left|\int_a^b f(x)\,\varrho^{(n)}(x)\,dx\right| &\leqq \sqrt{\int_a^b f^2\,dx \cdot \int_a^b \varrho^{(n)}(x)^2\,dx} \\ &= \sqrt{\varepsilon_n(\varphi)}\sqrt{\int_a^b f^2\,dx};\end{aligned} \tag{16}$$

$$\begin{aligned}\left|\int_a^b \{f(x)-\varphi(x)\}\,r^{(n)}(x)\,dx\right| &\leqq \sqrt{\int_a^b \{f(x)-\varphi(x)\}^2\,dx \int_a^b r^{(n)}(x)^2\,dx} \\ &\leqq \sqrt{\delta(f,\varphi)}\sqrt{\int_a^b f^2\,dx},\end{aligned} \tag{17}$$

since

$$\int_a^b r^{(n)}(x)^2\,dx = \int_a^b f^2\,dx - \sum_{i=1}^{n} f_i^2 \leqq \int_a^b f^2\,dx.$$

The formulas (15), (16), (17) prove the statement 4.

Theorem 1 follows immediately from the statements 2, 3, 4. Thus, if a positive number ε is given, we first choose a step function $\varphi(x)$ so that (step 3)

$$\delta(f,\varphi) < \left(\frac{\varepsilon}{2A}\right)^2, \qquad \text{where } A = \sqrt{\int_a^b f^2\,dx}.$$

The function $\varphi(x)$ being determined, we choose n so large that (step 2)

$$\varepsilon_n(\varphi) < \left(\frac{\varepsilon}{2A}\right)^2.$$

Then (step 4): $\varepsilon_n(f) < \varepsilon$, which proves Theorem 1.

3. We shall call the set (1) complete, if it is complete with respect to any function $f(x)$ in (L_2).

THEOREM 2. *The property of completeness of the set* (1) *is equivalent to the property that the formal Fourier's series of any function* $f(x)$ *in* (L_2) *may be integrated term by term, i. e.*

$$\int_{\xi_1}^{\xi_2} f(x)\,dx = \sum_{i=1}^{\infty} f_i \int_{\xi_1}^{\xi_2} u_i(x)\,dx, \tag{18}$$

where ξ_1, ξ_2 *are any numbers in* (a, b).*

To begin with, we can replace the equation of completeness by the more general one:

$$\sum_{i=1}^{\infty} f_i \varphi_i = \int_a^b f(x)\,\varphi(x)\,dx, \tag{19}$$

where $f(x)$ and $\varphi(x)$ are any functions in (L_2). This follows immediately from the equation (7) if we replace there $f(x)$ by $\alpha f(x) + \beta\varphi(x)$, α and β being arbitrary constants.

If now, the set (1) is complete, we deduce (18) from (19), if we set:

$$\varphi(x) = \varphi_{\xi_1, \xi_2}(x).$$

Suppose conversely that (18) is satisfied by any function $f(x)$ in (L_2). If we set in (18):

$$f(x) = \varphi_{\xi_1, \xi_2}(x),$$

we get

$$\begin{aligned} \xi_2 - \xi_1 = \int_{\xi_1}^{\xi_2} f(x)\,dx &= \int_a^b [\varphi_{\xi_1, \xi_2}(x)]^2\,dx \\ &= \sum_{i=1}^{\infty} \left[\int_a^b \varphi_{\xi_1, \xi_2}(x)\,u_i(x)\,dx\right]^2, \end{aligned} \tag{20}$$

which is simply the equation of completeness for the set (1) with respect to the elementary step function $\varphi_{\xi_1, \xi_2}(x)$. Hence theorem 2 follows from the theorem 1.

The formula (20) proves the following proposition:

THEOREM 3. *The identity*

$$\xi_2 - \xi_1 = \sum_{i=1}^{\infty} \left[\int_{\xi_1}^{\xi_2} u_i(x)\,dx\right]^2 \tag{21}$$

is a necessary and sufficient condition for the completeness of the set (1).

* This result was obtained by different method by C. Severini, Sulla teoria die chiusura dei sistemi di funzioni ortogonali, Rend. del Circ. Mat. di Palermo, 36 (1913), p. 182.

Theorem 2 leads immediately to the theorem of expansion:

THEOREM 4. *If the set* (1) *is complete, the Fourier's series*

$$\sum_{i=0}^{\infty} f_i u_i(x) \tag{22}$$

of the function $f(x)$ *represents the function* $f(x)$ *on every interval, on which it converges uniformly.**

4. Let us return now to the set of trigonometric functions

$$1,\ \cos x,\ \sin x,\ \cdots;\quad \cos nx,\ \sin nx,\ \cdots. \tag{23}$$

This set is orthogonal but not normalized on $(0,2\pi)$. Therefore the equation of completeness (7) and the identity (21) assume respectively the forms:

$$\frac{1}{\pi}\int_0^{2\pi} f^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

$$\frac{\xi_2-\xi_1}{\pi} = \frac{(\xi_2-\xi_1)^2}{2\pi^2} + \sum_{n=1}^{\infty}\frac{2-2\cos n(\xi_2-\xi_1)}{\pi^2 n^2}.$$

Setting $\xi_2-\xi_1 = x$, we get the equality

$$\sum_{n=1}^{\infty}\frac{1-\cos nx}{n^2} = \frac{\pi x}{2} - \frac{x^2}{4}, \tag{24}$$

as a necessary and sufficient condition for the completeness of the set (23). Now, the formula (24) which is equivalent to

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\pi - x}{2} \qquad (0<x<2\pi)$$

is well known from the times of Euler, and can be proved by elementary methods, for instance, by considering the expansion of the function $\log(1+z)$ in power series on the circle $|z| = 1$.†

* This statement contains more than the usual proposition: If the set (1) is closed (i. e. $f_i = 0$ $(i = 1, 2, \cdots)$ implies $f(x) \equiv 0$, the series (22) represents $f(x)$, provided it converges uniformly on (a, b).

† Cfr. also A. Kneser, Die Integralgleichungen, (2nd ed. 1922), pp. 15–17.

DARTMOUTH COLLEGE,
February 1926.

TENSORS WHOSE COMPONENTS ARE ABSOLUTE CONSTANTS.

By Tracy Yerkes Thomas.*

On looking over some of my notes, written in Zurich over a year ago, I find a little result concerning the determination of tensors C whose components are absolute constants, i. e. the same in all coördinate systems, which I would like to present in the following paper.

By hypothesis the tensor C transforms by the equations

$$(1) \qquad C^{rs\cdots w}_{\alpha\beta\cdots\lambda}\frac{\partial x^i}{\partial \bar{x}^r}\frac{\partial x^j}{\partial \bar{x}^s}\cdots\frac{\partial x^t}{\partial \bar{x}^w} = C^{ij\cdots t}_{\mu\nu\cdots\zeta}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\cdots\frac{\partial x^\zeta}{\partial \bar{x}^\lambda}$$

under a transformation $x = f(\bar{x})$, and hence under the one parameter family of transformations

$$(2) \qquad x^i = \bar{x}^i + \varepsilon\,\xi^i(\bar{x}^1, \cdots, \bar{x}^n)$$

where each ξ^i is an arbitrary function of the arguments $(\bar{x}^1, \cdots, \bar{x}^n)$. From (2) we have

$$\frac{\partial x^i}{\partial \bar{x}^r} = \delta^i_r + \varepsilon\frac{\partial \xi^i}{\partial \bar{x}^r}.$$

We shall assume that the tensor C involves p contravariant indices and q covariant indices. Then when we substitute these values of the derivatives into (1) we obtain an expression in ε and its powers up to the greater of the quantities (p, q). Each of the coefficients in this expression must vanish identically but it will be sufficient to consider the term in ε which leads to the equations†

$$(3) \qquad C^{\sigma j\cdots t}_{\alpha\beta\cdots\lambda}\delta^i_\tau + \cdots + C^{ij\cdots\sigma}_{\alpha\beta\cdots\lambda}\delta^t_\tau = C^{ij\cdots t}_{\tau\beta\cdots\lambda}\delta^\sigma_\alpha + \cdots + C^{ij\cdots t}_{\alpha\beta\cdots\tau}\delta^\sigma_\lambda.$$

If we put $\sigma = \tau$ in (3) and sum over the values of these indices we find immediately that

$$(4) \qquad \boxed{C^{ij\cdots t}_{\alpha\beta\cdots\lambda} = 0;\ \ p \neq q}.$$

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† For a treatment of infinitesimal transformations of tensors see a paper by T. Y. Thomas and A. D. Michal on the differential invariants of affinely connected manifolds to appear in these Annals.

In case $p = q$ we observe that a solution of (3) is given by

$$\delta^i_\alpha \, \delta^j_\beta \cdots \delta^t_\lambda. \tag{5}$$

This suggests that the general solution of (3) is

$$\Phi^{ij\cdots t}_{\alpha\beta\cdots\lambda} = A_1 \, \delta^i_\alpha \, \delta^j_\beta \cdots \delta^t_\lambda + \cdots + A_N \, \delta^h_\alpha \, \delta^i_\beta \cdots \delta^w_\lambda \tag{6}$$

which is formed by taking the sum of all the different expressions (5) consisting of all permutations $(h\,l\cdots w)$ of the indices $(i\,j\cdots t)$ when each has been multiplied by an arbitrary constant A_k. The total number of terms comprising (6) is

$$N = p! = q!$$

The proof that Φ is the most general solution of (3) may be made very readily by the method of induction. Assuming that any solution of (3) is of the form Φ when the upper and lower indices of C are $p-1 = q-1$ in number, we put $i = \tau$; $j = \tau$; $\cdots$; $t = \tau$ successively in the equations (3) and sum over all values of these indices. We then have, after a slight "relettering" has been made, the following system of equations

$$\left\{\begin{aligned}
&n C^{ij\cdots t}_{\alpha\beta\cdots\lambda} + C^{ji\cdots t}_{\alpha\beta\cdots\lambda} + \cdots + C^{tj\cdots i}_{\alpha\beta\cdots\lambda} = \Phi_1 \\
&C^{ij\cdots t}_{\alpha\beta\cdots\lambda} + n C^{ji\cdots t}_{\alpha\beta\cdots\lambda} + \cdots + C^{tj\cdots i}_{\alpha\beta\cdots\lambda} = \Phi_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&C^{ij\cdots t}_{\alpha\beta\cdots\lambda} + C^{ji\cdots t}_{\alpha\beta\cdots\lambda} + \cdots + n C^{tj\cdots i}_{\alpha\beta\cdots\lambda} = \Phi_p
\end{aligned}\right. \tag{7}$$

in which the quantities Φ_i in the right members denote particular expressions of the form Φ. Adding all equations of the system (7) we obtain

$$C^{ij\cdots t}_{\alpha\beta\cdots\lambda} + C^{ji\cdots t}_{\alpha\beta\cdots\lambda} + \cdots + C^{tj\cdots i}_{\alpha\beta\cdots\lambda} = \Psi \tag{8}$$

where Ψ is again of the form Φ. Then subtracting (8) from the first set of equations (7) we have as a consequence of our assumption that

$$\boxed{C^{ij\cdots t}_{\alpha\beta\cdots\lambda} = \Phi; \; p = q} \tag{9}$$

where Φ denotes an expression of the general form (6).

It remains to show that C is of the form (6) for the particular case $p = q = 1$. Here the equations (1) become

$$C^r_\alpha \frac{\partial x^i}{\partial \bar{x}^r} = C^i_\mu \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \tag{10}$$

and give

$$C^r_\alpha \, \delta^i_u \, \delta^v_r = C^i_\mu \, \delta^\mu_u \, \delta^v_\alpha \tag{11}$$

on account of the arbitrariness of the quantities $\partial x/\partial \bar{x}$ which appear in them. Putting $i = u$ in (11) we see that C^i_α has the form

$$C^i_\alpha = A \delta^i_\alpha; \quad A = \text{const.}$$

which completes the proof.

We may therefore say that *all tensors whose components are absolute constants are given either by* (4) *or by* (9). This completes the present note.

PRINCETON UNIVERSITY.

SOME CONVERGENCE PROOFS IN THE VECTOR ANALYSIS OF FUNCTION SPACE.*

By Dunham Jackson.

1. **Introduction.** In the space of ordinary geometrical experience, as well as in space of any finite number of dimensions, the formulas of transformation from one system of rectangular coördinate axes to another system having the same origin involve certain sums of N terms each, if N is the number of dimensions. A familiar notion of recent years is the interpretation of the theory of orthogonal systems of functions as a geometry with infinitely many coördinates. In this geometry a rotation of axes is naturally expressed by means of infinite series, instead of finite sums. For the discussion of the convergence of these series, and others arising out of them, one method, ultimately no doubt the most scientific method, is to set up hypotheses describing as closely as possible exactly those properties that are significant in the demonstrations. But as an introductory exercise there may be some interest in a study of special cases, serving in a way to realize concretely relations which in the abstract have a far wider range of validity. Such a special study is the object of this paper. A presentation of some theorems of convergence will be accompanied by an indication of their bearing on the definition of such concepts as those of gradient, divergence, and curl, in the vector analysis of function space.

2. **Properties of a set of orthogonal functions.** Let $\rho(x)$ be a continuous function of period 2π, which is everywhere positive. Let a succession of trigonometric sums $U_n(x)$, $n = 1, 2, \cdots$, be determined so that

$$\int_0^{2\pi} \rho(x)\, U_m(x)\, U_n(x)\, dx = 0 \quad (m \neq n), \qquad \int_0^{2\pi} \rho(x)\, [U_n(x)]^2\, dx = 1,$$

the order of $U_n(x)$ being in each case equal to the greatest integer contained in $\frac{1}{2}n$. The calculation of these sums is equivalent to the formation of a normalized orthogonal sequence of linear combinations of the functions

$$\sqrt{\rho(x)}, \qquad \sqrt{\rho(x)}\cos x, \qquad \sqrt{\rho(x)}\sin x, \qquad \sqrt{\rho(x)}\cos 2x, \cdots.$$

* Presented to the American Mathematical Society, under other titles, October 25, 1924, and April 11 and December 29, 1925; received in final form March 1926.

Let $u_n(x) = \sqrt{\varrho(x)}\, U_n(x)$; then

$$(1) \qquad \int_0^{2\pi} u_m(x)\, u_n(x)\, dx = 0 \quad (m \neq n), \qquad \int_0^{2\pi} [u_n(x)]^2\, dx = 1.$$

The function $\varrho(x)$, being continuous and everywhere positive, has a positive minimum value g. From the second of the equations (1) and the fact that $1/[\varrho(x)] \leqq 1/g$, it follows that

$$\int_0^{2\pi} [U_n(x)]^2\, dx = \int_0^{2\pi} \frac{1}{\varrho(x)} [u_n(x)]^2\, dx \leqq \frac{1}{g}.$$

Hence furthermore, since the integrand is always positive or zero,

$$\int_0^{x} [U_n(x)]^2\, dx \leqq \frac{1}{g}$$

for $0 \leqq x \leqq 2\pi$. Let δ_n be the constant term of the trigonometric sum $[U_n(x)]^2$, so that

$$[U_n(x)]^2 = \delta_n + \tau_n(x),$$

where $\tau_n(x)$ is a trigonometric sum of order n (at most) without constant term. From the fact that

$$\int_0^{2\pi} [U_n(x)]^2\, dx = 2\pi\, \delta_n$$

it is recognized that $0 \leqq \delta_n \leqq 1/(2\pi g)$, for all values of n. Since

$$\int_0^{x} [U_n(x)]^2\, dx = \delta_n x + \int_0^{x} \tau_n(x)\, dx,$$

it follows that

$$\left| \int_0^{x} \tau_n(x)\, dx \right| \leqq \delta_n x + \int_0^{x} [U_n(x)]^2\, dx \leqq \frac{2}{g}$$

for $0 \leqq x \leqq 2\pi$. But the quantity between bars is itself a trigonometric sum of the nth order, having $\tau_n(x)$ for its derivative. By an application of Bernstein's theorem, therefore, $|\tau_n(x)| \leqq 2n/g$, and

$$[U_n(x)]^2 \leqq \frac{1}{2\pi g} + \frac{2n}{g} \leqq C_1 n, \qquad |U_n(x)| \leqq \sqrt{C_1 n},$$

where C_1 is a quantity independent of n. The maximum of $\varrho(x)$ being denoted by G,

$$|u_n(x)| = |\sqrt{\varrho(x)}\, U_n(x)| \leqq \sqrt{GC_1}\, n,$$

or, if $\sqrt{GC_1} = C_2$, a number independent of n,

$$|u_n(x)| \leqq C_2 \sqrt{n}$$

for all values of n and x.

3. **Theorems on the development of given functions in series of orthogonal functions.** Let $f(x)$ be a given continuous function of period 2π. The following facts are well known, and readily verified:

Among all expressions of the form

$$\varphi_n(x) = \sum_{k=1}^{n} a_k u_k(x),$$

for a given value of n, there is just one for which the integral

$$\int_0^{2\pi} [f(x) - \varphi_n(x)]^2\, dx$$

is a minimum, the minimizing coefficients being given by the formula

$$a_k = \int_0^{2\pi} f(t)\, u_k(t)\, dt. \tag{2}$$

When the coefficients are thus determined, the corresponding sum $\varphi_n(x)$ being denoted by $s_n(x)$,

$$\int_0^{2\pi} [f(x) - s_n(x)]^2\, dx = \int_0^{2\pi} [f(x)]^2\, dx - \sum_{k=1}^{n} a_k^2,$$

from which it follows (since the left-hand member is non-negative) that

$$\sum_{k=1}^{n} a_k^2 \leqq \int_0^{2\pi} [f(x)]^2\, dx$$

for all values of n, and that the infinite series $\sum a_k^2$ is convergent.

The sum of the series is equal to the integral on the right. For

$$\int_0^{2\pi} [f(x) - s_n(x)]^2\, dx = \int_0^{2\pi} \varrho(x) \left[\frac{f(x)}{\sqrt{\varrho(x)}} - S_n(x)\right]^2 dx,$$

where

$$S_n(x) = \frac{s_n(x)}{\sqrt{\varrho(x)}} = \sum_{k=1}^{n} a_k U_k(x)$$

is a trigonometric sum; by Weierstrass's theorem, applied to the continuous function $F(x) = f(x)/\sqrt{\varrho(x)}$, there are trigonometric sums $T_n(x)$ for which $F(x) - T_n(x)$ is uniformly small, so that

$$\int_0^{2\pi} \varrho(x)\,[F(x) - T_n(x)]^2\,dx$$

approaches zero with increasing n; and the corresponding integral with $S_n(x)$ in place of $T_n(x)$ will approach zero with equal or greater rapidity, by reason of the minimizing property of the coefficients that define $S_n(x)$.

If $g(x)$ is another function satisfying the conditions imposed on $f(x)$, and

$$\sigma_n(x) = \sum_{k=1}^{n} b_k u_k(x)$$

the corresponding linear combination of the u's,

$$\int_0^{2\pi} [f(x) - s_n(x)]\,[g(x) - \sigma_n(x)]\,dx = \int_0^{2\pi} f(x)g(x)\,dx - \sum_{k=1}^{n} a_k b_k;$$

by Schwarz's inequality,

$$\left[\int_0^{2\pi} [f(x) - s_n(x)]\,[g(x) - \sigma_n(x)]\,dx\right]^2 \leqq \int_0^{2\pi} [f(x) - s_n(x)]^2\,dx \cdot \int_0^{2\pi} [g(x) - \sigma_n(x)]^2\,dx;$$

as each factor on the right approaches zero when n increases indefinitely, the left-hand member does the same, and

$$\sum_{k=1}^{\infty} a_k b_k = \int_0^{2\pi} f(x)g(x)\,dx.$$

Finally, if the series $\sum a_k u_k(x)$ for a given function $f(x)$ is uniformly convergent, it has $f(x)$ for its sum. For if the sum of the series is $\psi(x)$, and if $f(x) - \psi(x) = \omega(x)$, then

$$\gamma_k = \int_0^{2\pi} \omega(x)\,u_k(x)\,dx = 0, \qquad \int_0^{2\pi} [\omega(x)]^2\,dx = \sum_{k=1}^{\infty} \gamma_k^2 = 0,$$

and hence $\omega(x) = 0$ identically.

4. An inequality for the coefficients. Let it be assumed henceforth, unless otherwise stated, that the functions $\varrho(x)$ and $f(x)$ have continuous second derivatives for all values of x. Let M be a common upper bound for $|f(x)|$, $|f'(x)|$, and $|f''(x)|$, and, if $\varrho_1(x) = 1/\sqrt{\varrho(x)}$, let G_1 be a common upper bound for $\varrho_1(x)$, $|\varrho_1'(x)|$, and $|\varrho_1''(x)|$. Let $F(x)$ have the same meaning as in the preceding section. Then

$$|F''(x)| = \left|\frac{d^2}{dx^2}[f(x)\varrho_1(x)]\right| = |f''\varrho_1 + 2f'\varrho_1' + f\varrho_1''| \leqq 4G_1M.$$

When n is a positive even integer, let $m = \frac{1}{2}n-1$; and when n is odd, let $m = \frac{1}{2}(n-1)$. Since $F'(x)$ satisfies a Lipschitz condition with coefficient $4G_1M$, there exists for each positive integral value of m (and so for $n \geqq 3$) a trigonometric sum $T_n(x)$, of the mth order at most, such that

$$|F(x) - T_n(x)| \leqq \frac{KG_1M}{m^2} \leqq \frac{C_3M}{n^2},$$

where* K is an absolute constant (independent of m, n, x, $f(x)$, and $\varrho(x)$); in the last member, which is explicit enough for the following work, C_3 is independent of $f(x)$, though it depends on $\varrho(x)$ through the factor G_1.

The number of terms in $T_n(x)$, when sines and cosines are counted separately, is at most $2m+1 \leqq n$. Each of these terms can be expressed as a linear combination of some or all of the functions $U_k(x)$, $k = 1, 2, \cdots, n$. So $t_n(x) = \sqrt{\varrho(x)}\, T_n(x)$ is a linear combination of $u_1(x), \cdots, u_n(x)$, and

$$\int_0^{2\pi}[f(x) - t_n(x)]^2\,dx = \int_0^{2\pi}\varrho(x)[F(x) - T_n(x)]^2\,dx \leqq \frac{C_4M^2}{n^4},$$

where C_4, like the other C's to be used below, is independent of $f(x)$, as well as of n and x. By the minimizing property of the coefficients in $s_n(x)$ it follows then that

$$\int_0^{2\pi}[f(x) - s_n(x)]^2\,dx \leqq \frac{C_4M^2}{n^4}.$$

In consequence of the relations cited in the preceding section,

$$\int_0^{2\pi}[f(x) - s_n(x)]^2\,dx = \sum_{k=1}^{\infty}a_k^2 - \sum_{k=1}^{n}a_k^2 = \sum_{k=n+1}^{\infty}a_k^2.$$

* Cf., e. g., D. Jackson, On approximation by trigonometric sums and polynomials, Trans. Amer. Math. Soc., vol. 13 (1912), pp. 491–515; see p. 496, Theorem III.

Hence

$$a_{n+1}^2 \leqq \sum_{k=n+1}^{\infty} a_k^2 \leqq \frac{C_4 M^2}{n^4}, \qquad |a_{n+1}| \leqq \frac{\sqrt{C_4}\, M}{n^2} \leqq \frac{C_5 M}{(n+1)^2}.$$

The argument has involved a supposition that $n \geqq 3$, $n+1 \geqq 4$. But it is clear from the definition (2) that a relation of similar form, possibly with a different factor in place of C_5, can be written down for a_1, a_2 and a_3 (and could have been written down immediately, without reference to the reasoning of the present section, for any finite number of the coefficients). So it can be asserted that *there is a constant C, independent of $f(x)$, though depending on $\varrho(x)$, such that*

$$|a_n| \leqq \frac{CM}{n^2}$$

for all values of n from $n = 1$ on.

With the concluding result of § 2, this implies incidentally that $|a_n u_n(x)| \leqq C_6 M/n^{3/2}$, and hence that* *the series $\sum a_k u_k(x)$ converges uniformly to the value $f(x)$.*

5. **Equations of transformation connecting two orthogonal systems.** Let $\sigma(x)$ be another function which, like $\varrho(x)$, has the period 2π, a continuous second derivative, and a positive minimum. Let $v_1(x), v_2(x), \cdots$, be the normalized orthogonal system corresponding to $\sigma(x)$, as the system $u_1(x), u_2(x), \cdots$, corresponds to $\varrho(x)$. Each function $v_k(x)$, being equal to a trigonometric sum multiplied by $\sqrt{\sigma(x)}$, has a continuous second derivative, and therefore can be expanded in a uniformly convergent series of the form

$$v_k(x) = \sum_i p_{ki} u_i(x). \tag{3}$$

For each value of k, in consequence of the facts summarized in § 3,

$$\sum_i p_{ki}^2 = \int_0^{2\pi} [v_k(x)]^2 \, dx = 1,$$

and if $k \neq l$

$$\sum_i p_{ki} p_{li} = \int_0^{2\pi} v_k(x) \, v_l(x) \, dx = 0.$$

* Cf. D. Jackson, Note on the convergence of weighted trigonometic series, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 259–263; W. Stekloff, Sur une application de la théorie de fermeture au problème du développement des fonctions arbitraires en séries procédant suivant les polynomes de Tchébicheff, Mémoires de l'Acad. Imp. des Sci. Petrograd, ser. 8, Classe phys. math., vol. 33 (1914), No. 8.

Similarly, expansion of the u's in terms of the v's leads to relations of the form

$$(4)\quad u_i(x) = \sum_k q_{ik} v_k(x), \qquad \sum_k q_{ik}^2 = 1, \qquad \sum_k q_{ik} q_{jk} = 0 \qquad (i \neq j).$$

Here the coefficient q_{ik} in the expansion of $u_i(x)$ in terms of the v's is given by the formula

$$q_{ik} = \int_0^{2\pi} u_i(x)\, v_k(x)\, dx;$$

as this integral is the same as the one that defines p_{ki},

$$q_{ik} = p_{ki},$$

and the last two equations of (4) can be rewritten in the form

$$(5)\quad \sum_k p_{ki}^2 = 1, \qquad \sum_k p_{ki} p_{kj} = 0 \qquad (i \neq j).$$

There is of course nothing new in the form of these relations; the point is that their validity is assured by the hypotheses that have been adopted.

6. **Equations of transformation for vectors and for tensors of the second rank.** Let $f(x)$ be a continuous function of period 2π (the hypothesis with regard to the existence of $f''(x)$ being for the moment superfluous), and let

$$a_i = \int_0^{2\pi} f(x)\, u_i(x)\, dx, \qquad b_k = \int_0^{2\pi} f(x)\, v_k(x)\, dx.$$

Since the series expressing $v_k(x)$ in terms of the u's is uniformly convergent, and remains so after multiplication by $f(x)$,

$$(6)\quad b_k = \int_0^{2\pi} f(x) \left[\sum_i p_{ki} u_i(x)\right] dx = \sum_i p_{ki} a_i.$$

The function $f(x)$ may be regarded as constituting a vector in function space; the equations (3), for the successive values of k, are the equations of transformation of a fundamental system of unit vectors; the a's and the b's measure the components of $f(x)$, as resolved according to the two fundamental systems; and the identity of the coefficients in (6) with those in (3) shows that the components of a (more or less) arbitrary vector are transformed like the constituent vectors of a fundamental system, as would be expected by analogy from space of a finite number of dimensions.

As a result of § 4, applied to the expansion (3), $|p_{ki}| \leqq c_k/i^2$, where c_k is independent of i. By § 2, $|u_i(x)| \leqq C_2\sqrt{i}$, so that $|p_{ki}u_i(x)| \leqq C_2 c_k/i^{3/2}$. Hence the series

$$v_k(x) = \sum_i p_{ki}u_i(x), \qquad v_l(y) = \sum_j p_{lj}u_j(y),$$

are absolutely convergent, and can be multiplied together:

$$v_k(x)v_l(y) = \sum_i \sum_j p_{ki}p_{lj}u_i(x)u_j(y). \tag{7}$$

For the sake of an application to be made presently, it is to be noted that the double series is uniformly as well as absolutely convergent. The identity (7), considered for $k = 1, 2, \cdots$, $l = 1, 2, \cdots$, may be regarded as defining the equations of transformation of a tensor of the second rank.

Let $f(x, y)$ now be a function of two variables, continuous with regard to both together, and of period 2π in each. Let

$$a_{ij} = \int_0^{2\pi}\int_0^{2\pi} f(x, y)u_i(x)u_j(y)\,dx\,dy,$$

$$b_{kl} = \int_0^{2\pi}\int_0^{2\pi} f(x, y)v_k(x)v_l(y)\,dx\,dy.$$

If the integral defining b_{kl} is evaluated by means of (7), it is found that

$$b_{kl} = \sum_i \sum_j p_{ki}p_{lj}a_{ij},$$

which may be taken to mean that the doubly infinite matrix of the quantities a_{ij} has tensor character.

It is to be shown below that certain differential operators follow the same equations of transformation.

7. **Analogue of the curl.** As a first illustration, let

$$\Phi(x) = \varphi(y), \qquad y = f(x), \tag{8}$$

where $f(x)$ is a function of period 2π, with a continuous second derivative, and $\varphi(y)$ has a continuous first derivative for all values of y. A function of x may be typified in function space either by a point or by a vector. It is convenient here to think of $f(x)$ as a point, and of $\Phi(x)$ as a vector. Then the definition (8) makes Φ a vector point function in function space.

Let

$$a_i = \int_0^{2\pi} f(x)u_i(x)\,dx, \qquad b_k = \int_0^{2\pi} f(x)v_k(x)\,dx,$$

$$A_i = \int_0^{2\pi} \Phi(x)u_i(x)\,dx, \qquad B_k = \int_0^{2\pi} \Phi(x)v_k(x)\,dx. \tag{9}$$

It is known that

$$f(x) = \sum_i a_i\, u_i(x) = \sum_k b_k\, v_k(x),$$

the series being uniformly convergent. Through the dependence of Φ on $f(x)$, each of the quantities A_i, B_k is a function of the infinitely many variables $a_1, a_2, \cdots$, and likewise of the set of variables $b_1, b_2, \cdots$. Such a partial derivative as $\partial A_i/\partial a_j$ can be interpreted immediately, without further reference to the technical theory of functions of infinitely many variables. The requirement of the continuity of $f''(x)$, to be sure, restricts the a's and the b's to an extent that is not readily specified in terms of these variables themselves. (The necessary condition $|a_i| \leqq CM/i^2$, of § 4, is not sufficient.) But if the a's have been given a set of admissible values, the replacement of a_j by $a_j + \Delta a_j$, for the sake of defining a corresponding partial derivative, does not impair the continuity of the second derivative of the sum of the series, as it has the effect merely of adding the single term $\Delta a_j\, u_j(x)$. In fact, when $x, a_1, a_2, \cdots$, are regarded as independent variables, and similarly $x, b_1, b_2, \cdots$, it is recognized at once that

$$\frac{\partial}{\partial a_j} f(x) = u_j(x), \qquad \frac{\partial}{\partial b_l} f(x) = v_l(x),$$

$$\frac{\partial}{\partial a_j} \Phi(x) = \varphi'(y)\, u_j(x), \qquad \frac{\partial}{\partial b_l} \Phi(x) = \varphi'(y)\, v_l(x).$$

In the integral defining A_i, furthermore, the conditions for differentiating with regard to a_j under the sign of integration are satisfied, the integrand being for the purpose in hand essentially a function of the two variables a_j and x. The result is that

$$(10) \qquad \frac{\partial A_i}{\partial a_j} = \int_0^{2\pi} u_i(x)\, \frac{\partial}{\partial a_j} \Phi(x)\, dx = \int_0^{2\pi} \varphi'[f(x)]\, u_i(x)\, u_j(x)\, dx.$$

Similarly,

$$(11) \qquad \frac{\partial B_k}{\partial b_l} = \int_0^{2\pi} \varphi'[f(x)]\, v_k(x)\, v_l(x)\, dx.$$

Finally, $v_k\, v_l$ has the uniformly convergent expansion

$$v_k(x)\, v_l(x) = \sum_i \sum_j p_{ki}\, p_{lj}\, u_i(x)\, u_j(x).$$

On substitution in (11) and comparison with (10), this gives

$$(12) \qquad \frac{\partial B_k}{\partial b_l} = \sum_i \sum_j p_{ki}\, p_{lj}\, \frac{\partial A_i}{\partial a_j}.$$

The matrix of the quantities $\partial A_i/\partial a_j$ thus has the character of a tensor.

A similar calculation can be carried through if the definition of Φ is replaced by

$$\Phi(x) = \varphi(y') = \varphi[f'(x)],$$

the function φ having the same character as before, while $\varrho(x)$, $\sigma(x)$, and $f(x)$ are supposed now to possess continuous *third* derivatives. In the first place, it can be shown by reasoning analogous to that of § 4, and by reference to a corresponding theorem on trigonometric approximation, that $|a_n| \leqq k_1/n^3$, where k_1 is independent of n. On the other hand, Bernstein's theorem, applied to the relation $|U_n(x)| \leqq \sqrt{C_1}\, n^{1/2}$, shows that $|U_n'(x)| \leqq \sqrt{C_1}\, n^{3/2}$, whence it follows further that $|u_n'(x)| \leqq k_2\, n^{3/2}$, $|a_n u_n'(x)| \leqq k_3/n^{3/2}$. So $f'(x)$ is represented by the uniformly convergent series $\sum_i a_i u_i'(x)$. In the same way, $f'(x) = \sum_k b_k v_k'(x)$. Also, the representation of $v_l(x)$ in the form $\sum_j p_{lj} u_j(x)$ may be differentiated term by term for each value of l, yielding a series which is absolutely-uniformly convergent in each case. Hence

$$v_k(x)\, v_l'(x) = \sum_i \sum_j p_{ki}\, p_{lj}\, u_i(x)\, u_j'(x),$$

still with uniform convergence of the series of absolute values. The calculation then follows the same lines as before, the relations being

$$\frac{\partial}{\partial a_j} f'(x) = u_j'(x), \qquad \frac{\partial}{\partial b_l} f'(x) = v_l'(x),$$

$$\frac{\partial}{\partial a_j} \Phi(x) = \varphi'(y')\, u_j'(x), \qquad \frac{\partial}{\partial b_l} \Phi(x) = \varphi'(y')\, v_l'(x),$$

$$\frac{\partial A_i}{\partial a_j} = \int_0^{2\pi} \varphi'[f'(x)]\, u_i(x)\, u_j'(x)\, dx,$$

$$\frac{\partial B_k}{\partial b_l} = \int_0^{2\pi} \varphi'[f'(x)]\, v_k(x)\, v_l'(x)\, dx = \sum_i \sum_j p_{ki}\, p_{lj} \frac{\partial A_i}{\partial a_j}.$$

If $\Phi(x)$, more generally, has the form

$$\Phi(x) = \varphi(x, y, y'), \qquad y = f(x),$$

where φ has a continuous derivative with regard to each argument, while $f(x)$, $\varrho(x)$, $\sigma(x)$ are as in the preceding paragraph, the changes introduced are illustrated by the relations

$$\frac{\partial}{\partial a_j} \Phi(x) = \varphi_y(x, y, y')\, u_j(x) + \varphi_{y'}(x, y, y')\, u_j'(x),$$

$$\frac{\partial A_i}{\partial a_j} = \int_0^{2\pi} \varphi_y(x, y, y')\, u_i(x)\, u_j(x)\, dx + \int_0^{2\pi} \varphi_{y'}(x, y, y')\, u_i(x)\, u_j'(x)\, dx,$$

and (12) is verified again.

It is clear that higher derivatives may be admitted without causing any new difficulty, provided that $\varrho(x)$, $\sigma(x)$, and $f(x)$ have a sufficient number of continuous derivatives.

A more elaborate example, designed for a purpose which will appear in the next section, is

$$\Phi(x) = \int_0^{2\pi} K(x, t)\, \varphi[f(t)]\, dt, \tag{13}$$

where the function φ has for the moment a continuous first derivative with regard to its argument, $f(t)$ is of period 2π and has a continuous second derivative, and $K(x, t)$, of period 2π with regard to each variable, is continuous in both together. The quantities a_i, b_k, A_i, B_k are defined again by the formulas (9), interpreted now in terms of the present definition of Φ. The relations

$$\begin{aligned} \frac{\partial}{\partial a_j}\Phi(x) &= \int_0^{2\pi} K(x, t)\, \varphi'[f(t)]\, u_j(t)\, dt, \\ \frac{\partial A_i}{\partial a_j} &= \int_0^{2\pi}\int_0^{2\pi} K(x, t)\, \varphi'[f(t)]\, u_i(x)\, u_j(t)\, dt\, dx, \end{aligned} \tag{14}$$

together with the corresponding expressions for $\partial\Phi/\partial b_l$ and $\partial B_k/\partial b_l$, and the uniformly convergent development

$$v_k(x)\, v_l(t) = \sum_i \sum_j p_{ki}\, p_{lj}\, u_i(x)\, u_j(t),$$

lead to (12) once more as the general equation of transformation.

The more general form

$$\Phi(x) = \int_0^{2\pi} K(x, t)\, \varphi(t, y, y')\, dt, \qquad y = f(t),$$

or a still more complicated one involving higher derivatives, can be discussed in the same way, if $f(x)$, $\varrho(x)$, and $\sigma(x)$ are sufficiently differentiable.

The recurring tensor equation for the transformation of $\partial A_i/\partial a_j$, though its verification depends each time on the explicit form of the vector point function, or functional, to which it is applied, indicates that the matrix of the partial derivatives has an invariant character that may reasonably be supposed to have some ulterior significance. It generalizes the curl of three-dimensional vector analysis. The analogy is still more apparent in the skew-symmetric tensor having $(\partial A_i/\partial a_j) - (\partial A_j/\partial a_i)$ as its general element. The transformation

$$\frac{\partial B_k}{\partial b_l} - \frac{\partial B_l}{\partial b_k} = \sum_i \sum_j p_{ki}\, p_{lj} \left(\frac{\partial A_i}{\partial a_j} - \frac{\partial A_j}{\partial a_i} \right)$$

may be deduced from (12) at once, by suitable manipulation of the subscripts.

8. **Analogue of the divergence.** The developments of the preceding section inevitably suggest consideration of the series

$$\sum_i \frac{\partial A_i}{\partial a_i}, \tag{15}$$

which results from "contraction" of the derivative tensor. Here the laying of the analytical foundation is at first not so facile. For if $\Phi(x)$ is taken identical with $f(x)$, to give the simplest imaginable non-trivial case, $\partial A_i/\partial a_i = 1$, and the series (15) is divergent. If $\Phi(x) = \varphi[f(x)]$,

$$\frac{\partial A_i}{\partial a_i} = \int_0^{2\pi} \varphi'[f(x)]\,[u_i(x)]^2\, dx,$$

which does not approach zero, in general, as i increases; when $\varrho(x) \equiv 1$, for example, the definition of the u's will naturally be so arranged that $u_i(x) = (1/\sqrt{\pi}) \cos jx$ or $(1/\sqrt{\pi}) \sin jx$, for $i > 1$, with a suitable relation between j and i, and then

$$\lim_{i=\infty} \frac{\partial A_i}{\partial a_i} = \frac{1}{2\pi} \lim_{j=\infty} \int_0^{2\pi} \varphi'[f(x)](1 \pm \cos 2jx)\, dx = \frac{1}{2\pi} \int_0^{2\pi} \varphi'[f(x)]\, dx,$$

which is in general not zero.

The definition (13), however, leads to something constructive once more. Let it be supposed now that φ has a continuous *third* derivative, and, for simplicity of statement, that $K(x, t)$ has continuous partial derivatives of the first four orders, though not all of these will be used; the function $f(t)$, which serves as independent variable in the definition of the functional $\Phi(x)$, is to have merely a continuous second derivative, as before. Let

$$\psi_i(t) = \int_0^{2\pi} K(x, t)\, u_i(x)\, dx.$$

For each value of t, the quantity $\psi_i(t)$ is the ith coefficient in the expansion of $K(x, t)$ in series of the functions $u_i(x)$. As $K(x, t)$, $K_x(x, t)$, $K_{xx}(x, t)$ are supposed periodic in each variable and continuous in both, they are bounded. If M_1 is a common upper bound for their absolute values, it follows from § 4 that

$$|\psi_i(t)| \leqq \frac{CM_1}{i^2};$$

the factor C is independent of the form of $K(x, t)$ as a function of x, and so, in particular, independent of t. Furthermore,

$$\psi_i'(t) = \int_0^{2\pi} K_t(x, t)\, u_i(x)\, dx, \qquad \psi_i''(t) = \int_0^{2\pi} K_{tt}(x, t)\, u_i(x)\, dx.$$

If M_2 is a common upper bound for the absolute values of the nine functions K, K_x, K_{xx}, K_t, K_{tx}, K_{txx}, K_{tt}, K_{ttx}, K_{ttxx}, it appears more comprehensively that

$$|\psi_i(t)| \leqq \frac{CM_2}{i^2}, \qquad |\psi_i'(t)| \leqq \frac{CM_2}{i^2}, \qquad |\psi_i''(t)| \leqq \frac{CM_2}{i^2},$$

for all values of t and i. From (14),

$$\frac{\partial A_i}{\partial a_j} = \int_0^{2\pi} \varphi'[f(t)]\, \psi_i(t)\, u_j(t)\, dt.$$

Since φ has a continuous third derivative with regard to its argument, while $f(t)$ has a continuous second derivative, $\varphi'[f(t)]$ has a continuous second derivative with regard to t. If M_3 is a common upper bound for the absolute values of $\varphi'[f(t)]$ and of its first and second derivatives with regard to t, and if $\varphi'[f(t)]\, \psi_i(t) = \omega_i(t)$, then

$$|\omega_i(t)| \leqq \frac{CM_2 M_3}{i^2},$$

$$|\omega_i'(t)| \leqq \left|\psi_i(t) \frac{d}{dt}\{\varphi'[f(t)]\}\right| + |\varphi'[f(t)]\, \psi_i'(t)| \leqq \frac{2CM_2 M_3}{i^2},$$

$$|\omega_i''(t)| \leqq \frac{4CM_2 M_3}{i^2}.$$

So $4CM_2M_3/i^2$ is a common upper bound for $|\omega_i(t)|$, $|\omega_i'(t)|$, and $|\omega_i''(t)|$. Consequently, by another application of § 4,

$$\left|\frac{\partial A_i}{\partial a_j}\right| = \left|\int_0^{2\pi} \omega_i(t)\, u_j(t)\, dt\right| \leqq \frac{4C^2 M_2 M_3}{i^2 j^2} = \frac{Q}{i^2 j^2},$$

where the essential property of Q is merely that it is independent of i and j.

The last relation shows incidentally, of course, that $|\partial A_i/\partial a_i| \leqq Q/i^4$, and so that the series (15) is absolutely convergent. But further than that, it means that $\partial A_i/\partial a_j$ is the general term of an absolutely convergent double series. On the other hand, by Schwarz's inequality,

$$\left(\sum_k |p_{ki}\, p_{kj}|\right)^2 \leqq \left(\sum_k p_{ki}^2\right)\left(\sum_k p_{kj}^2\right) = 1,$$

$$\sum_k |p_{ki}\, p_{kj}| \leqq 1,$$

for any values of i and j. So

$$\sum_k \left| p_{ki}\, p_{kj} \frac{\partial A_i}{\partial a_j} \right| \leqq \left| \frac{\partial A_i}{\partial a_j} \right|.$$

Hence it appears that the triple series

$$\sum_i \sum_j \sum_k p_{ki}\, p_{kj} \frac{\partial A_i}{\partial a_j}$$

is absolutely convergent. By (12), the double sum with regard to i and j, for fixed k, is equal to $\partial B_k / \partial b_k$. So the value of the triple series is

$$\sum_k \frac{\partial B_k}{\partial b_k},$$

which is thus seen to be convergent. By (5), on the other hand, the sum with regard to k, for fixed i and j, is 0 if $j \neq i$, and is $\partial A_i / \partial a_i$ if $j = i$, whereby the value of the triple series becomes $\sum_i (\partial A_i / \partial a_i)$. For the case in hand, then, the quantity (15) is defined, *and is invariant under transformation of axes*:

$$\sum_k \frac{\partial B_k}{\partial b_k} = \sum_i \frac{\partial A_i}{\partial a_i}.$$

It corresponds to the divergence in ordinary vector analysis.

It is naturally desirable to express this quantity in a form not explicitly dependent on any particular orthogonal system. The invariance being once established, such an expression is readily obtained. It is sufficient to carry through the calculation for the case $\varrho(x) \equiv 1$, with functions $u_i(x)$ defined by the equations

$$u_i(x) = \frac{1}{\sqrt{2\pi}}, \quad u_{2j}(x) = \frac{\cos jx}{\sqrt{\pi}}, \quad u_{2j+1}(x) = \frac{\sin jx}{\sqrt{\pi}} \quad (j \geqq 1);$$

by the demonstration just completed, any other weight function $\varrho(x)$ satisfying the hypotheses would necessarily lead to the same result.

In general (cf. (14)),

$$\frac{\partial A_i}{\partial a_i} = \int_0^{2\pi} \int_0^{2\pi} K(x, t)\, \varphi'[f(t)]\, u_i(x)\, u_i(t)\, dt\, dx.$$

In the simple case under discussion,

$$\frac{\partial A_1}{\partial a_1} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} K(x, t)\,\varphi'[f(t)]\,dt\,dx,$$

$$\frac{\partial A_{2j}}{\partial a_{2j}} + \frac{\partial A_{2j+1}}{\partial a_{2j+1}} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} K(x, t)\,\varphi'[f(t)]\cos j(x-t)\,dt\,dx.$$

All the functions involved being continuous, the successive integrations may be performed in either order. By integration by parts,

$$\int_0^{2\pi} K(x, t)\cos j(x-t)\,dx = -\frac{1}{j}\int_0^{2\pi} K_x(x, t)\sin j(x-t)\,dx;$$

the absence of another term on the right is due to the fact that $K(x, t)$ and $\sin j(x-t)$ have the period 2π with regard to x. Hence

$$\sum_i \frac{\partial A_i}{\partial a_i} = \frac{\partial A_1}{\partial a_1} - \frac{1}{\pi}\sum_j \int_0^{2\pi}\int_0^{2\pi} K_x(x, t)\,\varphi'[f(t)]\,\frac{\sin j(x-t)}{j}\,dx\,dt.$$

The series $\sum_j [\sin j(x-t)]/j$ is convergent for all values of x and t. For pairs of values (x, t) belonging to the square $0 \leqq x \leqq 2\pi$, $0 \leqq t \leqq 2\pi$, the function

$$\psi(x, t) = \sum_j \frac{\sin j(x-t)}{j}$$

may be alternatively defined by the equations

$$\psi(x, t) = \frac{1}{2}(t - x + \pi) \qquad \text{when } x > t,$$

$$\psi(x, t) = \frac{1}{2}(t - x - \pi) \qquad \text{when } x < t, \tag{16}$$

$$\psi(x, t) = 0 \qquad \text{when } x = t.$$

The series is not uniformly convergent over the square. But it is uniformly convergent over any closed sub-region of the square not containing a point of the line $x = t$; and its partial sums are uniformly bounded everywhere.* So it is readily seen to be integrable term by term, even without reference to theorems of the highest generality on the integration of non-uniformly convergent series. The properties justifying the termwise integration are not affected when the series is multiplied by the continuous factor $K_x(x, t)\,\varphi'[f(t)]$. Consequently

* Cf., e. g., D. Jackson, Über eine trigonometrische Summe, Rend. Circ. Mat. di Palermo, vol. 32 (1911), pp. 257–262.

$$\sum_i \frac{\partial A_i}{\partial a_i} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} K(x, t)\,\varphi'[f(t)]\,dx\,dt$$
$$-\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} K_x(x, t)\,\varphi'[f(t)]\,\Psi(x, t)\,dx\,dt.$$

When $\Psi(x, t)$ is regarded as defined by the equations (16), the right-hand member is intelligible without reference to any particular system of orthogonal functions; *it constitutes an invariant definition of the "divergence" for the case in hand.*

The substitution of $\varphi(t, y, y')$ for $\varphi(y)$ in the definition of $\Phi(x)$ introduces no essential difficulty, though the formulas, including the invariant formula for the divergence, are correspondingly longer. There is occasion incidentally to find an upper bound for the absolute value of

$$\int_0^{2\pi} \bar{\omega}_i(t)\,u_j'(t)\,dt,$$

where $\bar{\omega}_i(t) = \varphi_{y'}(t, y, y')\,\psi_i(t)$; but as an integration by parts gives

$$\int_0^{2\pi} \bar{\omega}_i(t)\,u_j'(t)\,dt = -\int_0^{2\pi} \bar{\omega}_i'(t)\,u_j(t)\,dt,$$

because of the periodicity of the functions, the order of magnitude of the result is seen to be a question merely of the existence of a sufficient number of continuous derivatives.

9. **Analogue of the gradient.** The same order of ideas naturally gives rise to a concept to which the name *gradient* can be applied. But this concept turns out to be long since familiar under another guise, in the case to which one first thinks of applying it, and more recently recognized of the form appropriate to the present discussion.*

Let

$$\Phi = \int_0^{2\pi} \varphi(t, y, y')\,dt, \qquad y = f(t),$$

where φ and f have a sufficient number of continuous derivatives, and $f(t)$ has the period 2π. (It will be more than sufficient if all the derivatives through the fourth order are continuous.) Considered as a functional with $f(x)$ as its argument, this Φ does not depend on x, and has the character, not of a vector, but of a scalar point function in function space. When the functions $u_i(x)$ are looked upon as a system of mutually orthogonal

* Cf. P. Lévy, Leçons d'analyse fonctionnelle, Paris, 1922, pp. 127–128.

unit vectors, the gradient of Φ is defined by formal analogy as the sum of the series

$$\sum_i \frac{\partial \Phi}{\partial a_i} u_i(x). \tag{17}$$

Now

$$\frac{\partial \Phi}{\partial a_i} = \int_0^{2\pi} [\varphi_y(t, y, y')\, u_i(t) + \varphi_{y'}(t, y, y')\, u_i'(t)]\, dt$$

$$= \int_0^{2\pi} \varphi_y(t, y, y')\, u_i(t)\, dt - \int_0^{2\pi} \frac{d}{dt} [\varphi_{y'}(t, y, y')]\, u_i(t)\, dt,$$

the last form resulting from an integration by parts. So $\partial \Phi / \partial a_i$ is the ith coefficient in the expansion of

$$\varphi_y(x, y, y') - \frac{d}{dx} \varphi_{y'}(x, y, y') \tag{18}$$

in series of the u's. The hypotheses being sufficient to insure convergence, the sum of the series is the "functional derivative" (18). As the form of (18) is independent of the choice of the system of u's, no further proof of invariance is needed.

The character of (18) as a gradient is more forcibly brought out by the statement, which may be verified without difficulty, that *the directional derivative of Φ in any direction* (in the space of the class of functions considered) *is equal to the magnitude of the vector* (18), *multiplied by the cosine of the angle between this vector and the direction in which the derivative is taken*, the geometrical terms being of course understood in the light of the definitions appropriate to function space.

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HARMONIC CUBICS*.

By Temple Rice Hollcroft.

1. **Introduction.** The purpose of this paper is to investigate some new properties of harmonic cubics and of the curves associated with them.

The equation of the cubic will be taken in the canonical form

$$x^3+y^3+z^3+6mxyz = 0.$$

It was first proved by Salmon† that the anharmonic ratio of the four tangents drawn from any point of a given cubic is constant. He also proved that the necessary and sufficient condition for this ratio to be harmonic or equianharmonic is that the invariant T or S vanish respectively.‡ To cubics for which $T = 0$ he gave the name harmonic cubics and to those for which $S = 0$, equianharmonic cubics. He proved further that the Hessian of the Hessian of a harmonic cubic is the harmonic cubic itself.

For the canonical form, $T = 0$ becomes

$$8m^6+20m^3-1 = 0$$

so that a cubic of the syzygetic pencil is harmonic when and only when its parameter m is a root of this equation. The six harmonic cubics occur in three pairs. In any pair, either of the cubics is the Hessian of the other.

The dual of Salmon's Theorem is that any tangent to a given class cubic cuts the cubic in four other points whose anharmonic ratio is constant. Also, dually, there are six and only six harmonic class cubics in the one parameter canonical system of class cubics associated with each syzygetic pencil.

The Hessian of a cubic of the syzygetic pencil with the parameter m is a cubic of the same pencil whose parameter is $-(1+2m^3)/6m^2$. Since the Hessian of the Hessian is still another cubic of the pencil, etc., the process of obtaining the Hessian may be considered merely as a transformation of the parameter and may be continued indefinitely and any finite number of distinct cubics of the pencil obtained from one such cubic except in case of the six harmonic cubics, the four equianharmonic cubics and the four degenerate cubics that belong to each syzygetic pencil.

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* Presented to the American Mathematical Society, May 1, 1926.

† Salmon, Higher Plane Curves, 1st ed. (1852), pp. 151–152.

‡ Salmon, loc. cit., 1st ed., pp. 191–193; 3rd ed., pp. 199–200.

2. Parameters. The equation

$$8m^6+20m^3-1=0$$

can be readily solved and the parameters of the six harmonic cubics obtained. This was done in 1871 by Durège.* His results are used in the same form by Wieleitner† in a table devised by Wiener in his classification of cubics. It appears that none of these noted that the irrational expression under the cube root sign is a perfect cube and that the parameters of the six harmonic cubics may therefore be given in the following simpler form:

$$m_1=\frac{1}{2}(\sqrt{3}-1), \qquad m_2=-\frac{1}{2}(\sqrt{3}+1),$$

$$m_3=\frac{\omega}{2}(\sqrt{3}-1), \qquad m_4=-\frac{\omega}{2}(\sqrt{3}+1),$$

$$m_5=\frac{\omega^2}{2}(\sqrt{3}-1), \qquad m_6=-\frac{\omega^2}{2}(\sqrt{3}+1).$$

In these, ω is an imaginary cube root of unity.

The pairs m_1, m_2; m_3, m_4; m_5, m_6 are such that in any pair, each is the parameter of the Hessian of the other. The members of the first pair are real and of the last two, conjugate imaginary.

3. The Cayleyan. The Cayleyan of a cubic with the parameter m is a class cubic in the dual canonical form with the parameter $(1-4m^3)/6m$. A process similar to that of obtaining the Hessian may be defined for the Cayleyan. It may be considered a transformation of the parameter changing $m^{(i)}$ into $m^{(i+1)}$, where

$$m^{(i+1)}=\frac{1-4m^{(i)3}}{6m^{(i)}}$$

with the further provision that each time the transformation is applied the curve is dualized, i. e., point coördinates are exchanged for line coördinates or vice versa. An order or class cubic obtained in this way will be denoted by the symbol (Cayleyan)^n, using the exponent to denote the number of times the transformation has been repeated. Thus the (Cayleyan)^n of an order (class) cubic is a class (order) cubic if n is odd or an order (class) cubic if n is even.

The symbol (Hessian)^n will be used to indicate the cubic which is the Hessian of the Hessian etc. repeated n times of a given cubic.

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* H. Durège, Die ebenen Curven dritter Ordnung (1871), p. 328.

† H. Wieleitner, Theorie der ebenen algebraischen Kurven höherer Ordnung (1905), p. 243.

The following theorem may now be proved:

THEOREM I. *The (Cayleyan)² of a harmonic cubic is the original cubic.* Denote the parameter of a cubic by m and of its Cayleyan by m'. Then

$$m' = \frac{1-4m^3}{6m}.$$

If the (Cayleyan)² is the original cubic, we must have also

$$m = \frac{1-4m'^3}{6m'}.$$

Eliminating m' from the two expressions, there results

$$(8m^3+1)(8m^6+20m^3-1) = 0.$$

This condition in m is identically the same as the condition that the (Hessian)² be the original cubic. It is satisfied by ten cubics, six harmonic and four degenerate. The parameters of but three degenerate cubics appear in the above condition, but the condition (as is also true for the Hessian) is really of the tenth degree in m with one infinite root. If we make the parameters of the cubic and Cayleyan homogeneous of the form $m = a_1/a_2$, this readily appears in that the condition is of degree ten in a_2 and contains a_2 as a factor.

Then for a harmonic cubic and for no other proper cubic the (Hessian)² and the (Cayleyan)² coincide with each other and with the original cubic.

The following further theorem holds for the Cayleyan.

THEOREM II. *The Cayleyan of a harmonic cubic is a harmonic class cubic whose parameter is the parameter of the given cubic muliplied by ω^s where s is the power of ω in the parameter of the given cubic.*

This is proved by substituting the values of the harmonic parameters m_i in the expression $(1-4m_i^3)/6m_i$. In case the parameter of the harmonic cubic is real, $s = 0$ and the Cayleyan has the same parameter as the cubic.

Thus, although the Hessian transformation repeated twice gives the same parameter as the Cayleyan transformation repeated twice, these identical results are reached by different routes. For example, if each application of a transformation is indicated by an arrow, these routes may be represented as follows:

Hessian transformation	*Cayleyan transformation*
$m_1 \to m_2 \to m_1$	$m_1 \to m_1 \to m_1$
$m_3 \to m_4 \to m_3$	$m_3 \to m_5 \to m_3$
$m_6 \to m_5 \to m_6$	$m_6 \to m_4 \to m_6$ etc.

The six harmonic class cubics in the system dual to the syzygetic pencil and the six harmonic cubics of the pencil are thus definitely associated in pairs such that either parameter in a pair is the parameter of the Cayleyan of the cubic defined by the other parameter

4. **Special cubics.** Associated with each harmonic cubic, there exists an indefinite number of cubics none of which are harmonic whose parameters can be found in terms of the parameters of the given harmonic cubic. Each harmonic cubic with the parameter m_i, $i = 1, \cdots, 6$, considered as a Hessian is the Hessian of three distinct cubics only one of which is harmonic. The other two cubics have the parameters

$$m = m_i^2 \pm \sqrt{m_i(m_i^3 - 1)}.$$

These two cubics which have a given harmonic Hessian and are not themselves harmonic are the Hessians of six cubics each of which has the characteristic that its $(\text{Hessian})^2$ is harmonic. These six cubics are the Hessians of eighteen cubics for each of which the $(\text{Hessian})^3$ is harmonic. This process may be continued indefinitely and any finite number of cubics of the pencil obtained for each of which the process of obtaining the Hessian leads to a harmonic cubic and thereafter continuously gives one of two harmonic cubics. There are thus obtained $2 \cdot 3^{n-1}$ cubics of the pencil whose $(\text{Hessian})^n$ is a given harmonic cubic.

Consider a pair of harmonic cubics with parameters m_i and m_{i+1}, each the Hessian of the other. The set of $2 \cdot 3^{n-1}$ cubics whose $(\text{Hessian})^n$ has the parameter m_i does not include any of the $2 \cdot 3^{n-2}$ cubics whose $(\text{Hessian})^{n-1}$ has the parameter m_{i+1} although the $(\text{Hessian})^{n+1}$ of all cubics of the first set has the parameter m_{i+1}. In general, for the same value of n or any two values of n the $2 \cdot 3^{n-1}$ cubics whose $(\text{Hessian})^n$ is one harmonic cubic of a pair has no cubics in common with the set of $2 \cdot 3^{n-1}$ cubics whose $(\text{Hessian})^n$ is the other harmonic cubic of the pair, although the $(\text{Hessian})^{n+1}$ of either set is the $(\text{Hessian})^n$ of the other. Then, considering all six harmonic cubics as Hessians, for any given n there are $4 \cdot 3^n$ distinct cubics of the syzygetic pencil whose $(\text{Hessian})^n$ is harmonic.

All these cubics have definite values of the parameter that can be found in terms of the parameter of the given harmonic cubic for any n as follows: Solve the cubic equation

$$2m^3 + 6m_i m^2 + 1 = 0$$

which defines m as the parameter of a cubic whose Hessian has the harmonic parameter m_i. One of the values of m is the parameter which

together with m_i forms a pair. Removing the factor containing this root from the cubic equation, there results the quadratic equation

$$m^2 - 2m_i^2 m + m_i = 0$$

whose two roots, given above, are the parameters of the two non-harmonic cubics that have this harmonic Hessian. Substitute each of these two roots for m_i in the original cubic equation and solve each of the two resulting cubic equations. Substitute each of these six roots for m_i as before, solve each of these six cubic equations, etc. The solutions of the $(n-1)$st set of $2 \cdot 3^{n-2}$ cubic equations will be the parameters of the $2 \cdot 3^{n-1}$ cubics for each of which the $(\text{Hessian})^n$ is the harmonic cubic whose parameter is m_i.

Similar results hold for the Cayleyan. When the Cayleyan is harmonic with the parameter m_i, one of the three cubics of which it is the Cayleyan is harmonic. The parameters of the other two cubics are the roots of the quadratic equation

$$2m^2 + 2m_i m + a m_i + b = 0$$

wherein

$a = b = 1$ when $i = 1$ or 2, i. e., when m_i is real;
$a = \omega, b = \omega^2$ when $i = 3$ or 4, i. e., when m_i contains ω;
$a = \omega^2, b = \omega$ when $i = 5$ or 6, i. e., when m_i contains ω^2.

As in the case of the Hessian, there are $4 \cdot 3^n$ cubics of the syzygetic pencil for which the $(\text{Cayleyan})^n$ is a harmonic class or order cubic according as n is odd or even. For a given harmonic Cayleyan, the parameters of the $2 \cdot 3^{n-1}$ cubics can be found for any n by solving successive cubic equations.

The set of $4 \cdot 3^n$ cubics whose $(\text{Hessian})^n$ is harmonic and the set of $4 \cdot 3^n$ cubics whose $(\text{Cayleyan})^n$ is harmonic have no cubics in common either for the same or for different values of n. This follows from the results of Section 7 where it will be proved that when the Hessian and the Cayleyan are harmonic they either have a harmonic cubic or no cubic in common. Therefore the two non-harmonic cubics belonging to any harmonic Hessian are distinct from the two non-harmonic cubics belonging to any harmonic Cayleyan.

5. **Equianharmonic cubics.** Considering the four equianharmonic cubics as Hessians, denote their parameters (the roots of the equation $m(m^3 - 1) = 0$) by m'' and find the parameters of the three cubics belonging to each.

For $m'' = 0$, m satisfies the equation $2m^3+1 = 0$.

For m'' any cube root of unity, $m = -m''$ and $\frac{1}{2}m''(1\pm\sqrt{-7})$.

As for harmonic cubics, it can be shown in a similar way that in any syzygetic pencil there are $4\cdot 3^n$ proper cubics for which the $(\text{Hessian})^n$ is equianharmonic or whose $(\text{Hessian})^{n+1}$ is degenerate. Likewise, as before, there are $4\cdot 3^n$ cubics for which the $(\text{Cayleyan})^n$ is an equianharmonic class or order cubic according as n is odd or even. These same $4\cdot 3^n$ cubics may also be considered as a set of proper cubics whose $(\text{Cayleyan})^{n+1}$ is degenerate. (The fact that the number $4\cdot 3^n$ found here is the same as that found for harmonic cubics is merely a coincidence).

6. **Theorems.** In addition to the two theorems given in Section 3, the following theorems define special cubics or properties of cubics of the syzygetic pencil.

THEOREM III. *The condition that the Hessian of a cubic be the cubic itself is satisfied by four cubics, all of which are degenerate.*

These are the cubics with parameter m such that $m = \infty$ and $8m^3+1 = 0$. This theorem has been long known. It is given here with the others for completeness.

THEOREM IV. *The condition that the parameter of a cubic equal the parameter of its Cayleyan is satisfied by four real cubics, two harmonic and two degenerate.*

In addition to being satisfied by $m = \infty$, the condition leads to the equation

$$4m^3+6m^2-1 = 0$$

which is satisfied by $m = -\frac{1}{2}$ and $\frac{1}{2}(-1\pm\sqrt{3})$. This theorem and Theorem II partially overlap.

THEOREM V. *The condition that the parameter of the Hessian of a cubic f be equal to the parameter of the Cayleyan of f is satisfied by six cubics of which two are the two real equianharmonic cubics, two are degenerate and two imaginary.*

The condition leads to the equation

$$m(4m^4-2m^3-m-1) = 0$$

which is satisfied by $m = 0, 1, -\frac{1}{2}\pm\frac{1}{2}\sqrt{-2}$. The condition is also satisfied by $m = \infty$.

THEOREM VI. *The condition that the parameter of the Cayleyan of the Hessian of a cubic f equal the parameter of the Hessian of the dualized Cayleyan of f is satisfied by fourteen distinct cubics of which six are harmonic, four equianharmonic and four degenerate; that is, by all the unique cubics of the syzygetic pencil and by no others.*

Let m' and m'' be the parameters of the Hessian and the Cayleyan of f respectively. Then $m' = -(1+2m^3)/6m^2$ and $m'' = (1-4m^3)/6m$. Now express the condition

$$(1-4m'^3)/6m' = -(1+2m''^3)/6m''^2$$

in terms of m. This condition, in addition to being satisfied by $m = 0$ and $m = \infty$, gives the equation

$$(8m^6+20m^3-1)(m^3-1)(8m^3+1)^2 = 0.$$

This equation is satisfied by all the harmonic, the remaining three equianharmonic, and the remaining three degenerate cubics of the pencil.

Many other theorems similar to the above can be found. In general, they involve equations that can not be solved except by approximation and are not satisfied by any of the fourteen unique cubics except the two degenerate cubics with the parameters $m = -\frac{1}{2}$ and $m = \infty$. These two degenerate cubics are the only cubics of the syzygetic pencil for which the parameters of the (Hessian)^n, the (Cayleyan)^n and the cubic are all equal for all values of n.

The proofs of the following theorems follow readily from the algebraic expression of the statements.

THEOREM VII. *The product of any two parameters $m_i\, m_j$ of harmonic cubics such that $i+j$ is odd, is the parameter of one of the three degenerate cubics whose parameters are finite.*

THEOREM VIII. *The negative sum of the parameters of any harmonic cubic and its Hessian is the parameter of an equianharmonic cubic.*

THEOREM IX. *The product of the parameter m_i of any harmonic cubic and the parameter of its Cayleyan is always the real number $\frac{1}{2}(2 \pm \sqrt{3})$, in which the negative sign holds for i odd and the positive sign for i even.*

THEOREM X. *The sum of the equations of any two of the three equianharmonic cubics whose parameters are cube roots of unity is the equation of a degenerate cubic whose parameter is finite.*

7. **Conditions on a given Hessian and a given Cayleyan.** Assume that an order cubic and a class cubic are given. What is the condition on their parameters that the order cubic be the Hessian and the class cubic the Cayleyan of the same cubic or of the same two cubics? The solution of this problem involves the two equations

$$2m^3+6k_1m^2+1 = 0 \tag{1}$$

$$4m^3+6k_2m-1 = 0. \tag{2}$$

The roots of equation (1) are the parameters of the three cubics whose Hessian has the parameter k_1. The roots of equation (2) are the parameters of the three cubics whose Cayleyan has the parameter k_2.

The discriminant of (1) is

$$8k_1^3+1 = 0$$

and the discriminant of (2) is the same function of k_2. Then if two of the curves that have a given Hessian (Cayleyan) coincide, the Hessian (Cayleyan) consists of three inflectional lines (critic centers) and the cubic whose parameter is a double root of equation (1) is equianharmonic and of equation (2), degenerate.

The condition that equations (1) and (2) have a common root is the resultant

$$24k_1^2k_2^2+4k_1^3+4k_2^3-6k_1k_2+1 = 0.$$

Since this equation is symmetrical in k_1 and k_2, any statement concerning k_1 will hold equally well for k_2, that is, the Hessian and the Cayleyan bear the same relation so far as their parameters are concerned to the one cubic to which both belong. The resultant tells us further that to a given cubic considered as a Hessian there correspond three and only three class cubics that may be associated with it as Cayleyans, and conversely.

In case the Hessian and the Cayleyan are both harmonic cubics, the resultant is satisfied when and only when $k_1 = m_i$ and $k_2 = m_j$, where m_i and m_j are harmonic parameters such that $i+j = 3$ when both parameters are real and $i+j = 9$ when both parameters are imaginary. The cubic common to both is always harmonic.

For example, substitute $k_1 = m_1 = \frac{1}{2}(\sqrt{3}-1)$ in the resultant. The cubic in k_2 has one root $k_2 = m_2 = -\frac{1}{2}(\sqrt{3}+1)$ and the remaining quadratic has the two roots

$$k_2 = \frac{1}{4}\left[7\sqrt{3}-11 \pm 3\sqrt{2(7\sqrt{3}+12)}\right].$$

These two values of k_2 may be considered either the parameters of the Cayleyans of the two non-harmonic cubics whose Hessian has the harmonic parameter m_1 or the parameters of the Hessians of the two non-harmonic cubics whose Cayleyan has the harmonic parameter m_1.

Using the other harmonic parameters in the same way, we can obtain the parameters of all non-harmonic Hessians or Cayleyans that have cubics in common with harmonic Cayleyans or Hessians respectively.

For a given k_2, the discriminant of the resultant considered as a cubic in k_1 is the cube of the discriminant of (2). Eliminate k_2 from this and

the resultant aud we obtain the cube of the discriminant of (1). Then two of the three Cayleyans associated with a given Hessian coincide when and only when both the given Hessian and the Cayleyans are degenerate.

If we put $k_1 = k_2$ in the resultant, we obtain the parameters of the Hessians of the cubics discussed in Theorem V.

All the above statements concerning Hessians hold equally well for Cayleyans.

The condition that equations (1) and (2) have two roots in common is, in addition to the above resultant,

$$2k_1^2 + k_2 = 0.$$

Eliminating k_1 and k_2 from this last condition and from (1) and (2), we obtain

$$(m^3 - 1)(8m^3 + 1) = 0.$$

Then a Hessian and a Cayleyan can have two cubics in common when and only when all the cubics belonging to both are either equianharmonic or degenerate and when both the Hessian and the Cayleyan are degenerate. This may also be shown by eliminating k_1 or k_2 from the above condition and the resultant.

It is evident that (1) and (2) can not have three roots in common. Therefore the three cubics that have a common Hessian can not have a common Cayleyan and conversely.

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